KYUNGPOOK Math. J. 56(2016), 777-789
http://dx.doi.org/10.5666/KMJ.2016.56.3.777
pISSN 1225-6951 eISSN 0454-8124


# Oscillation of Second-Order Nonlinear Forced Functional Dynamic Equations with Damping Term on Time Scales

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Abstract. In this paper, we establish some new oscillation criteria for the second-order forced nonlinear functional dynamic equations with damping term

$$(r(t)x^{\Delta}(t))^{\Delta} + q(\sigma(t))x^{\Delta}(t) + p(t)f(x(\tau(t))) = e(t),$$

and

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\Delta}(t) + p(t)f(x(\sigma(t))) = e(t),$$

on a time scale $\mathbb{T}$, where $r(t)$, $p(t)$ and $q(t)$ are real-valued right-dense continuous (rd-continuous) functions [1] defined on $\mathbb{T}$ with $p(t) < 0$ and $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing differentiable function and $\lim_{t\to\infty} \tau(t) = \infty$. No restriction is imposed on the forcing term $e(t)$ to satisfy Kartsatos condition. Our results generalize and extend some pervious results [5, 8, 10, 11, 12] and can be applied to some oscillation problems that not discussed before. Finally, we give some examples to illustrate our main results.

## 1. Introduction

The theory of time scales was introduced by Hilger [4] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary closed subset of the reals. When time scale equals to the reals or to the integers, it represents the classical theories of differential and difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important

* Corresponding Author.
Received November 26, 2013; revised May 23, 2014; accepted September 19, 2014.
2010 Mathematics Subject Classification: 34C10, 34K11, 39A10.
Key words and phrases: Oscillation, forced second order delay dynamic equations, time scales.

applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}^n$ (the space of the harmonic numbers). For an introduction to time scale calculus and dynamic equations, see Bohner and Peterson books [1, 2].

In this paper, we will consider the following second-order nonlinear forced functional dynamic equations with damping term of the form

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\Delta}(t) + p(t)f(x(\sigma(t))) = e(t),\ t \in \mathbb{T},\ t \geq t_0, \tag{1.1}$$

and

$$(r(t)x^{\Delta}(t))^{\Delta} + q(\sigma(t))x^{\Delta}(t) + p(t)f(x(\tau(t))) = e(t),\ t \in \mathbb{T},\ t \geq t_0, \tag{1.2}$$

where $\mathbb{T}$ is an unbounded above time scale with $t_0 \in \mathbb{T}$; $r(t)$, $p(t)$, $q(t)$ and $e(t)$ are real-valued right-dense continuous functions on $\mathbb{T}$ with $p(t) < 0$. The function $\tau : \mathbb{T} \to \mathbb{T}$, satisfies $\tau(t) \to \infty$ as $t \to \infty$ and $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ whenever $x \neq 0$.

By a solution of (1.1) or (1.2), we mean a nontrivial real-valued function $x$ satisfies (1.1) or (1.2) for $t \in \mathbb{T}$. A solution $x$ of (1.1) or (1.2) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq. (1.1) or (1.2) is said to be oscillatory if all of its nonconstant solutions defined for all large $t$ are oscillatory.

Equation (1.1) includes the second order forced nonlinear differential equation

$$(r(t)x^{'}(t))^{'} + q(t)x^{'}(t) + p(t)f(x(t)) = e(t), \tag{1.3}$$

and when $f(x) = |x(t)|^{\nu} sgn\ x(t)$, equation (1.3) takes the form:

$$(r(t)x^{'}(t))^{'} + q(t)x^{'}(t) + p(t)|x(t)|^{\nu} sgn\ x(t) = e(t) \ \ with \ \ \nu > 0, \tag{1.4}$$

which is the forced Emden-Fowler equation.

The oscillatory behavior of equation (1.3) has been studied by many authors. Kartsatos [6, 7] assumed that $e(t)$ is the second derivative of an oscillatory function $h(t)$. Under certain conditions, he found that the forced equation

$$x^{''}(t) + q(t)f(x(t)) = e(t), \tag{1.5}$$

would remain oscillatory if the unforced equation is oscillatory. Later, many authors such as [9, 13] investigate the oscillatory behavior of (1.5) by using Kartsatos technique.

Recently, without imposing the Kartsatos condition $e(t) = h^n(t)$ (the nth derivative of an oscillatory function $h(t)$), authors [8, 10, 11, 12] studied the oscillation of the forced equation. In fact Sun and J. S. Wong [11] obtained some new oscillation criteria for the super linear ($\nu > 1$) equation (1.3) including equation (1.4) without imposing the Kartsatos condition. They say nothing about the oscillation of equation(1.3) with $0 < \nu < 1$. Our purpose in this paper is to investigate the oscillation of equations (1.1) and (1.2), including the results of Sun and Wong [11]. We believe that our approach is simpler and more general than that of Sun and Wong [11]. We are also answer the question in [11] for the oscillation of equation (1.3) when $0 < \nu < 1$.

## 2. Main Results

In this section, we establish some sufficient conditions for the oscillation of equations (1.1) and (1.2). Let $D = \{(t,s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$, $D_0 = \{(t,s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$. We say that the function $H \in C_{rd}(D, \mathbb{R})$ belongs to the class $\Im$, if

($H_1$) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ on $D_0$,

($H_2$) $H$ has a non positive continuous $\Delta$-partial derivative $H^{\Delta_s}(t,s)$ and a non negative continuous second-order $\Delta$-partial derivative $H^{\Delta^2_{s^2}}(t,s)$ with respect to the second variable,

($H_3$) $H^{\Delta_s}(t,t) = 0$, $\lim_{t\to\infty} \frac{H^{\Delta_s}(t,t_0)}{H(t,t_0)} = O(1)$.

**I- Oscillatory behavior of solutions of Eq. (1.1)**:

**Theorem 2.1.** *Assume that there exist two positive constants $c$ and $\nu$ such that either*

$$|f(x)| \geq c|x|^{\nu}, \qquad \nu > 1,$$

*or*

$$|f(x)| \leq c|x|^{\nu}, \qquad 0 < \nu < 1.$$

*If there exists a kernel function $H(t,s)$ satisfying ($H_1$)-($H_3$) such that*

$$[(H^{\Delta_s}(t,s)r(s))^{\Delta_s} - (H(t,\sigma(s))q(s))^{\Delta_s}] \geq 0, \tag{2.1}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) - P(t,s)]\Delta s = \infty, \tag{2.2}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) - P(t,s)]\Delta s = -\infty, \tag{2.3}$$

*where*
$P(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}} [(H^{\Delta_s}(t,s)r(s))^{\Delta_s} - (H(t,\sigma(s))q(s))^{\Delta s}]^{\frac{\nu}{\nu-1}} [cH(t,\sigma(s))|p(s)|]^{\frac{1}{1-\nu}}$,
*then equation (1.1) is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t) > 0$

for $t \geq t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Multiplying Eq. (1.1) by $H(t,\sigma(s))$ for $t \geq t_0$ and integrating from $t_0$ to $t$, we get

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = \int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s +$$

$$\int_{t_0}^{t} H(t,\sigma(s))q(s)x^{\Delta}(s)\Delta s + \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\sigma(s)))\Delta s. \tag{2.4}$$

Using integration by parts two times, we have

$$\int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s = -H(t,t_0)r(t_0)x^{\Delta}(t_0) - \int_{t_0}^{t} H^{\Delta_s}(t,s)r(s)x^{\Delta}(s)\Delta s.$$
$$= -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0) +$$

$$\int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta} x(\sigma(s))\Delta s, \tag{2.5}$$

and

$$\int_{t_0}^{t} H(t,\sigma(s))q(s)x^{\Delta}(s)\Delta s = -H(t,\sigma(t_0))q(t_0)x(t_0) - \int_{t_0}^{t} (H(t,\sigma(s))q(s))^{\Delta_s} x(\sigma(s))\Delta s. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = M(t,t_0) - \int_{t_0}^{t} (H(t,\sigma(s))q(s))^{\Delta_s} x(\sigma(s))\Delta s$$
$$+ \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s + \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\sigma(s)))\Delta s,$$

where $M(t,t_0) = -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0) - H(t,\sigma(t_0))q(t_0)x(t_0)$.

Case I: For $|f(x)| \geq c|x|^{\nu}$, $\nu > 1$ and $p(t) < 0$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leq M(t,t_0) + \int_{t_0}^{t} [((H^{\Delta_s}(t,s)r(s))^{\Delta_s} - (H(t,\sigma(s))q(s))^{\Delta_s})x(\sigma(s)) -$$

$$cH(t,\sigma(s))|p(s)|x^{\nu}(\sigma(s))]\Delta s. \tag{2.7}$$

Set $F(x) = ax - bx^{\nu}$, for $x > 0$, $a \geq 0$, $b > 0$. If $\nu > 1$, then $F(x)$ has the maximum $F_{max} = (\nu - 1)\nu^{\frac{\nu}{1-\nu}} a^{\frac{\nu}{\nu-1}} b^{\frac{1}{1-\nu}}$, (see [3]). From (2.7), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leq M(t,t_0) + \int_{t_0}^{t} P(t,s)\Delta s,$$

where
$P(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}[(H^{\Delta_s}(t,s)r(s))^{\Delta_s} - (H(t,\sigma(s))q(s))^{\Delta s}]^{\frac{\nu}{\nu-1}}[cH(t,\sigma(s))|p(s)|]^{\frac{1}{1-\nu}}$.

Consequently,

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,\sigma(s))e(s) - P(t,s)]\Delta s \leq \frac{M(t,t_0)}{H(t,t_0)}.$$

Taking lim sup as $t \to \infty$, we get a contradiction to (2.2).

Case II: For $|f(x)| \leq c|x|^{\nu}$, $0 < \nu < 1$ and $p(t) < 0$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geq M(t,t_0)+\int_{t_0}^{t}[((H^{\Delta_s}(t,s)r(s))^{\Delta_s}-(H(t,\sigma(s))q(s))^{\Delta_s})x(\sigma(s))-$$

$$cH(t,\sigma(s))|p(s)|x^{\nu}(\sigma(s))]\Delta s. \tag{2.8}$$

Set $F(x) = ax - bx^{\nu}$, for $x > 0$, $a \geq 0$, $b > 0$. If $0 < \nu < 1$, then $F(x)$ has the minimum $F_{min} = (\nu-1)\nu^{\frac{\nu}{1-\nu}}a^{\frac{\nu}{\nu-1}}b^{\frac{1}{1-\nu}}$, (see [3]). From (2.8), we have

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,\sigma(s))e(s) - P(t,s)]\Delta s \geq \frac{M(t,t_0)}{H(t,t_0)}.$$

Taking lim inf as $t \to \infty$, we get a contradiction to (2.3).
The proof is completed. □

**Remark.** Theorem 2.2 of [8] and Theorem 3.4 of [12] are special cases of our Theorem 2.1 when $q(t) = 0$ and $r(t) = 1$.

### II - Oscillatory behavior of solutions of Eq. (1.2):

In the following, we establish oscillation criteria for Eq. (1.2) when $\tau(t) \leq t$ and $\tau(t) \geq t$. The oscillation of this equation does not discussed before.

**Theorem 2.2.** *Assume that $\tau(t) \leq t$ and there exist two positive constants $c$ and $\nu$ such that*

$$|f(x)| \leq c|x|^{\nu}, \qquad 0 < \nu < 1.$$

*If there exists a kernel function $H(t,s)$ satisfying $(H_1) - (H_3)$ such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geq 0, \qquad (H(t,s)q(s))^{\Delta s} \leq 0, \tag{2.9}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{t_0}^{\tau(t)} Q(t,s)\Delta s] = \infty, \tag{2.10}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{t_0}^{\tau(t)} Q(t,s)\Delta s] = -\infty, \tag{2.11}$$

*where* $Q(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}+$ $|(H(t,s)q(s))^{\Delta s}|]^{\frac{\nu}{\nu-1}}[cH(t,\sigma(\tau^*(s)))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}]^{\frac{1}{1-\nu}}$, $\tau^*(t)$ *and* $\sigma^*(t)$ *are the inverse functions of* $\tau(t)$ *and* $\sigma(t)$ *respectively, then equation (1.2) is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1.2). Suppose that $x(t) > 0$ for $t \geq t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Multiplying Eq. (1.2) by $H(t,\sigma(s))$ for $t \geq t_0$ and integrating from $t_0$ to $t$, we get

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = \int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s+$$
$$\int_{t_0}^{t} H(t,\sigma(s))q(\sigma(s))x^{\Delta}(s)\Delta s + \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s. \tag{2.12}$$

Using integration by parts two times, we have

$$\int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s = -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0)$$
$$+\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s, \tag{2.13}$$

and

$$\int_{t_0}^{t} H(t,\sigma(s))q(\sigma(s))x^{\Delta}(s)\Delta s = -H(t,t_0)q(t_0)x(t_0) - \int_{t_0}^{t}(H(t,s)q(s))^{\Delta_s}x(s)\Delta s. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = N(t,t_0) - \int_{t_0}^{t}(H(t,s)q(s))^{\Delta_s}x(s)\Delta s+$$
$$\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s + \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s,$$

where $N(t,t_0) = -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0) - H(t,t_0)q(t_0)x(t_0)$.

Since $|f(x)| \leq c|x|^{\nu}$, $0 < \nu < 1$ and $p(t) < 0$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geq N(t,t_0) + \int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s-$$
$$\int_{t_0}^{t}(H(t,s)q(s))^{\Delta_s}x(s)\Delta s - \int_{t_0}^{t} cH(t,\sigma(s))|p(s)|x^{\nu}(\tau(s))\Delta s. \tag{2.15}$$

Since

$$\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s=\int_{\sigma(t_0)}^{\sigma(t)}(H^{\Delta}(t,\sigma^*(\theta))r(\sigma^*(\theta)))^{\Delta}(\sigma^*(\theta))^{\Delta}x(\theta)\Delta\theta,$$

and

$$\int_{t_0}^{t}cH(t,\sigma(s))|p(s)|x^{\nu}(\tau(s))\Delta s=\int_{\tau(t_0)}^{\tau(t)}cH(t,\sigma(\tau^*(\xi)))|p(\tau^*(\xi))|(\tau^*(\xi))^{\Delta}x^{\nu}(\xi)\Delta\xi,$$

then (2.15), becomes

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geq N(t,t_0)+\int_{\sigma(t_0)}^{\sigma(t)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}x(s)\Delta s+$$

$$\int_{t_0}^{t}|(H(t,s)q(s))^{\Delta_s}|x(s)\Delta s-c\int_{\tau(t_0)}^{\tau(t)}H(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)\Delta s. \tag{2.16}$$

Since $\tau(t)\leq t\leq\sigma(t)$, we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geq K(t,t_0)+\int_{t_0}^{\tau(t)}[((H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}+$$

$$|H(t,s)q(s))^{\Delta_s}|)x(s)-cH(t,\sigma(\tau^*(s)))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)]\Delta s, \tag{2.17}$$

where

$$K(t,t_0)=N(t,t_0)-\int_{t_0}^{\sigma(t_0)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}x(s)\Delta s-$$

$$c\int_{\tau(t_0)}^{t_0}H(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)\Delta s.$$

Set $F(x)=ax-bx^{\nu}$, for $x>0$, $a\geq 0$, $b>0$. If $0<\nu<1$, then $F(x)$ has the minimum $F_{min}=(\nu-1)\nu^{\frac{\nu}{1-\nu}}a^{\frac{\nu}{\nu-1}}b^{\frac{1}{1-\nu}}$. From (2.17), we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geq K(t,t_0)+\int_{t_0}^{\tau(t)}Q(t,s)\Delta s.$$

i.e.,

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s-\int_{t_0}^{\tau(t)}Q(t,s)\Delta s\geq K(t,t_0). \tag{2.18}$$

Thus, multiplying (2.18) by $H^{-1}(t,t_0)$ and taking the Lower limit as $t\to\infty$, we get a contradiction with (2.11). The proof is completed. □

**Theorem 2.3.** *Assume that $\tau(t)\geq t$ and there exist two positive constants $c$ and $\nu$ such that*

$$|f(x)|\leq c|x|^{\nu}, \qquad 0<\nu<1.$$

*If there exists a kernel function $H(t,s)$ satisfying $(H_1)$-$(H_3)$ such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\geq 0, \qquad ((H(t,s)q(s))^{\Delta s}\leq 0, \tag{2.19}$$

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_t^{\tau(t)} cH(t,\sigma(\tau^*(s)))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}s^{\nu k}\Delta s<\infty, \tag{2.20}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s+\int_{\tau(t_0)}^{t} Q(t,s)\Delta s]=\infty, \tag{2.21}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s-\int_{\tau(t_0)}^{t} Q(t,s)\Delta s]=-\infty, \tag{2.22}$$

*where $Q(t,s)$, $\tau^*(t)$ and $\sigma^*(t)$ are the same as in Theorem 2.2, then all solutions of equation (1.2) satisfying $x(t)=O(t^k)$ are oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1.2). Suppose that $x(t)>0$ for $t\geq t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 2.2 to get (2.16), i.e.,

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geq N(t,t_0)+\int_{\sigma(t_0)}^{\sigma(t)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}x(s)\Delta s+$$
$$\int_{t_0}^{t}|(H(t,s)q(s))^{\Delta_s}|x(s)\Delta s-c\int_{\tau(t_0)}^{\tau(t)}H(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)\Delta s.$$

Since $\tau(t)\geq t$, we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geq L(t,t_0)+\int_{\tau(t_0)}^{t}[((H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}+$$
$$|(H(t,s)q(s))^{\Delta_s})|x(s)-cH(t,\sigma(\tau^*(s)))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)]\Delta s-$$
$$\int_t^{\tau(t)}cH(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s}x^{\nu}(s)\Delta s,$$

where

$$L(t,t_0)=N(t,t_0)+\int_{\sigma(t_0)}^{\tau(t_0)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta_s}x(s)\Delta s.$$

From Theorem 2.2, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geq L(t,t_0) + \int_{\tau(t_0)}^{t} Q(t,s)\Delta s - \int_{t}^{\tau(t)} cH(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s} x^{\nu}(s)\Delta s.$$

Since $x(t) \leq Mt^k$ for some constant $M > 0$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\tau(t_0)}^{t} Q(t,s)\Delta s \geq L(t,t_0) - cM^{\nu} \int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s))|p(\tau^*(s))|(\tau^*(s))^{\Delta_s} s^{\nu k}(s)\Delta s. \tag{2.23}$$

Thus, multiplying (2.23) by $H^{-1}(t,t_0)$ and taking the Lower limit as $t \to \infty$, we get a contradiction with (2.22). The proof is completed. □

## 3. Examples

**Example 3.1.** Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$-x^{''}(t) + x^{'}(t) - t^m x^{\nu}(t) = t^{\alpha}\cos t, \tag{3.1}$$

where $m \geq 0$, $\alpha > 0$ and $0 < \nu < 1$. Here, $r(t) = -1$, $q(t) = 1$, $p(t) = -t^m$, $f(x) = x^{\nu}$, $0 < \nu < 1$ with $c = 1$ and $e(t) = t^{\alpha}\cos t$. To apply Theorem 2.1, take $H(t,s) = (t-s)$. Therefore, we have

$$[(H^{'}(t,s)r(s))^{'} - (H(t,s)q(t))^{'}] = 1 > 0.$$

Since $P(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}(t-s)^{\frac{1}{1-\nu}} s^{\frac{m}{1-\nu}}$, then

$$\begin{aligned}\int_0^t P(t,s)\,ds &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} \int_0^t (t-s)^{\frac{1}{1-\nu}} s^{\frac{m}{1-\nu}}\,ds \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} t^{\frac{m+1}{1-\nu}+1} \int_0^1 (1-u)^{\frac{1}{1-\nu}} u^{\frac{m}{1-\nu}}\,du \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} B(\frac{1}{1-\nu}+1, \frac{m}{1-\nu}+1) t^{\frac{m+1}{1-\nu}+1},\end{aligned}$$

where $B(\frac{1}{1-\nu}+1, \frac{m}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)s^{\alpha}\cos s\,ds = t^{\alpha+2}\int_0^1 (1-u)u^{\alpha}\cos ut\,du = t^{\alpha+2} I_{1,\alpha}(t),$$

where $I_{1,\alpha}(t)$ has the asymptotic formula

$$I_{1,\alpha}(t) = \Gamma(2)t^{-2}\cos(t-\pi) + o(t^{-2}) \text{ as } t \to \infty.$$

Consequently, Eq. (3.1) is oscillatory if $\alpha > \frac{m+1}{1-\nu} + 1$.

**Remark.** The result of [11] can not be applied to equation (3.1) for $r(t) = -1 < 0$ and $0 < \nu < 1$. But, according to Theorem 2.1, when $\mathbb{T} = \mathbb{R}$ and $H(t,s) = (t-s)$, this equation is oscillatory.

**Example 3.2.** Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$(tx^{'}(t))^{'} + x^{'}(t) - t^m x^{\nu}(t) = t^{\alpha}\cos t, \tag{3.2}$$

where $m > 0$, $\alpha > 0$ and $0 < \nu < 1$. Here, $r(t) = t$, $q(t) = 1$, $p(t) = -t^m$, $f(x) = x^{\nu}$, $0 < \nu < 1$ with $c = 1$ and $e(t) = t^{\alpha}\cos t$. To apply Theorem 2.1, take $H(t,s) = (t-s)^{\beta}$ with $\beta > 1$. Therefore, we have

$$[(H^{'}(t,s)r(s))^{'} - (H(t,s)q(t))^{'}] = \beta(\beta-1)(t-s)^{\beta-2}s > 0.$$

Since $P(t,s) = (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}(t-s)^{\frac{\beta+(2-\beta)\nu}{1-\nu}}s^{\frac{m-\nu}{1-\nu}}$, then

$$\begin{aligned}\int_0^t P(t,s)\,ds &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}\int_0^t (t-s)^{\frac{\beta+(2-\beta)\nu}{1-\nu}}s^{\frac{m-\nu}{1-\nu}}\,ds\\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}t^{\frac{m+\beta+(1-\beta)\nu}{1-\nu}+1}\int_0^1 (1-u)^{\frac{\beta+(2-\beta)\nu}{1-\nu}}u^{\frac{m-\nu}{1-\nu}}\,du\\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}B(\frac{\beta+(2-\beta)\nu}{1-\nu}+1, \frac{m-\nu}{1-\nu}+1)t^{\frac{m+\beta+(1-\beta)\nu}{1-\nu}+1},\end{aligned}$$

where $B(\frac{\beta+(2-\beta)\nu}{1-\nu}+1, \frac{m-\nu}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)^{\beta}s^{\alpha}\cos s\,ds = t^{\beta+\alpha+1}\int_0^1 (1-u)^{\beta}u^{\alpha}\cos ut\,du = t^{\beta+\alpha+1}I_{\beta,\alpha}(t),$$

where $I_{\beta,\alpha}(t)$ has the asymptotic formula

$$I_{\beta,\alpha}(t) = \Gamma(\beta+1)t^{-\beta-1}\cos(t-(\beta+1)\frac{\pi}{2}) + o(t^{-\beta-1}) \text{ as } t \to \infty.$$

Consequently, Eq. (3.2) is oscillatory if $\alpha > \frac{m+\beta+(1-\beta)\nu}{1-\nu} + 1$.

**Example 3.3.** Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$x^{''}(t) - t^m x^{\nu}(t-\tau) = t^{\alpha}\cos t, \tag{3.3}$$

where $m \geq 0$, $\alpha > 0$ and $0 < \nu < 1$. Here, $r(t) = 1$, $q(t) = 0$, $p(t) = -t^m$, $f(x) = x^{\nu}$, $0 < \nu < 1$ with $c = 1$ and $e(t) = t^{\alpha}\cos t$. To apply Theorem 2.2, take $H(t,s) = (t-s)^{\beta}$ with $\beta > 1$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'} = \beta(\beta-1)(t-s)^{\beta-2} > 0, \qquad (H(t,s)q(t))^{'} = 0.$$

Since $Q(t,s) = (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}(t-s)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s-\tau)^{\frac{\beta}{1-\nu}}(s+\tau)^{\frac{m}{1-\nu}}$, then

$$\begin{aligned}\int_0^{t-\tau} Q(t,s)\, ds &= (\nu-1)\frac{\nu}{\beta(\beta-1)}^{\frac{\nu}{1-\nu}} \int_\tau^t (t-s+\tau)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s)^{\frac{\beta}{1-\nu}} s^{\frac{m}{1-\nu}}\, ds \\ &\geq (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} t^{\frac{(2-\beta)\nu}{1-\nu}} \int_0^t (t-s)^{\frac{\beta}{1-\nu}} s^{\frac{m}{1-\nu}}\, ds \\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} t^{\frac{m+\beta+(2-\beta)\nu}{1-\nu}+1} \int_0^1 (1-u)^{\frac{\beta}{1-\nu}} u^{\frac{m}{1-\nu}}\, du \\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} B(\frac{\beta}{1-\nu}+1, \frac{m}{1-\nu}+1) t^{\frac{m+\beta+(2-\beta)\nu}{1-\nu}+1},\end{aligned}$$

where $B(\frac{\beta}{1-\nu}+1, \frac{m}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)^\beta s^\alpha \cos s\, ds = t^{\beta+\alpha+1}\int_0^1 (1-u)^\beta u^\alpha \cos ut\, du = t^{\beta+\alpha+1} I_{\beta,\alpha}(t),$$

where $I_{\beta,\alpha}(t)$ has the asymptotic formula

$$I_{\beta,\alpha}(t) = \Gamma(\beta+1)t^{-\beta-1}\cos(t-(\beta+1)\frac{\pi}{2}) + o(t^{-\beta-1}) \text{ as } t \to \infty.$$

Consequently, Eq. (3.3) is oscillatory if $\alpha > \frac{m+\beta+(2-\beta)\nu}{1-\nu}+1$.

**Remark.** The results of [8, 10, 11, 12] can not be applied to equation (3.3) for $\tau(t) \neq t$ ($\tau(t) \leq t$). But, according to Theorem 2.2, when $\mathbb{T} = \mathbb{R}$ and $H(t,s) = (t-s)^\beta$ with $\beta > 1$, this equation is oscillatory.

**Example 3.4.** Consider the equation ($\mathbb{T} = \mathbb{R}$)

$$(-tx'(t))' - t^5 x^\nu(t) = t^\alpha \cos t, \tag{3.4}$$

where $\alpha > 0$ and $0 < \nu < 1$. Here, $r(t) = -t$, $q(t) = 0$, $p(t) = -t^5$, $f(x) = x^\nu$, $0 < \nu < 1$ with $c = 1$ and $e(t) = t^\alpha \cos t$. To apply Theorem 2.1, take $H(t,s) = (t-s)$. Therefore, we have

$$(H'(t,s)r(s))' = 1 > 0, \qquad (H(t,s)q(t)' = 0.$$

Since $P(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}(t-s)^{\frac{1}{1-\nu}} s^{\frac{5}{1-\nu}}$, then

$$\begin{aligned}\int_0^t P(t,s)ds &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} \int_0^t (t-s)^{\frac{1}{1-\nu}} s^{\frac{5}{1-\nu}} ds \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} t^{\frac{6}{1-\nu}+1}\int_0^1 (1-u)^{\frac{1}{1-\nu}} u^{\frac{5}{1-\nu}} du \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} B(\frac{1}{1-\nu}+1, \frac{5}{1-\nu}+1) t^{\frac{6}{1-\nu}+1},\end{aligned}$$

where $B(\frac{1}{1-\nu}+1, \frac{5}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)s^\alpha \cos s\, ds = t^{\alpha+2}\int_0^1 (1-u)u^\alpha \cos ut\, du = t^{\alpha+2} I_{1,\alpha}(t),$$

where $I_{1,\alpha}(t)$ has the asymptotic formula

$$I_{1,\alpha}(t) = \Gamma(2)t^{-2}\cos(t-\pi) + o(t^{-2}) \text{ as } t \to \infty.$$

Consequently, Eq. (3.4) is oscillatory if $\alpha > \frac{6}{1-\nu}+1$.

**Remark.** The results of [8, 10, 12] can not be applied to equation (3.4) for $r(t) = -t < 0$. But, according to Theorem 2.1, when $\mathbb{T} = \mathbb{R}$ and $H(t,s) = (t-s)$, this equation is oscillatory.

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