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The Geometry of the Space of Symmetric Bilinear Forms on $\mathbb{R}^2$ with Octagonal Norm

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Abstract. Let $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. It is the two dimensional real predual of Lorentz sequence space. In this paper we classify the smooth points of the unit ball of the space of symmetric bilinear forms on $d_*(1,w)^2$. We also show that the unit sphere of the space of symmetric bilinear forms on $d_*(1,w)^2$ is the disjoint union of the sets of smooth points, extreme points and the set A as follows:

$$S_{\mathcal{L}_s(^2d_*(1,w)^2)} = smB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup A,$$

where the set A consists of $ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ with $(a = b = 0, c = \pm\frac{1}{1+w^2})$, $(a \neq b, ab \geq 0, c = 0)$, $(a = b, 0 < ac, 0 < |c| < |a|)$, $(a \neq |c|, a = -b, 0 < ac, 0 < |c|)$, $(a = \frac{1-w}{1+w}, b = 0, c = \frac{1}{1+w})$, $(a = \frac{1+w+w(w^2-3)c}{1+w^2}, b = \frac{w-1+(1-3w^2)c}{w(1+w^2)}, \frac{1}{2+2w} < c < \frac{1}{(1+w)^2(1-w)}, c \neq \frac{1}{1+2w-w^2})$, $(a = \frac{1+w(1+w)c}{1+w}, b = \frac{-1+(1+w)c}{w(1+w)}, 0 < c < \frac{1}{2+2w})$ or $(a = \frac{1-w(1+w)c}{1+w}, b = \frac{1-(1+w)c}{1+w}, \frac{1}{1+w} < c < \frac{1}{(1+w)^2(1-w)})$.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $extB_E$ and smB_E the sets of extreme and smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous symmetric bilinear form L on the product $E \times E$ such that $P(x) = L(x,x)$ for every $x \in E$. We

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denote by $\mathcal{L}_s(^2E)$ the Banach space of all continuous symmetric bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x,y)|$. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of polynomials on a Banach space, we refer to [7].

In 1998, Choi and the author [3] characterized the smooth points of the unit ball of $\mathcal{P}(^2l_2^2)$ and in 1999, Choi and the author [5] characterized the smooth points of the unit ball of $\mathcal{P}(^2l_1^2)$ and studied smooth polynomials of $\mathcal{P}(^2l_1)$. In 2009, the author [10] classified the smooth symmetric bilinear forms of $\mathcal{L}_s(^2l_\infty^2)$. We refer to ([1], [3–6], [8–19] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. Let $0 < w < 1$ be fixed. We denote the two dimensional real predual of Lorentz sequence space by

$$d_*(1,w)^2 := \{(x,y) \in \mathbb{R}^2 : \|(x,y)\|_{d_*} := \max\{|x|,\ |y|,\ \frac{|x|+|y|}{1+w}\ \}\}.$$

In fact, $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. We will denote by $T((x_1,x_2),(y_1,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ a symmetric bilinear form on $d_*(1,w)^2$. Recently, the author [12] computed the norm of $T \in \mathcal{L}_s(^2d_*(1,w)^2)$ in terms of their real coefficients and determined all the extreme symmetric bilinear forms of the unit ball of $\mathcal{L}_s(^2d_*(1,w)^2)$. In this paper, using results of the previous work [12], we classify the smooth symmetric bilinear forms of the unit ball of the space $\mathcal{L}_s(^2d_*(1,w)^2)$. We also show that the unit sphere $S_{\mathcal{L}_s(^2d_*(1,w)^2)}$ is the disjoint union of the sets of smooth points, extreme points and the set A as follows:

$$S_{\mathcal{L}_s(^2d_*(1,w)^2)} = smB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup A,$$

where A consists of $ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ with $(a = b = 0, c = \pm\frac{1}{1+w^2})$, $(a \neq b, ab \geq 0, c = 0)$, $(a = b, 0 < ac, 0 < |c| < |a|)$, $(a \neq |c|, a = -b, 0 < ac, 0 < |c|)$, $(a = \frac{1-w}{1+w}, b = 0, c = \frac{1}{1+w})$, $(a = \frac{1+w+w(w^2-3)c}{1+w^2}, b = \frac{w-1+(1-3w^2)c}{w(1+w^2)}, \frac{1}{2+2w} < c < \frac{1}{(1+w)^2(1-w)}, c \neq \frac{1}{1+2w-w^2})$, $(a = \frac{1+w(1+w)c}{1+w}, b = \frac{-1+(1+w)c}{w(1+w)}, 0 < c < \frac{1}{2+2w})$ or $(a = \frac{1-w(1+w)c}{1+w}, b = \frac{1-(1+w)c}{1+w}, \frac{1}{1+w} < c < \frac{1}{(1+w)^2(1-w)})$.

2. The Results

Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$ for some reals a,b,c. By substituting $((x_1,y_1),(x_2,y_2))$ in T for $((x_1,y_1),(-x_2,-y_2))$ or $((x_1,-y_1),(x_2,-y_2))$ or $((y_1,x_1),(y_2,x_2))$, we may assume that $|b| \leq a,\ c \geq 0$.

Theorem 2.1.([12, Theorem 2.1]) *Let* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$ *with* $|b| \leq a,\ c \geq 0$. *Then*

$$\|T\| = \max\{bw^2 + 2cw + a,\ a - bw^2,\ (a+b)w + c(1+w^2),\ (a-b)w + c(1-w^2)\}.$$

In fact, we have the following:
Case 1: $b \geq 0$
Subcase 1: $c > a$
If $w \leq \frac{c-a}{c-b}$*, then* $\|T\| = (a+b)w + c(1+w^2)$.
If $w > \frac{c-a}{c-b}$*, then* $\|T\| = bw^2 + 2cw + a$.
Subcase 2: If $c \leq a$, $\|T\| = bw^2 + 2cw + a$.
Case 2: $b < 0$
Subcase 1: $c < |b|$
If $w \leq \frac{c}{|b|}$*, then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a-b)w + c(1-w^2)\}$.
If $w > \frac{c}{|b|}$*, then* $\|T\| = \max\{a - bw^2,\ (a-b)w + c(1-w^2)\}$.
Subcase 2: $c \geq |b|$
If $w \leq \frac{|b|}{c}$*, then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a-b)w + c(1-w^2)\}$.
If $w > \frac{|b|}{c}$*, then* $\|T\| = \max\{bw^2 + 2cw + a,\ (a+b)w + c(1+w^2)\}$.

By Theorem 2.1, if $\|T\| = 1$, then $|a| \leq 1,\ |b| \leq 1,\ |c| \leq \frac{1}{1+w^2}$.

Theorem 2.2. *Let* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2d_*(1,w)^2)$*. Then the following are equivalent:*
(1) $ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ *is a smooth point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$ *;*
(2) $-(ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1))$ *is a smooth point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$*;*
(3) $ax_1x_2 + by_1y_2 - c(x_1y_2 + x_2y_1)$ *is a smooth point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$*;*
(4) $bx_1x_2 + ay_1y_2 + c(x_1y_2 + x_2y_1)$ *is a smooth point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$*.*

Proof. Let $S((x_1,y_1),(x_2,y_2)) := T((u_1,v_1),(u_2,v_2))$ for some $((u_1,v_1),(u_2,v_2)) = ((x_1,y_1),(-x_2,-y_2))$ or $((x_1,-y_1),(x_2,-y_2))$ or $((y_1,x_1),(y_2,x_2))$. Then $S \in \mathcal{L}_s({}^2d_*(1,w)^2)$, and T is smooth if and only if S is smooth. □

Theorem 2.3.([12, Theorem 2.3]) *Let* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2d_*(1,w)^2)$*. Then*
(a) *Let* $w < \sqrt{2}-1$*. Then* T *is an extreme point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$ *if and only if*

$$
\begin{aligned}
T \in \Big\{ & \pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2), \pm\frac{1}{(1+w)^2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)], \\
& \pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2 + x_2y_1)], \\
& \pm\frac{1}{1+w^2}[x_1x_2 - y_1y_2 \pm w(x_1y_2 + x_2y_1)], \pm\frac{1}{1+w^2}[wx_1x_2 - wy_1y_2 \pm (x_1y_2 + x_2y_1)], \\
& \pm\frac{1}{(1+w)^2(1-w)}[(1-w-w^2)x_1x_2 - wy_1y_2 \pm (x_1y_2 + x_2y_1)], \\
& \pm\frac{1}{(1+w)^2(1-w)}[wx_1x_2 - (1-w-w^2)y_1y_2 \pm (x_1y_2 + x_2y_1)]\Big\}.
\end{aligned}
$$

(b) *Let* $w = \sqrt{2}-1$*. Then* T *is an extreme point of* $\mathcal{L}_s({}^2d_*(1,w)^2)$ *if and only if*

$$
T \in \Big\{\pm x_1x_2, \pm y_1y_2, \pm\frac{2+\sqrt{2}}{4}(x_1x_2 + y_1y_2), \pm\frac{1}{2}[x_1x_2 + y_1y_2 \pm (x_1y_2 + x_2y_1)],
$$

$$\pm\frac{\sqrt{2}}{4}[x_1x_2 + y_1y_2 \pm (\sqrt{2}+1)(x_1y_2 + x_2y_1)],$$
$$\pm\frac{\sqrt{2}}{4}[(\sqrt{2}+1)(x_1y_2 - x_2y_1) \pm (x_1y_2 + x_2y_1)]\}.$$

(c) *Let $w > \sqrt{2}-1$. Then T is an extreme point of $\mathcal{L}_s({}^2d_*(1,w)^2)$ if and only if*

$$T \in \{\pm x_1x_2, \pm y_1y_2, \pm\frac{1}{1+w^2}(x_1x_2+y_1y_2), \pm\frac{1}{(1+w)^2}[x_1x_2+y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+w^2}[x_1x_2 - y_1y_2 \pm \frac{1-w}{1+w}(x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{1+w^2}[\frac{1-w}{1+w}(x_1x_2 - y_1y_2) \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{2+2w}[(2+w)x_1x_2 - \frac{1}{w}y_1y_2 \pm (x_1y_2+x_2y_1)],$$
$$\pm\frac{1}{2+2w}[\frac{1}{w}x_1x_2 - (2+w)y_1y_2 \pm (x_1y_2+x_2y_1)]\}.$$

Theorem 2.4. *Let $f \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ and $\alpha = f(x_1x_2), \beta = f(y_1y_2), \gamma = f(x_1y_2+x_2y_1)$.*

(a) *Let $w < \sqrt{2}-1$. Then*

$$\begin{aligned}\|f\| &= \max\{|\alpha|, |\beta|, \frac{1}{1+w^2}|\alpha+\beta|, \frac{1}{(1+w)^2}(|\alpha+\beta|+|\gamma|),\\ &\frac{1}{1+2w-w^2}(|\alpha-\beta|+|\gamma|), \frac{1}{1+w^2}(|\alpha-\beta|+w|\gamma|),\\ &\frac{1}{1+w^2}(w|\alpha-\beta|+|\gamma|), \frac{1}{(1+w)^2(1-w)}(|(1-w-w^2)\alpha - w\beta|+|\gamma|),\\ &\frac{1}{(1+w)^2(1-w)}(|w\alpha-(1-w-w^2)\beta|+|\gamma|)\}.\end{aligned}$$

(b) *Let $w = \sqrt{2}-1$. Then*

$$\begin{aligned}\|f\| &= \max\{|\alpha|, |\beta|, \frac{2+\sqrt{2}}{4}|\alpha+\beta|, \frac{1}{2}(|\alpha+\beta|+|\gamma|), \frac{\sqrt{2}}{4}(|\alpha-\beta|+(\sqrt{2}+1)|\gamma|),\\ &\frac{\sqrt{2}}{4}((\sqrt{2}+1)|\alpha-\beta|+|\gamma|)\}.\end{aligned}$$

(c) *Let $\sqrt{2}-1 < w$. Then*

$$\begin{aligned}\|f\| &= \max\{|\alpha|, |\beta|, \frac{1}{1+w^2}|\alpha+\beta|, \frac{1}{(1+w)^2}(|\alpha+\beta|+|\gamma|),\\ &\frac{1}{1+2w-w^2}(|\alpha-\beta|+|\gamma|), \frac{1}{1+w^2}(|\alpha-\beta|+\frac{1-w}{1+w}|\gamma|),\end{aligned}$$

$$\frac{1}{1+w^2}(\frac{1-w}{1+w}|\alpha-\beta|+|\gamma|), \frac{1}{2+2w}(|(2+w)\alpha-\frac{1}{w}\beta|+|\gamma|),$$
$$\frac{1}{2+2w}(|\frac{1}{w}\alpha-(2+w)\beta|+|\gamma|)\}.$$

Proof. It follows from Theorem 2.3 since

$$\|f\| = \sup\{|f(T)| : T \in extB_{\mathcal{L}_s(^2d_*(1,w)^2)}\}. \qquad \Box$$

Theorem 2.5. *Let* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s(^2d_*(1,w)^2)$ *with* $|b| < a,\ c > 0$. *Let* $S = \{bw^2 + 2cw + a,\ a - bw^2,\ (a+b)w + c(1+w^2),\ (a-b)w + c(1-w^2)\}$. *Then* $T \in smB_{\mathcal{L}_s(^2d_*(1,w)^2)}$ *if and only if there exists a unique* $l \in S$ *such that* $l = 1$.

Proof. ($\Rightarrow$): For $(u_1,v_1),(u_2,v_2) \in S_{d_*(1,w)^2}$, let $\delta_{(u_1,v_1),(u_2,v_2)} \in \mathcal{L}_s(^2d_*(1,w)^2)^*$ such that $\delta_{(u_1,v_1),(u_2,v_2)}(L) = L((u_1,v_1),(u_2,v_2))$ for $L \in \mathcal{L}_s(^2d_*(1,w)^2)$. Then $\|\delta_{(u_1,v_1),(u_2,v_2)}\| \leq 1$. Note that, by Theorem 2.4, $1 = \|\delta_{(1,w),(1,w)}\| = \|\delta_{(1,-w),(1,w)}\| = \|\delta_{(1,w),(w,1)}\| = \|\delta_{(1,-w),(w,1)}\|$. Obviously,

$$\delta_{(1,w),(1,w)}(T) = bw^2 + 2cw + a, \delta_{(1,-w),(1,w)}(T) = a - bw^2,$$
$$\delta_{(1,w),(w,1)}(T) = (a+b)w + c(1+w^2), \delta_{(1,-w),(w,1)}(T) = (a-b)w + c(1-w^2).$$

Hence, if $T \in smB_{\mathcal{L}_s(^2d_*(1,w)^2)}$, then, by Theorem 2.1, there exists a unique $l \in S$ such that $l = 1$.

($\Leftarrow$): Let $f \in \mathcal{L}_s(^2d_*(1,w)^2)^*$ such that $1 = \|f\| = f(T)$ with $\alpha = f(x_1x_2), \beta = f(y_1y_2), \gamma = f(x_1y_2 + x_2y_1)$.

Case 1: $l = bw^2 + 2cw + a = 1$

Then

$$(*)\quad bw^2 + 2cw + a = 1 = a\alpha + b\beta + c\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**)\quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{1}{2wn})(x_1y_2 + x_2y_1)\| \\ &= \|ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{2n})(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{1}{2wn})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\alpha - \frac{1}{2wn}\gamma|$, hence $\alpha = \frac{1}{2w}\gamma$ and $1 \geq f(ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{2n})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\beta - \frac{w}{2}\gamma|$, hence $\beta = \frac{w}{2}\gamma$. It follows that, by $(*)$,

$$1 = a\alpha + b\beta + c\gamma = \frac{\gamma}{2w}(bw^2 + 2cw + a) = \frac{\gamma}{2w}.$$

Therefore, $\alpha = 1, \beta = w^2, \gamma = 2w$, hence $f = \delta_{(1,w),(1,w)}$ is uniquely determined.

Case 2: $l = a - bw^2 = 1$

Then

$$(*)\quad a - bw^2 = 1 = a\alpha + b\beta + c\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**)\quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + (b \pm \frac{1}{nw^2})y_1y_2 + c(x_1y_2 + x_2y_1)\| \\ &= \|ax_1x_2 + by_1y_2 + (c \pm \frac{1}{n})(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + (b \pm \frac{1}{nw^2})y_1y_2 + c(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\alpha + \frac{1}{w^2}\beta|$, hence $\alpha = -\frac{1}{w^2}\beta$ and $1 \geq f(ax_1x_2 + by_1y_2 + (c \pm \frac{1}{n})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\gamma|$, hence $\gamma = 0$. It follows that, by $(*)$,

$$w^2 = w^2(a\alpha + b\beta + c\gamma) = \beta(bw^2 - a) = -\beta.$$

Therefore, $\alpha = 1, \beta = -w^2, \gamma = 0$, hence $f = \delta_{(1,-w),(1,w)}$ is uniquely determined.

Case 3: $l = (a + b)w + c(1 + w^2) = 1$

Then

$$(*)\quad (a + b)w + c(1 + w^2) = 1 = a\alpha + b\beta + c\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**)\quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{w}{n(1 + w^2)})(x_1y_2 + x_2y_1)\| \\ &= \|ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{n(1 + w^2)})(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{w}{n(1+w^2)})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\alpha - \frac{w}{1+w^2}\gamma|$, hence $\alpha = \frac{w}{1+w^2}\gamma$ and $1 \geq f(ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{n(1+w^2)})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\beta - \frac{w}{1+w^2}\gamma|$, hence $\beta = \frac{w}{1+w^2}\gamma$. It follows that, by $(*)$,

$$1 = a\alpha + b\beta + c\gamma = \frac{\gamma}{1 + w^2}((a + b)w + c(1 + w^2)) = \frac{\gamma}{1 + w^2}.$$

Therefore, $\alpha = \beta = w, \gamma = 1 + w^2$, hence $f = \delta_{(1,w),(w,1)}$ is uniquely determined.

Case 4: $l = (a - b)w + c(1 - w^2) = 1$

Then

$$(*)\quad (a - b)w + c(1 - w^2) = 1 = a\alpha + b\beta + c\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**)\quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{w}{n(1 - w^2)})(x_1y_2 + x_2y_1)\| \\ &= \|ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \pm \frac{w}{n(1 - w^2)})(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{w}{n(1-w^2)})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\alpha - \frac{w}{1-w^2}\gamma|$.

Hence $\alpha = \frac{w}{1-w^2}\gamma$ and $1 \geq f(ax_1x_2+(b\pm\frac{1}{n})y_1y_2+(c\pm\frac{w}{n(1-w^2)})(x_1y_2+x_2y_1)) = 1+\frac{1}{n}|\beta+\frac{w}{1-w^2}\gamma|$, hence $\beta = -\frac{w}{1-w^2}\gamma$. It follows that, by $(*)$,

$$1 = a\alpha + b\beta + c\gamma = \frac{\gamma}{1-w^2}((a-b)w + c(1-w^2)) = \frac{\gamma}{1-w^2}.$$

Therefore, $\alpha = w, \beta = -w, \gamma = 1-w^2$, hence $f = \delta_{(1,-w),(w,1)}$ is uniquely determined. □

We are in position to classify the smooth symmetric bilinear forms of the unit ball of $\mathcal{L}_s({}^2d_*(1,w)^2)$.

Theorem 2.6. *Let* $T = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in S_{\mathcal{L}_s({}^2d_*(1,w)^2)}$. *Then* $T \notin smB_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ *if and only if* $(|a| = 1, b = 0 = c)$, $(a = b = 0, c = \pm\frac{1}{1+w^2})$, $(ab \geq 0, c = 0)$, $(a = b, 0 < ac, 0 < |c| \leq |a|)$, $(a = -b, 0 < ac, 0 < |c|)$, $(a = \frac{1-w}{1+w}, b = 0, c = \frac{1}{1+w})$, $(a = \frac{1+w+w(w^2-3)c}{1+w^2}, b = \frac{w-1+(1-3w^2)c}{w(1+w^2)}, \frac{1}{2+2w} \leq c \leq \frac{1}{(1+w)^2(1-w)})$, $(a = \frac{1+w(1+w)c}{1+w}, b = \frac{-1+(1+w)c}{w(1+w)}, 0 < c < \frac{1}{2+2w})$ *or* $(a = \frac{1-w(1+w)c}{1+w}, b = \frac{1-(1+w)c}{1+w}, \frac{1}{1+w} < c < \frac{1}{(1+w)^2(1-w)})$.

Proof. Without loss of generality, we may assume that $|b| \leq a,\ c \geq 0$. Let $T = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in S_{\mathcal{L}_s({}^2d_*(1,w)^2)}$ and let $f \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $1 = \|f\| = f(T)$ with $\alpha = f(x_1x_2), \beta = f(y_1y_2), \gamma = f(x_1y_2 + x_2y_1)$. If $a = 1$, then $T = x_1x_2$. We claim that T is not smooth. Indeed, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = 0, g(x_1y_2 + x_2y_1) = 0, h(x_1x_2) = 1, h(y_1y_2) = 0, h(x_1y_2 + x_2y_1) = w^2$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth. If $a = 0$, then $T = \frac{1}{1+w^2}(x_1y_2+x_2y_1)$. We claim that T is not smooth. Indeed, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 0 = g(y_1y_2), g(x_1y_2 + x_2y_1) = 1 + w^2, h(x_1x_2) = w = h(y_1y_2), h(x_1y_2 + x_2y_1) = 1 + w^2$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth. Suppose that $0 < a < 1$. We will consider the three cases $(c = 0)$ or $(a = |b|, c > 0)$ or $(|b| < a, c > 0)$.

Case 1: c=0

We claim that $b \neq 0$ since if not, then $a = 1$, which is impossible. If $b > 0$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = w^2, g(x_1y_2 + x_2y_1) = 2w, h(x_1x_2) = 1, h(y_1y_2) = w^2, h(x_1y_2 + x_2y_1) = 0$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth. In particular, extreme $\pm\frac{1}{1+w^2}(x_1x_2 + y_1y_2)$ is not smooth. If $b < 0$, then $T = ax_1x_2 - |b|y_1y_2$ and $1 = \|T\| = a + bw^2$. We will show that T is smooth. By Theorem 2.1,

$$(*) \quad a + bw^2 = 1 = a\alpha + b\beta.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**) \quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + (b \pm \frac{1}{nw^2})y_1y_2\| \\ &= \|ax_1x_2 + by_1y_2 \pm \frac{1}{n}(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + (b \pm \frac{1}{nw^2})y_1y_2) = 1 + \frac{1}{n}|\alpha + \frac{1}{w^2}\beta|$, hence $\alpha = -\frac{1}{w^2}\beta$ and $1 \geq f(ax_1x_2 + by_1y_2 \pm \frac{1}{n}(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\gamma|$, hence $\gamma = 0$. It follows that, by $(*)$,

$$1 = a\alpha + b\beta = -\frac{\beta}{w^2}(a - bw^2) = -\frac{\beta}{w^2}.$$

Therefore, $\alpha = 1, \beta = -w^2, \gamma = 0$, hence $f = \delta_{(1,w),(1,w)}$ is uniquely determined.

Case 2: $a = |b|, c > 0$

Then $(a = b, c > 0)$ or $(a = -b, c > 0)$. First suppose that $a = b, c > 0$. If $c > a$, then we claim that T is smooth. By Theorem 2.1,

$$(*) \quad 2aw + c(1 + w^2) = 1 = a\alpha + a\beta + c\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}(**) \quad 1 &= \|(a \pm \frac{1}{n})x_1x_2 + (a \mp \frac{1}{n})y_1y_2 + c(x_1y_2 + x_2y_1)\| \\ &= \|ax_1x_2 + (a \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{n(1+w^2)})(x_1y_2 + x_2y_1)\|.\end{aligned}$$

From $(**)$, $1 \geq f((a \pm \frac{1}{n})x_1x_2 + (a \mp \frac{1}{n})y_1y_2 + c(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\alpha - \beta|$, hence $\alpha = \beta$ and $1 \geq f(ax_1x_2 + (a \pm \frac{1}{n})y_1y_2 + (c \mp \frac{w}{n(1+w^2)})(x_1y_2 + x_2y_1)) = 1 + \frac{1}{n}|\beta - \frac{w}{1+w^2}\gamma|$, hence $\beta = \frac{w}{1+w^2}\gamma$. It follows that, by $(*)$,

$$1 = a\alpha + a\beta + c\gamma = \frac{\gamma}{1+w^2}(2aw + c(1+w^2)) = \frac{\gamma}{1+w^2}.$$

Therefore, $\alpha = w = \beta, \gamma = 1 + w^2$, hence $f = \delta_{(1,w),(1,w)}$ is uniquely determined.

If $c \leq a$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = w^2, g(x_1y_2 + x_2y_1) = 2w, h(x_1x_2) = \frac{1+w^2}{2} = h(y_1y_2), h(x_1y_2 + x_2y_1) = 2w$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth. In particular, extreme $\frac{1}{(1+w)^2}(x_1x_2 + y_1y_2 + (x_1y_2 + x_2y_1))$ is not smooth. Suppose that $a = -b, c > 0$. We claim that T is not smooth.

If $c < |b|, w \leq \frac{c}{|b|}$ and $w \geq \sqrt{2}-1$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = w^2, g(x_1y_2 + x_2y_1) = 2w, h(x_1x_2) = \frac{1-w^2}{2}, h(y_1y_2) = -\frac{(1-w^2)}{2}, h(x_1y_2 + x_2y_1) = 2w$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth.

If $c < |b|, w \leq \frac{c}{|b|}$ and $w < \sqrt{2}-1$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = w, g(y_1y_2) = -w, g(x_1y_2 + x_2y_1) = 1 - w^2, h(x_1x_2) = w - \epsilon, h(y_1y_2) = -(w - \epsilon), h(x_1y_2 + x_2y_1) = 1 - w^2$ for a sufficiently small $\epsilon > 0$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth. In particular, extreme $\frac{1}{1+w^2}(x_1x_2 - y_1y_2 + w(x_1y_2 + x_2y_1))$ is not smooth.

If $c < |b|, w > \frac{c}{|b|}$, then $w > \sqrt{2} - 1$. Let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = w, g(y_1y_2) = -w, g(x_1y_2 + x_2y_1) = 1 - w^2, h(x_1x_2) = w - \epsilon, h(y_1y_2) = -(w-\epsilon,), h(x_1y_2+x_2y_1) = 1-w^2$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) =$

$h(T)$, which implies that T is not smooth. In particular, extreme $\frac{1}{1+w^2}(x_1x_2 - y_1y_2 + \frac{1-w}{1+w}(x_1y_2 + x_2y_1))$ is not smooth.

If $c > |b|, w \leq \frac{|b|}{c}$ and $w < \sqrt{2}-1$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = w, g(y_1y_2) = -w, g(x_1y_2 + x_2y_1) = 1 - w^2, h(x_1x_2) = w - \epsilon, h(y_1y_2) = -(w - \epsilon,), h(x_1y_2 + x_2y_1) = 1 - w^2$ for a sufficiently small $\epsilon > 0$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth.

If $c > |b|, w \leq \frac{|b|}{c}$ and $w \geq \sqrt{2}-1$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = w^2, g(x_1y_2+x_2y_1) = 2w, h(x_1x_2) = \frac{1-w^2}{2}, h(y_1y_2) = -\frac{(1-w^2)}{2}, h(x_1y_2 + x_2y_1) = 2w$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth.

If $c > |b|, w > \frac{|b|}{c}$ and $\frac{|b|}{c} > \frac{1-w}{1+w}$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = 1, g(y_1y_2) = w^2, g(x_1y_2+x_2y_1) = 2w, h(x_1x_2) = \frac{1-w^2}{2}, h(y_1y_2) = -\frac{(1-w^2)}{2}, h(x_1y_2 + x_2y_1) = 2w$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth.

If $c > |b|, w > \frac{|b|}{c}$ and $\frac{|b|}{c} < \frac{1-w}{1+w}$, let $g, h \in \mathcal{L}_s({}^2d_*(1,w)^2)^*$ such that $g(x_1x_2) = w, g(y_1y_2) = w, g(x_1y_2+x_2y_1) = 1+w^2, h(x_1x_2) = w-\epsilon = h(y_1y_2), h(x_1y_2+x_2y_1) = 1 + w^2$ for a sufficiently small $\epsilon > 0$. Theorem 2.4 shows that $\|g\| = 1 = \|h\| = g(T) = h(T)$, which implies that T is not smooth.

If $c = |b|$, then $T = \frac{1}{1+2w-w^2}[x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)]$ is an extreme point, so it is not smooth. Indeed, let $2w < g_\epsilon(x_1y_2 + x_2y_1) < 1 - w^2$ and $g_\epsilon(x_1x_2) = \frac{1+2w-w^2-\gamma}{2}, g_\epsilon(y_1y_2) = -\alpha$. Theorem 2.4 shows that $\|g_\epsilon\| = 1 = g_\epsilon(T)$, which implies that T is not smooth.

Case 3: $|b| < a$ and $c > 0$

Suppose that T is not smooth. If $b \geq 0$, then, by Theorem 2.5, $w = \frac{c-a}{c-b}$ and $a < c$. Then

$$bw^2 + 2cw + a = 1 = (a + b)w + c(1 + w^2).$$

Simple computation shows that $T = (\frac{1-w}{1+w})x_1x_2 + \frac{1}{1+w}(x_1y_2 + x_2y_1)$. Suppose that $b < 0$. If $w \leq \frac{c}{|b|}$, then, by Theorem 2.5,

$$bw^2 + 2cw + a = 1 = (a - b)w + c(1 - w^2).$$

Simple computation shows that $T = (\frac{1+w+w(w^2-3)c}{1+w^2})x_1x_2 + (\frac{w-1+(1-3w^2)c}{w(1+w^2)})y_1y_2 + c(x_1y_2 + x_2y_1)$ for $\frac{1}{2+2w} \leq c < \frac{1}{1+2w-w^2}$. In particular, if $c = \frac{1}{2+2w}$, then extreme $\frac{1}{2+2w}[(2 + w)x_1x_2 - \frac{1}{w}y_1y_2 + (x_1y_2 + x_2y_1)]$ is not smooth.

If $w > \frac{c}{|b|}$, then, by Theorem 2.5,

$$a - bw^2 = 1 = (a - b)w + c(1 - w^2).$$

Simple computation shows that $T = (\frac{1+w(1+w)c}{1+w})x_1x_2 + (\frac{-1+(1+w)c}{w(1+w)})y_1y_2 + c(x_1y_2 + x_2y_1)$ for $0 < c < \frac{1}{2+2w}$.

If $w \leq \frac{|b|}{c}$, then, by Theorem 2.5,

$$bw^2 + 2cw + a = 1 = (a-b)w + c(1-w^2).$$

Simple computation shows that $T = (\frac{1+w+w(w^2-3)c}{1+w^2})x_1x_2 + (\frac{w-1+(1-3w^2)c}{w(1+w^2)})y_1y_2 + c(x_1y_2 + x_2y_1)$ for $\frac{1}{1+2w-w^2} \leq c \leq \frac{1}{(1+w)^2(1-w)}$. In particular, if $c = \frac{1}{(1+w)^2(1-w)}$, then extreme $\frac{1}{(1+w)^2(1-w)}((1-w-w^2)x_1x_2 - wy_1y_2 + (x_1y_2 + x_2y_1))$ is not smooth is not smooth.

If $w > \frac{|b|}{c}$, then, by Theorem 2.5,

$$bw^2 + 2cw + a = 1 = (a+b)w + c(1+w^2).$$

Simple computation shows that $T = (\frac{1-w(1+w)c}{1+w})x_1x_2 + (\frac{1-(1+w)c}{1+w})y_1y_2 + c(x_1y_2 + x_2y_1)$ for $\frac{1}{1+w} < c < \frac{1}{(1+w)^2(1-w)}$. Therefore, it completes the proof. □

We show that the unit sphere $S_{\mathcal{L}_s(^2d_*(1,w)^2)}$ is the disjoint union of three nonempty subsets as follows:

Theorem 2.7.

$$S_{\mathcal{L}_s(^2d_*(1,w)^2)} = smB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup extB_{\mathcal{L}_s(^2d_*(1,w)^2)} \bigcup A,$$

where A consists of $ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ with $(a = b = 0, c = \pm\frac{1}{1+w^2})$, $(a \neq b, ab \geq 0, c = 0)$, $(a = b, 0 < ac, 0 < |c| < |a|)$, $(a \neq |c|, a = -b, 0 < ac, 0 < |c|)$, $(a = \frac{1-w}{1+w}, b = 0, c = \frac{1}{1+w})$, $(a = \frac{1+w+w(w^2-3)c}{1+w^2}, b = \frac{w-1+(1-3w^2)c}{w(1+w^2)}, \frac{1}{2+2w} < c < \frac{1}{(1+w)^2(1-w)}, c \neq \frac{1}{1+2w-w^2})$, $(a = \frac{1+w(1+w)c}{1+w}, b = \frac{-1+(1+w)c}{w(1+w)}, 0 < c < \frac{1}{2+2w})$ *or* $(a = \frac{1-w(1+w)c}{1+w}, b = \frac{1-(1+w)c}{1+w}, \frac{1}{1+w} < c < \frac{1}{(1+w)^2(1-w)})$.

Proof. Note that in the proof of Theorem 2.6 it has shown that every extreme symmetric bilinear form is not smooth. It follows from Theorems 2.3 and 2.6. □

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