Uniqueness of Entire Functions that Share an Entire Function of Smaller Order with One of Their Linear Differential Polynomials

XIAO-MIN LI*

Department of Mathematics, Ocean University Of China, Qingdao, Shandong 266100, P. R. China
e-mail : xmli01267@gmail.com

HONG-XUN YI

Department of Mathematics, Shandong University, Jinan, Shandong 250100, P. R. China
e-mail : hxyi@sdu.edu.cn

ABSTRACT. We prove a uniqueness theorem of entire functions sharing an entire function of smaller order with their linear differential polynomials. The results in this paper improve the corresponding results given by Gundersen-Yang[4], Chang-Zhu[3], and others. Some examples are provided to show that the results in this paper are best possible.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in Nevanlinna theory of meromorphic functions as explained, e.g., in [5], [7] and [11]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f - a$ and $g - a$ have

the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see[12]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (see [12]). In this paper, we also need the following definition.

Definition 1.1. For a nonconstant entire function f, the order $\sigma(f)$, lower order $\mu(f)$ and hyper-order $\sigma_2(f)$ are defined as

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} = \limsup_{r \to \infty} \frac{\log\log M(r,f)}{\log r},$$

$$\mu(f) = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r} = \liminf_{r \to \infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r \to \infty} \frac{\log\log\log M(r,f)}{\log r},$$

where and what follows, $M(r,f) = \max_{|z|=r}\{|f(z)|\}$.

In 1977, Rubel-Yang [8] proved that if an entire function f shares two distinct finite complex numbers CM with its derivative f', then $f = f'$. How is the relation between f and f', if an entire function f shares one finite complex number a CM with its derivative f' ? In 1996, Brück [2] made a conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f) < \infty$ is the hyper-order of f such that $\sigma_2(f)$ is not a positive integer, and if f and f' share one finite complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. For the case that $a = 0$, the above conjecture had been proved by Brück [2]. Brück [2] also proved that the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r,f)$ without any growth restriction. In 2005, Al-Khaladi [1] showed that the conjecture remains true for meromorphic functions f such that $N(r, 1/f') = S(r,f)$. But the conjecture is still an open question by now. In this direction, we recall the following result proved by Gundersen-Yang [4], which shows that the above conjecture is true for $a \neq 0$, provided that f satisfies the additional assumption $\sigma(f) < \infty$:

Theorem A ([4], Theorem 1) *Let f be a nonconstant entire function of finite order, and let $a \neq 0$ be a finite complex number. If f and f' share a CM, then $f' - a = c(f - a)$, for some nonzero constant c.*

Later on, Chang-Zhu [3] proved the following result to improve Theorem A:

Theorem B ([3], Theorem 1) *Let f be an entire function such that $\sigma(f) < \infty$, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $f' - a$ share 0 CM, then $f' - a = c(f - a)$ for some nonzero constant c.*

Consider the following linear differential polynomial related to f

$$(1.1) \qquad L[f] = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f,$$

where and what follows, k is a positive integer and $a_0, a_1, \cdots a_{k-1}$ are complex numbers.

We will prove the following result to improve Theorems A and B:

Theorem 1.2. *Let f be a nonconstant entire function such that $\sigma(f) < \infty$, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $L[f] - a$ share 0 CM, where $L[f]$ is defined as in (1.1), then $\sigma(f) = 1$ and one of the following two cases will occur:*

  (i) $L[f] - a = c(f - a)$, *where c is some nonzero constant*

  (ii) f *is a solution of the equation* $L[f] - a = (f - a)e^{p_1 z + p_0}$ *such that* $\sigma(f) = \mu(f) = 1$, *where not all* $a_0, a_1, \cdots, a_{k-1}$ *are zeros, $p_1 \neq 0$ and p_0 are complex numbers.*

From Theorem 1.2 we get the following corollary:

Corollary 1.3. *Let f be a nonconstant entire function such that $\sigma(f) < \infty$, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $f^{(k)} - a$ share 0 CM, where k is a positive integer, then f is a solution of $f^{(k)} - a = c(f - a)$ such that $\sigma(f) = 1$, where c is some nonzero constant.*

Proceeding as in the proof of Theorem 1.2 in Section 3 of this paper, we get the following theorem:

Theorem 1.4. *Let f be a nonconstant entire function such that $\sigma(f) < \infty$, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \mu(f)$. If $f - a$ and $L[f] - a$ share 0 CM, where $L[f]$ is defined as in (1.1), then $\sigma(f) = \mu(f) = 1$ and one of the conclusions (i)-(ii) of Theorem 1.2 still holds.*

From Theorem 1.4 we get the following corollary:

Corollary 1.5. *Let f be a nonconstant entire function such that $\sigma(f) < \infty$, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \mu(f)$. If $f - a$ and $f^{(k)} - a$ share 0 CM, where k is a positive integer, then f is a solution of $f^{(k)} - a = c(f - a)$ such that $\sigma(f) = \mu(f) = 1$, where c is some nonzero constant.*

Example 1.6. Let $f = 1 - 2e^z$ and $L[f] = f' - f$. Then $\mu(f) = \sigma(f) = 1$ and $L[f](z) - 1 = (f(z) - 1) \cdot e^{-z}$. This example shows that the conclusion (ii) of Theorem 1.2 may occur.

Example 1.7.([3]) Let $f(z) = e^{2z} - (z - 1)e^z$ and $a(z) = e^{2z} - ze^z$. Then $f - a$ and $f' - a$ share 0 CM and $\mu(f) = \sigma(f) = \sigma(a) = \mu(f) = 1$, but $f'(z) - a(z) = (f(z) - a(z))e^z$. This example shows that the condition "$\sigma(a) < \sigma(f)$" in Corollary 1.3 and the condition "$\sigma(a) < \mu(f)$" in Corollary 1.5 are best possible.

In 1995, Yi-Yang[12] posed the following question:

Question 1.8. ([12], p.398) Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex constant. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM,

where n and $m\,(n < m)$ are distinct positive integers not all even or odd, then can we get the result $f = f^{(n)}$?

Regarding Question 1.8, Gundersen-Yang [4] proved the following result:

Theorem E ([4], Theorem 2) *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite complex number, and let n be a positive integer. If a is shared by f, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f = f'$.*

We will prove the following result to improve and complement Theorem E:

Theorem 1.9. *Let f and a be two nonconstant entire functions such that $\sigma(a) < \sigma(f) < \infty$. If $f - a$, $f' - a$ and $L[f] - a$ share 0 CM, where $L[f]$ is defined as in (1.1), then one of the following four cases will occur:*

  (i) $f(z) = \gamma_1 e^z$, *where $\gamma_1 \neq 0$ is a constant.*

  (ii) $f(z) = \gamma_2 e^{cz} - [a(1-c)]/c$, *where $n \geq 2$, $\gamma_2 \neq 0$ is a constant.*

  (iii) $L[f] = f'$ *and* $f' - a = c(f - a)$, *where c is some nonzero constant.*

  (iv) $f(z) = \gamma_3 e^{cz}$, *where γ_3 and c are two nonzero constants.*

2. Preliminaries

In this section, we introduce some important results that will be used to prove the main results in this paper. First of all we introduce Wiman-Valiron theory. For this purpose, we first introduce the following notions: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Next we define by $\mu(r) = \max\{|a_n|r^n : n = 0, 1, 2, \cdots\}$ the maximum term of f, and define by $\nu(r, f) = \max\{m : \mu(r) = |a_m|r^m\}$ the central index of f, see, e.g., the reference [7, p.50].

Lemma 2.1. ([7], Corollary 2.3.4) *Let f be a transcendental meromorphic function and k be a positive integer. Then $m(r, f^{(k)}/f) = O\{\log(rT(r, f)\}$, outside of a possible exceptional set E of finite linear measure, and if f is of finite order of growth, then $m(r, f^{(k)}/f) = O(\log r)$.*

Lemma 2.2. ([6], Satz 4.5) *Let f be an entire function of infinite order, with the lower order $\mu(f)$ and order $\sigma(f)$. Then $\mu(f) = \liminf\limits_{r \to \infty} \frac{\log \nu(r,f)}{\log r}$ and $\sigma(f) = \limsup\limits_{r \to \infty} \frac{\log \nu(r,f)}{\log r}$.*

Lemma 2.3. (see [7], Lemma 1.1.2) *Let $g : (0, +\infty) \longrightarrow R$, $h : (0, +\infty) \longrightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set F of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.*

Lemma 2.4. ([6], Satz 4.4) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, let $\mu(r, f)$ be the maximum term of f, and let $\nu(r, f)$ be the central index. Then for $0 < r < R$*

we have

$$M(r, f) < \mu(r, f) \left\{ \nu(R, f) + \frac{R}{R - r} \right\}.$$

Lemma 2.5. ([9]) *Let f be a meromorphic function and k a positive integer. If f is a solution of the differential equation $a_0 f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f = 0$, where $a_0, a_1, \cdots, a_k$ are complex numbers with $a_0 \neq 0$, then $T(r, f) = O(r)$. Moreover, if f is transcendental, then $r = O\{T(r, f)\}$.*

Lemma 2.6. ([7], Remark of Corollary 2.3.5 or [12], Corollary of Theorem 1.21) *Let f be a transcendental meromorphic function, and let n be a positive integer, then $\sigma(f) = \sigma(f^{(n)})$.*

Lemma 2.7. ([10], Theorem 1.1) *Let f be a nonconstant entire function, let a be a nonzero small function relative to f, and let $k \geq 2$ be a positive integer. If $f - a$, $f' - a$ and $L[f] - a$ share 0 CM, where $L[f]$ is defined as in (1.1), and if $f' \not\equiv f$, then a reduces to a constant and $a_0 + c^{k-1} + \sum_{j=1}^{k-1} a_j c^{j-1} = 1$ for some constant $c \neq 0$ such that $f = \gamma e^{cz} - a(1 - c)/c$, where γ is a nonzero constant.*

3. Proof of Theorems

Proof of Theorem 1.2. From the condition that $f - a$ and $L[f] - a$ share 0 CM we get

$$(3.1) \qquad \frac{L[f] - a}{f - a} = e^Q,$$

where Q is an entire function. From the condition $\sigma(a) < \sigma(f)$ we know that $\sigma(f) > 0$, which implies that f is a transcendental entire function. From (3.1), Lemma 2.1 and the condition $\sigma(a) < \sigma(f) < \infty$, we get

$$(3.2) \qquad T(r, e^Q) \leq 2T(r, f) + O(\log r),$$

as $r \longrightarrow \infty$. From (3.2) and Definition 1.1 we get $\sigma(e^Q) < \infty$, which implies that Q is a polynomial. We discuss the following two cases.

    Case 1. Suppose that

$$(3.3) \qquad \liminf_{r \longrightarrow \infty} \frac{\log \nu(r, f)}{\log r} > 1.$$

Then from (3.3) and Lemma 2.2 we get

$$(3.4) \qquad \mu(f) = \liminf_{r \longrightarrow \infty} \frac{\log \nu(r, f)}{\log r} > 1.$$

From the condition that f is a nonconstant entire function, we have

$$(3.5) \hspace{3cm} M(r, f) \longrightarrow \infty,$$

as $r \longrightarrow \infty$. Let

$$(3.6) \hspace{3cm} M(r, f) = |f(z_r)|,$$

where $z_r = re^{i\theta(r)}$, and $\theta(r) \in [0, 2\pi)$. From (3.6) and the Wiman-Valiron theory (see [7], Theorem 3.2), we see that there exist subsets $F_j \subset (1, \infty)$ $(1 \leq j \leq n)$ with finite logarithmic measure, i.e., $\int_{F_j} \frac{dt}{t} < \infty$, such that for some point $z_r = re^{i\theta(r)}$ $(\theta(r) \in [0, 2\pi))$ satisfying $|z_r| = r \notin F_j$ and $M(r, f) = |f(z_r)|$, we have

$$(3.7) \hspace{2cm} \frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^j \{1 + o(1)\} \quad (1 \leq j \leq n),$$

as $r \notin \cup_{j=1}^n F_j$ and $r \longrightarrow \infty$. By Definition 1.1, Lemma 2.3, Definition 1.1.1 and Theorem 1.1.3 from [13], and the assumtion $\sigma(a) < \sigma(f)$ we know that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^n F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that

$$(3.8) \hspace{2.5cm} \lim_{r_n \to \infty} \frac{\log \log M(r_n, f)}{\log r_n} = \sigma(f)$$

and

$$(3.9) \hspace{2.5cm} \lim_{r_n \to \infty} \frac{M(r_n, a)}{M(r_n, f)} = 0.$$

Since

$$(3.10) \hspace{2cm} \frac{L[f](z) - a(z)}{f(z) - a(z)} = \frac{\frac{L[f](z)}{f(z)} - \frac{a(z)}{f(z)}}{1 - \frac{a(z)}{f(z)}},$$

from (1.1), (3.3), (3.5)-(3.10) we get

$$(3.11) \hspace{2cm} \frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} = \left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^k \{1 + o(1)\},$$

as $r_n \longrightarrow +\infty$. From (3.11) we have

$$(3.12) \hspace{1.5cm} \log \left|\frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})}\right| = k\{\log \nu(r_n, f) - \log r_n\} + o(1),$$

as $r_n \longrightarrow +\infty$. Let

$$(3.13) \hspace{2cm} Q = p_m z^m + p_{m-1} z^{m-1} + \cdots + p_1 z + p_0,$$

where p_0, p_1, $\cdots$, p_{m-1}, p_m are complex numbers with $p_m \neq 0$. It follows from (3.13) that $\lim\limits_{|z|\to\infty} |Q(z)|/|p_m z^m| = 1$ and $|Q(z)|/|p_m z^m| > 1/e$, as $|z| > r_0$, where r_0 is a sufficiently large positive number. Combining this with (1.1), we get
(3.14)
$$m \log |z| + \log |p_m| - 1 \leq \log |Q| = \log |\log e^Q| \leq |\log\log e^Q| = \left| \log\log \frac{L[f] - a}{f - a} \right|,$$

as $|z| \longrightarrow +\infty$. From (3.12), (3.14), Lemma 2.2 and the condition $\sigma(f) < \infty$ we get

$$
\begin{aligned}
&m \log |z_{r_n}| + \log |p_m| - 1 \\
\leq\ & \left| \log\log \frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \right| \\
=\ & \left| \log \left| \log \frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \right| + i \arg \left(\log \frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \right) \right| \\
\leq\ & \left| \log \left| \log \frac{L[f](z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \right| \right| + 2\pi \\
\leq\ & \log\log \nu(r_n, f) + \log\log r_n + O(1) \leq 2\log\log r_n + O(1),
\end{aligned}
$$

i.e.,
(3.15)
$$m \log |z_{r_n}| + \log |p_m| - 1 \leq \log\log \nu(r_n, f) + \log\log r_n + O(1) \leq 2\log\log r_n + O(1),$$

as $r_n \longrightarrow +\infty$. This is impossible. Thus Q is a constant, and so (3.11) is rewritten as

(3.16)
$$\left(\frac{\nu(r_n, f)}{z_{r_n}} \right)^k \{1 + o(1)\} = c,$$

as $r_n \longrightarrow +\infty$, where c is some nonzero constant. From (3.16) we get

(3.17)
$$\lim_{r_n \to \infty} \frac{\log \nu(r_n, f)}{\log r_n} = 1.$$

On the other hand, by Lemma 2.4 we know that

(3.18) $$M(r_n, f) < \mu(r_n)\{\nu(2r_n, f) + 2\} = |a_{\nu(r_n,f)}| r_n^{\nu(r_n,f)} \cdot \{\nu(2r_n, f) + 2\}.$$

Since $|a_j| < M_1$ for all nonnegative integers j and some constant $M_1 > 0$, we get from (3.18) that

(3.19) $$\log\log M(r_n, f) \leq \log \nu(r_n, f) + \log\log \nu(2r_n, f) + \log\log r_n + C_1,$$

where $C_1 > 0$ is a suitable constant. From Lemma 2.2 and the condition $\sigma(f) < \infty$ we get

(3.20) $$\log \nu(2r_n, f) < \{1 + \sigma(f)\}(\log r_n + \log 2),$$

as $r_n \longrightarrow \infty$. From (3.17), (3.19) and (3.20) we get

$$(3.21) \quad \log \log M(r_n, f) \le \log \nu(r_n, f) + 2 \log \log r_n + O(1) \le \log \nu(r_n, f)\{1 + o(1)\},$$

as $r_n \longrightarrow \infty$. From (3.21) we get

$$(3.22) \qquad\qquad \frac{\log \log M(r_n, f)}{\log r_n} \le \frac{\log \nu(r_n, f)}{\log r_n}.$$

From (3.8), (3.17) and (3.22) we get

$$(3.23) \qquad\qquad\qquad \sigma(f) \le 1.$$

From (3.4), (3.23) and the fact $\mu(f) \le \sigma(f)$ we get a contradiction.

    Case 2.   Suppose that

$$(3.24) \qquad\qquad\qquad \liminf_{r \longrightarrow \infty} \frac{\log \nu(r, f)}{\log r} \le 1.$$

Then from (3.24) and Lemma 2.2 we get

$$(3.25) \qquad\qquad\qquad \mu(f) \le 1.$$

We discuss the following two subcases.

    Subcase 2.1.   Suppose that

$$(3.26) \qquad\qquad\qquad \sigma(f) > 1.$$

By (3.26), Definition 1.1, Lemma 2.3, Definition 1.1.1 and Theorem 1.1.3 from [13] and the assumption $\sigma(a) < \sigma(f)$ we know that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^{n} F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.8) and (3.9) hold. Next in the same manner as in Case 1 we prove that Q is a constant such that (3.16) holds. Proceeding as in Case 1 we get (3.21)-(3.23). From (3.23) and (3.26) we get a contradiction.

    Subcase 2.2.   Suppose that

$$(3.27) \qquad\qquad\qquad \sigma(f) \le 1.$$

    we will prove

$$(3.28) \qquad\qquad\qquad \sigma(f) \ge 1.$$

Suppose that

$$(3.29) \qquad\qquad\qquad \sigma(f) < 1.$$

Then from (3.2), (3.29), Definition 1.1 and the condition $\sigma(a) < \sigma(f)$ we get $\sigma(e^Q) \leq \sigma(f) < 1$, which implies that Q, and so e^Q is a constant. Thus (3.1) is rewritten as

$$(3.30) \qquad \frac{L[f] - a}{f - a} = c,$$

where c is some nonzero constant. From (1.1), Lemma 2.3, Definition 1.1.1 and Theorem 1.1.3 from [13], and the assumption $\sigma(a) < \sigma(f)$ we know that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^n F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.8) and (3.9) hold, and such that

$$(3.31)$$
$$\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^k \{1 + o(1)\} + a_{k-1}\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^{k-1}\{1 + o(1)\} + \cdots + a_1\left(\frac{\nu(r_n, f)}{z_{r_n}}\right) + a_0 = c,$$

as $r_n \longrightarrow \infty$. Moreover, from Lemma 2.2 we get

$$(3.32) \qquad \nu(r_n, f) \leq r_n^{\sigma(f) + \varepsilon_0},$$

as $r_n \geq R_0$, where $\varepsilon_0 = (1 - \sigma(f))/2$ and R_0 is a sufficiently large positive number. From (3.29) and (3.32) we get

$$(3.33) \qquad \lim_{r_n \longrightarrow \infty} \left|\frac{\nu(r_n, f)}{z_{r_n}}\right|^j \leq \lim_{r_n \to \infty} r_n^{\frac{j(\sigma(f) - 1)}{2}} = 0 \, (1 \leq j \leq k).$$

From (3.31) and (3.33) we get

$$(3.34) \qquad a_0 = c.$$

By (1.1) and (3.34) we know that (3.30) is rewritten as

$$(3.35) \qquad f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' = (1 - c)a.$$

If $c = 1$, then it follows from (3.35) that $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' = 0$. This together with Lemma 2.5 implies that $\sigma(f) = \mu(f) = 1$, which contradicts (3.29). Next we suppose that $c \neq 1$. By rewriting (3.35) we get

$$(3.36) \qquad a_1 = \frac{(1 - c)a}{f'} - \frac{a_2 f''}{f'} - \frac{a_3 f^{(3)}}{f'} - \cdots - \frac{a_{k-1}f^{(k-1)}}{f'} - \frac{f^{(k)}}{f'}.$$

By Lemma 2.6 we know that $\sigma(f') = \sigma(f)$, this together with (3.29) and the condition $\sigma(a) < \sigma(f)$ implies

$$(3.37) \qquad \sigma(a) < \sigma(f') < 1.$$

From (3.37), Definition 1.1 and Lemma 2.3 Definition 1.1.1 and Theorem 1.1.3 from [13], we know that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$

satisfying $M(r_n, f') = |f'(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^{n} F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that

$$(3.38) \qquad\qquad \lim_{r_n \to \infty} \frac{\log \log M(r_n, f')}{\log r_n} = \sigma(f')$$

and

$$(3.39) \qquad\qquad \lim_{r_n \to \infty} \frac{M(r_n, a)}{M(r_n, f')} = 0.$$

From (3.36)-(3.39) we have
(3.40)
$$a_1 = \frac{(1-c)a(z_{r_n})}{M(r_n, f')} - \sum_{j=2}^{k-1} a_j \left(\frac{\nu(r_n, f')}{z_{r_n}} \right)^{j-1} \{1 + o(1)\} - \left(\frac{\nu(r_n, f')}{z_{r_n}} \right)^{k-1} \{1 + o(1)\},$$

as $r_n \longrightarrow \infty$. From (3.37) and in the same manner as in the proof of (3.33) we get

$$(3.41) \qquad\qquad \lim_{r_n \to \infty} \left| \frac{\nu(r_n, f')}{z_{r_n}} \right|^j = 0, \quad 1 \leq j \leq k-1.$$

From (3.39)-(3.41) we get

$$\begin{aligned} |a_1| &\leq \lim_{r_n \to \infty} \left| \frac{(1-c)a(z_{r_n})}{M(r_n, f')} \right| + \sum_{j=2}^{k-1} |2a_j| \lim_{r_n \to \infty} \left| \frac{\nu(r_n, f')}{z_{r_n}} \right|^{j-1} \\ &\quad + 2 \lim_{r_n \to \infty} \left| \frac{\nu(r_n, f')}{z_{r_n}} \right|^{k-1} = 0, \end{aligned}$$

which implies $a_1 = 0$. Similarly we have $a_j = 0$ for $2 \leq j \leq k-1$. Thus (3.35) can be rewritten as $f^{(k)} = (1-c)a$. This together with Lemma 2.6 implies $\sigma(f) = \sigma(f^{(k)}) = \sigma(a)$, which contradicts the condition $\sigma(a) < \sigma(f)$. (3.28) is thus completely proved. From (3.27) and (3.28) we get

$$(3.42) \qquad\qquad\qquad\qquad \sigma(f) = 1.$$

From (3.2), (3.42) and Definition 1.2 we get $\sigma(e^Q) \leq 1$. If Q is a constant, from (3.1) we get the conclusion (i) of Theorem 1.1. Next we suppose that Q is a polynomial with degree $\deg(Q) = 1$. Then

$$(3.43) \qquad\qquad\qquad\qquad Q(z) = p_1 z + p_0,$$

where $p_1 \neq 0$ and p_0 are complex numbers. First of all, we will prove

$$(3.44) \qquad\qquad\qquad\qquad \mu(f) = 1.$$

In fact, from (3.42) and $\mu(f) \leq \sigma(f)$ we get

(3.45) $$\mu(f) \leq 1.$$

Suppose that

(3.46) $$\mu(f) < 1.$$

From (3.2) and Definition 1.1 we know that there exists an infinite sequence of positive numbers r_n such that

(3.47) $$\lim_{r_n \to \infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f).$$

From (3.2) we get

(3.48) $$\mu(e^Q) \leq \lim_{r_n \to \infty} \frac{\log T(r_n, e^Q)}{\log r_n} \leq \lim_{r_n \to \infty} \frac{\log T(r_n, f)}{\log r_n}.$$

From (3.46)-(3.48) we get

(3.49) $$\mu(e^Q) < 1.$$

Since $\mu(e^Q) = \sigma(e^Q) = \deg(Q)$, from (3.43) we get $\mu(e^Q) = 1$, which contradicts (3.49). Thus $\mu(f) \geq 1$, this together with (3.45) gives (3.44). Secondly we will prove that not all $a_0, a_1, \cdots a_{k-2}$ and a_{k-1} are zero. Assume the contrary, i.e., suppose that $a_j = 0$ for $0 \leq j \leq k - 1$. Then it follows from (1.1) and (3.43) that (3.1) can be rewritten as

(3.50) $$f^{(k)}(z) - a(z) = (f(z) - a(z))e^{p_1 z + p_0}.$$

From Definition 1.1, Lemma 2.3 Definition 1.1.1 and Theorem 1.1.3 from [13], and the assumption $\sigma(a) < \sigma(f)$ we know that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \cup_{j=1}^n F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.8) and (3.9) hold. From (3.8), (3.9) and (3.50) we get

(3.51) $$\left(\frac{\nu(r_n, f)}{z_{r_n}} \right)^k \{1 + o(1)\} = e^{p_1 z_{r_n} + p_0},$$

as $r_n \longrightarrow \infty$. From (3.51) we get

$$
\begin{aligned}
|p_1| r_n - |p_0| &= |p_1||z_{r_n}| - |p_0| \leq |p_1 z_{r_n} + p_0| \leq |\log e^{p_1 z_{r_n} + p_0}| + O(1) \\
&\leq k|\log \nu(r_n, f) - \log r_n| + O(1) \leq k\{\sigma(f) + 2\} \log r_n + O(1),
\end{aligned}
$$

as $r_n \longrightarrow \infty$. From this and $p_1 \neq 0$ we get a contradiction. Thus from (3.42)-(3.44) we get the conclusion (ii) of Theorem 1.2. This completes the proof of Theorem 1.1.

Proof of Theorem 1.3. From Theorem B and the assumptions of Theorem 1.3 we know that there exists some nonzero constant c_1 such that

$$(3.52) \qquad f' - a = c_1(f - a),$$

where c_1 is a nonzero constant. If $c_1 = 1$, from (3.52) get $f' = f$, which reveals the conclusion (i) of Theorem 1.3. Next we suppose that $c_1 \neq 1$. From Theorem 1.2 we know that $\sigma(f) = 1$ and one of the conclusions (i) and (ii) of Theorem 1.2 will hold. We discuss the following two cases.

Case 1.   Suppose that there exists some nonzero constant c_2 such that

$$(3.53) \qquad L[f] - a = c_2(f - a).$$

From (1.1), (3.52) and (3.53) we get
(3.54)
$$L[f] = (c_1^k + a_{k-1}c_1^{k-1} + \cdots + a_1c_1 + a_0)f + (1 - c_1)(a^{(k-1)} + d_{k-2}a^{(k-2)} + \cdots + d_1a' + d_0a),$$

where $d_0, d_1, \cdots, d_{k-3}, d_{k-2}$ are constants. By substituting (3.54) into (3.53) we get

$$
\begin{aligned}
& (c_1^k + a_{k-1}c_1^{k-1} + \cdots + a_1c_1 + a_0 - c_2)f \\
= {} & (c_1 - 1)(a^{(k-1)} + d_{k-2}a^{(k-2)} + \cdots + d_1a' + d_0a) + (1 - c_2)a. \qquad (3.55)
\end{aligned}
$$

If $c_1^k + a_{k-1}c_1^{k-1} + \cdots + a_1c_1 + a_0 - c_2 \not\equiv 0$, from (3.55) we get

$$(3.56) \qquad f = \frac{(c_1 - 1)(a^{(k-1)} + d_{k-2}a^{(k-2)} + \cdots + d_1a' + d_0a) + (1 - c_2)a}{c_1^k + a_{k-1}c_1^{k-1} + \cdots + a_1c_1 + a_0 - c_2}.$$

From (3.56), Lemma 2.1 and the condition $\sigma(a) < \infty$ we get

$$(3.57) \qquad T(r, f) \leq 2T(r, a) + O(\log r).$$

From (3.57) and Definition 1.1 we get $\sigma(f) \leq \sigma(a)$, which contradicts the condition $\sigma(a) < \sigma(f)$. Thus $c_1^k + a_{k-1}c_1^{k-1} + \cdots + a_1c_1 + a_0 - c_2 = 0$, and so (3.55) is rewritten as

$$(3.58) \qquad (c_1 - 1)(a^{(k-1)} + d_{k-2}a^{(k-2)} + \cdots + d_1a' + d_0a) + (1 - c_2)a = 0.$$

Suppose that $k \geq 2$. If a is a transcendental entire function, from (3.58) and Lemma 2.5 we get $\sigma(a) = 1$. Thus $\sigma(f) = \sigma(a)$, which contradicts the condition $\sigma(a) < \sigma(f)$. Thus a is a nonzero polynomial, and so

$$(3.59) \qquad T(r, a) = o\{T(r, f)\}.$$

From (3.59) and Lemma 2.7 we get the conclusions (i)-(ii) of Theorem 1.3.

Suppose that $k = 1$. Then from (1.1) we get

$$(3.60) \qquad L[f] = f' + a_0f.$$

From (3.52) we get $f' = a + c_1(f - a)$, which and (3.60) implies

(3.61)
$$L[f] = (c_1 + a_0)f + a - ac_1.$$

By substituting (3.61) into (3.53) we get

(3.62)
$$(c_1 + a_0 - c_2)f = a(c_1 - c_2).$$

If $c_1 + a_0 - c_2 \neq 0$, from (3.62) we get $f = a(c_1 - c_2)/(c_1 + a_0 - c_2)$, from which we get $\sigma(f) \leq \sigma(a)$, which contradicts the condition $\sigma(a) < \sigma(f)$. Thus $c_1 + a_0 - c_2 = 0$, and so it follows from (3.62) that $c_1 = c_2$ and $a_0 = 0$. Thus (3.60) is rewritten as $L[f] = f'$, which and (3.52) reveal the conclusion (iii) of Theorem 1.3.

Case 2. Suppose that

(3.63)
$$L[f](z) - a(z) = (f(z) - a(z)) \cdot e^{p_1 z + p_0}$$

such that $\sigma(f) = \mu(f) = 1$, where $p_1 \ (\neq 0)$ and p_0 are two finite complex numbers.

Suppose that $k \geq 2$. Then from (3.59) and Lemma 2.7 we get the conclusions (i)-(ii) of Theorem 1.3. Suppose that $k = 1$. Then (3.63) is rewritten as

(3.64)
$$f'(z) + a_0 f(z) - a(z) = (f(z) - a(z))e^{p_1 z + p_0}.$$

From (3.52) and (3.64) we get

(3.65)
$$\frac{(c_1 + a_0)f(z) - c_1 a(z)}{f(z) - a(z)} = e^{p_1 z + p_0}.$$

If $c_1 + a_0 = 0$, then (3.65) reveals the conclusion (iv) of Theorem 1.3. Next we suppose that

(3.66)
$$c_1 + a_0 \neq 0.$$

If $a = c_1 a/(c_1 + a_0)$, then (3.65) is rewritten as $e^{p_1 z + p_0} = c_1 + a_0$, which is impossible. Next we suppose that

(3.67)
$$a \not\equiv \frac{c_1 a}{c_1 + a_0}.$$

From (3.65) we know that $f/a - 1$ and $f/a - c_1/(c_1 + a_0)$ share 0 CM. From the (3.67) and $a \not\equiv 0$ we get $c_1/(c_1 + a_0) \neq 1$. Thus by (3.67), the condition $\sigma(a) < \sigma(f) < \infty$ and the second fundamental theorem we get

$$\begin{aligned} T\left(r, \frac{f}{a}\right) &\leq \overline{N}\left(r, \frac{f}{a}\right) + \overline{N}\left(r, \frac{1}{f/a - 1}\right) + \overline{N}\left(r, \frac{1}{f/a - c_1/(c_1 + a_0)}\right) \\ &\quad + S\left(r, \frac{f}{a}\right) \\ &\leq \overline{N}\left(r, \frac{1}{a}\right) + O(\log r) \leq T(r, a) + O(\log r), \end{aligned}$$

and so we have

$$(3.68) \qquad T(r, f) = T\left(r, \frac{f}{a} \cdot a\right) \leq T\left(r, \frac{f}{a}\right) + T(r, a) \leq 2T(r, a) + O(\log r).$$

From (3.68) and Definition 1.1 we get $\sigma(f) \leq \sigma(a)$, which contradicts the condition $\sigma(a) < \sigma(f)$. This completes the proof of Theorem 1.3

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