

Effect of Open Packing upon Vertex Removal

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ABSTRACT. In a graph $G = (V, E)$, a non-empty set $S \subseteq V$ is said to be an *open packing set* if no two vertices of S have a common neighbour in G . The maximum cardinality of an open packing set is called the *open packing number* and is denoted by ρ^o . In this paper, we examine the effect of ρ^o when G is modified by deleting a vertex.

1. Introduction

All graphs considered in this article are finite, undirected, with neither loops nor multiple edges. For graph theory terminology not presented here, we follow Chartrand and Lesniak [1].

The *open neighbourhood* of a vertex $v \in V$, denoted by $N(v)$, is defined to be $N(v) = \{x \in V : vx \in E\}$; the set of all vertices which are adjacent to v . The *closed neighbourhood* of a vertex $v \in V$ is denoted by $N[v]$ and is defined to be $N[v] = N(v) \cup \{v\}$. The subgraph induced by a set S of vertices of a graph G is denoted by $\langle S \rangle$ where $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{(u, v) \in E(G) : \text{both } u \text{ and } v \text{ are in } S\}$. By a *major vertex*, we mean a vertex that is adjacent to all other vertices of the graph. For an integer $l \geq 1$, we define the l -*corona* of a graph G to be the graph of order $(l+1)|V(G)|$ obtained from G by attaching a path of length l to each vertex of G so that the resulting paths are vertex-disjoint. The 1-*corona* of G is also called the *corona* of G . The *fan graph* denoted by F_n is the 1-point union of n copies of the cycle C_3 . That is, F_n is a graph of order $2n+1$ and size $3n$ obtained by attaching n copies of a triangle at a vertex.

A set S of vertices of G is an *open packing set* of G if the open neighbourhoods of the vertices of S are pairwise disjoint in G . The *lower open packing number* of G , denoted by $\rho_L^o(G)$, is the minimum cardinality of a maximal open packing set of G

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while the *open packing number* of G , denoted by $\rho^o(G)$, is the maximum cardinality among all open packing sets of G . An open packing set of G with cardinality $\rho_L^o(G)$ and $\rho^o(G)$ are respectively called the ρ_L^o -set and ρ^o -set of G . For basic results on open packing number see [3], [4] and [5].

The study of the effect of removal of a vertex or an edge on any graph theoretic parameter has interesting applications in the context of network. For instance, a detailed study of this kind associated with the concept of domination can be seen in [2]. As far as, the open packing number $\rho^o(G)$ is concerned, it may increase or decrease or remain unchanged when a vertex is removed from G . For example, in a star graph $K_{1,n}$ ($n \geq 3$), removal of the center vertex increases the value of ρ^o by $n - 2$ and the removal of any pendant vertex does not alter the value of ρ^o . Also, in the 1-corona of a complete graph, the removal of any pendant vertex decreases the value of ρ^o . So, one can partition $V(G)$ into three sets V^0 , V^+ and V^- , where

$$V^0(G) = \{v \in V : \rho^o(G - v) = \rho^o(G)\}$$

$$V^+(G) = \{v \in V : \rho^o(G - v) > \rho^o(G)\}$$

$$V^-(G) = \{v \in V : \rho^o(G - v) < \rho^o(G)\}$$

This paper initiates an investigation of the properties of these sets. We need the following results.

Theorem 1.1([5]). *If G is a connected graph on n vertices with $\Delta(G) = n - 1$, then $\rho^o(G) \leq 2$. Further, $\rho^o(G) = 2$ if and only if $\delta(G) = 1$ and $\rho_L^o(G) = 2$ if and only if G is a star.*

Theorem 1.2([5]). *Let G be a graph of diameter 2, then $\rho^o(G) \leq 2$.*

Theorem 1.3([3]). *If G is a connected graph of order $n \geq 3$, then $\rho^o(G) \leq \frac{2n}{3}$.*

Theorem 1.4([4]). *Let G be a connected graph of order $n \geq 3$. Then $\rho^o(G) = \frac{2n}{3}$ if and only if G is the 2-corona of a connected graph.*

2. Examples

Here, we determine the sets V^0 , V^- and V^+ for some common classes of graphs such as paths, cycles, complete multi-partite graphs and the Petersen graph. For this purpose, we need the following lemma.

Lemma 2.1. *If G is a disconnected graph with components G_1, G_2, \dots, G_r , then $\rho^o(G) = \sum_{i=1}^r \rho^o(G_i)$ and $\rho_L^o(G) = \sum_{i=1}^r \rho_L^o(G_i)$.*

Proof. Let S_i , where $1 \leq i \leq r$, be a maximal open packing set of the component G_i . In order to prove the lemma, we need to prove two facts. One is the set $S = \cup_{i=1}^r S_i$ is a maximal open packing set of G and the other one is the intersection of a maximal open packing set of G with the vertex set $V(G_i)$ of each component G_i is a maximal open packing set of G_i . First fact is obvious, for if there is a vertex $v \notin S$ of G , say in G_i , such that $S \cup \{v\}$ is an open packing set of G , then $S \cup \{v_i\}$ is also an

open packing set of G_i , contradicting the maximality of S_i . Now, if D is a maximal open packing set of G such that $D \cap V(G_i)$ is not a maximal open packing set of G_i for some i , then choose a vertex $v_i \notin D$ in G_i with $(D \cap V(G_i)) \cup \{v_i\}$ is an open packing set of G_i so that $D \cup \{v_i\}$ is an open packing set of G , again contradicting the maximality of D . \square

The exact values of ρ^o for paths and cycles are given in [3].

Proposition 2.2([3]). For $n \geq 2$, then $\rho^o(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\ \lfloor \frac{n+2}{2} \rfloor & \text{otherwise.} \end{cases}$

Proposition 2.3([3]). For $n \geq 3$, then $\rho^o(C_n) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \equiv 2 \pmod{4}, \\ \lfloor \frac{n}{2} \rfloor & \text{otherwise.} \end{cases}$

With the aid of the above propositions and the Lemma 2.1, one can determine the sets V^o , V^- and V^+ for paths as follows.

Proposition 2.4. For the path $P_n = (v_1, v_2, \dots, v_n)$ on $n \geq 2$ vertices, we have

- (i) If $n \equiv 0 \pmod{4}$, then $V^0 = \{v_r : r \equiv 0 \text{ or } 1 \pmod{4}\}$ and $V^+ = V - V^0$.
- (ii) If $n \equiv 1 \pmod{4}$, then $V^0 = \{v_r : r \equiv 0 \text{ or } 2 \pmod{4}\}$, $V^+ = \{v_r : r \equiv 3 \pmod{4}\}$ and $V^- = \{v_r : r \equiv 1 \pmod{4}\}$.
- (iii) If $n \equiv 2 \pmod{4}$, then $V^0 = \{v_r : r \equiv 0 \text{ or } 3 \pmod{4}\}$ and $V^- = V - V^0$.
- (iv) If $n \equiv 3 \pmod{4}$, then $V = V^0$.

Proof. Suppose $n \equiv 0 \pmod{4}$. Let $n = 4k$, for some positive integer k . Then by Proposition 2.2, we have $\rho^o(P_n) = 2k$. Now removal of the vertex v_r , where $r \equiv 0$ or $1 \pmod{4}$ will split the path P_n into two paths P_{4k_1} and P_{4k_2+3} , for some non-negative integers k_1 and k_2 such that $k_1 + k_2 = k - 1$. Therefore, it follows from Proposition 2.2 and Lemma 2.1, that $\rho^o(P_n - v_r) = \rho^o(P_{4k_1}) + \rho^o(P_{4k_2+3}) = 2k_1 + 2k_2 + 2 = 2(k_1 + k_2) + 2 = 2(k - 1) + 2 = 2k = \rho^o(P_n)$ and so $v_r \in V^0$. Also, if $r \equiv 2$ or $3 \pmod{4}$, then $P_n - v_r = P_{4k_1+1} \cup P_{4k_2+2}$, for some non-negative integer k_1 and k_2 such that $k_1 + k_2 = k - 1$. Therefore, $\rho^o(P_n - v_r) = \rho^o(P_{4k_1+1}) + \rho^o(P_{4k_2+2}) = 2k_1 + 1 + 2k_2 + 2 = 2(k_1 + k_2) + 3 = 2(k - 1) + 3 = 2k + 1 = \rho^o(P_n) + 1$ and thus $v_r \in V^+$. This proves (i) and the remaining can be proved in the similar fashion. \square

As did above, one can determine the sets V^0 , V^- and V^+ for cycles with the aid of Proposition 2.3 as follows.

Proposition 2.5. For the cycle $C_n = (v_1, v_2, \dots, v_n)$ on $n \geq 3$ vertices, we have

$$V = \begin{cases} V^0 & \text{if } n \equiv 0, 1 \pmod{4}, \\ V^+ & \text{otherwise.} \end{cases}$$

Proposition 2.6. *If G is the Petersen graph, then $V(G) = V^+(G)$.*

Proof. As the Petersen graph G is of diameter 2, it follows from Theorem 1.2 that $\rho^o(G) \leq 2$ and also any two adjacent vertices of G form an open packing set so that $\rho^o(G) = 2$. Further, for any vertex v in the Petersen graph, its open neighbourhood $N(v)$ forms an open packing set of $G - v$ so that $\rho^o(G - v) \geq 3$ and therefore $v \in V^+(G)$. Thus $V^+(G) = V(G)$. \square

Proposition 2.7. *Let G be a complete k -partite graph of order $n \geq 3$, which is not a star. Then $V(G) = V^0(G)$ when $k \neq 3$, and when $k = 3$, the set $V^+(G)$ is the union of the parts of G with exactly one vertex and $V^0(G) = V(G) - V^+(G)$.*

Proof. It is obvious that the value of $\rho^o(G)$ is 2 when $k = 2$ and it is 1 when $k \geq 3$. Let (V_1, V_2, \dots, V_k) be the partition of G . If $k = 2$, then each of V_1 and V_2 is of cardinality at least two so that removal of any vertex from G results in a bipartite graph whose ρ^o value is 2 and thus $V = V^0$. When $k = 3$, removal of a vertex that is the only vertex of a part results in a complete bipartite graph for which $\rho^o = 2$. Further, removal of a vertex that belongs to a part of cardinality more than one results in again a complete tripartite graph for which $\rho^o(G) = 1$. Thus V^+ is the union of parts of cardinality one and $V^0 = V - V^+$ as $V^- = \phi$. As, when $k \geq 4$, removal of any vertex gives a complete $(k - 1)$ -partite graph it follows that $V = V^0$. \square

3. Properties of the Sets

This section investigates the properties of the vertices in a graph G belonging to the sets $V^0(G)$, $V^+(G)$ and $V^-(G)$. Even if the removal of a vertex can increase the open packing number arbitrarily, we observe that the removal of a vertex can decrease the open packing number by at most one because of the fact that $S - \{v\}$ is an open packing set of $G - v$ whenever S is an open packing set of a graph G . That is,

Observation 3.1. *For any vertex v in a graph G , $\rho^o(G) - 1 \leq \rho^o(G - v)$.*

Theorem 3.2. *Let G be a graph of order at least two. A vertex $v \in V^-$ if and only if every ρ^o -set S of G contains v and $S - \{v\}$ is a ρ^o -set of $G - v$.*

Proof. Assume that $v \in V^-$. Suppose there is a ρ^o -set S of G such that $v \notin S$. Then S is an open packing set of $G - v$ so that $\rho^o(G - v) \geq \rho^o(G)$. Therefore $v \notin V^-$, which is a contradiction. Thus v lies in every ρ^o -set of G . Since $S - \{v\}$ is an open packing set of $G - v$ and $v \in V^-$, it follows from Observation 3.1 that $S - \{v\}$ is a ρ^o -set of $G - v$. Converse is obvious. \square

Corollary 3.3. *For any connected graph G of order $n \geq 4$, we have $|V^-| \leq \frac{2n}{3}$, with equality only when G is the 2-corona of a connected graph.*

Proof. As by Theorem 3.2 the set $V^-(G)$ is contained in any ρ^o -set of G , the inequality follows from Theorem 1.3. Further, $|V^-(G)| = \frac{2n}{3}$ implies that $\rho^o(G) = \frac{2n}{3}$ and so G is by Theorem 1.4 the 2-corona of a connected graph. On the other

hand, in a 2-corona G of a connected graph, the value of ρ^o gets reduced by 1 when a pendant vertex or a support vertex is removed whereas it remains unchanged when any of the other vertices is removed. Hence $|V^-(G)| = \frac{2|V(G)|}{3}$. \square

The following theorem provides a sufficient condition for a vertex v belonging to the set V^+ .

Theorem 3.4. *If w is a vertex of G having two neighbours of degree two that are adjacent, then $w \in V^+$.*

Proof. Let u and v be two neighbours of w that are adjacent such that $\deg u = \deg v = 2$. Let S be a ρ^o -set of G . Obviously, S contains at most one of the vertices u, v and w . Suppose $u \in S$. Certainly, S is also an open packing set of $G - w$. In fact, the set $S \cup \{v\}$ itself is an open packing set of $G - w$ as the edge uv is one of the components of $G - w$. Therefore, $\rho^o(G - w) \geq |S| + 1 = \rho^o(G) + 1$ and thus $w \in V^+$. The similar argument holds true when $v \in S$. Also, if $w \in S$, then the set $(S - \{w\}) \cup \{u, v\}$ is an open packing set of $G - w$ so that $\rho^o(G - w) > \rho^o(G)$. \square

Theorem 3.5. *If a vertex $v \in V^+$, then v is adjacent to at least two vertices in every ρ^o -set of $G - v$.*

Proof. If there exists a ρ^o -set S of $G - v$ such that the vertex v has at most one neighbour in S , then S is also an open packing set of G so that $\rho^o(G) \geq |S| = \rho^o(G - v)$, a contradiction to the assumption that $v \in V^+$. Hence the result follows. \square

Corollary 3.6. *A pendant vertex v lies either in V^- or in V^0 and further $v \in V^-$ if and only if v lies in every ρ^o -set of G .*

Proof. The fact that $v \in V^- \cup V^0$ is immediate in view of Theorem 3.5. Further, suppose a pendant vertex v lies in every ρ^o -set of G . Being v is a pendant vertex, every open packing set of $G - v$ is also an open packing set of G . Therefore, the vertex v must lie in $V^-(G)$; for otherwise we can have a ρ^o -set of G not containing v . Converse is already proved in Theorem 3.2. \square

Corollary 3.7. *If a pendant vertex v lies in $V^-(G)$, then it is the only pendant neighbour to its support.*

Proof. When the support vertex of v has a pendant neighbour other than v it is possible to get a ρ^o -set of G leaving the vertex v and so by Corollary 3.6, the vertex v cannot be in V^- . \square

Remark 3.8. The converse of each of Theorem 3.4 and Theorem 3.5 is not true. For example, the center vertex of the star $K_{1,n}$ ($n \geq 3$) lies in V^+ , whereas it does not lie on any triangle. On the other hand, in the path P_n , where $n \equiv 3(\text{mod } 4)$, the two neighbours of a support vertex u lie in every ρ^o -set of $G - u$, whereas u is not in the set V^+ .

4. Graphs of Diameter Two

In this section, we study the effect of $\rho^o(G)$ upon removal of vertices from graphs G of diameter two. More specifically, we determine the classes of graphs G of diameter two for which (i) $V(G) = V^+(G)$ (ii) $V(G) = V^-(G)$ and (iii) $V(G) = V^0(G)$. Recall that the value of $\rho^o(G)$ when G is of diameter two is either 1 or 2. So, the characterization of those classes of graphs with diameter two is done in two cases namely when $\rho^o(G) = 1$ and when $\rho^o(G) = 2$; that are stated respectively in Theorem A and Theorem B. In this connection, let us define for our convenience some terminologies. We define a vertex v to be the *private common neighbour* of a pair of distinct vertices (u, w) if $N(u) \cap N(w) = \{v\}$. The *private common neighbour set* of a vertex v is a subset $B \subseteq N(v)$ such that v is the private common neighbour of every pair of vertices of B . The *private common degree* of v is the cardinality of its private common neighbour set and is denoted by $pd(v)$. Note that $pd(v)$ is either 0 or at least two.

Theorem A. *Let G be a graph of diameter two with $\rho^o(G) = 1$. Then $V = V^0$ if and only if $pd(v) = 0$ for all $v \in V(G)$.*

Theorem B. *Let G be a graph of diameter two with $\rho^o(G) = 2$. Then*

- (a) *there is no graph G with $V(G) = V^-(G)$.*
- (b) *$V(G) = V^+(G)$ if and only if $pd(v) \geq 3$.*
- (c) *$V(G) = V^0(G)$ if and only if for any vertex $v \in V(G)$, either $pd(v) = 2$ or $pd(v) = 0$ with the property that there is an edge in $G - v$ not lying on a triangle in $G - v$.*

These two theorems are proved by the following lemmas.

Lemma 4.1. *Let G be a graph with diameter 2. Then a vertex $v \in V^+(G)$ if and only if its private common degree is at least 2 or 3 according as $\rho^o(G)$ is 1 or 2.*

Proof. Suppose $v \in V^+(G)$. Let S be ρ^o -set of $G - v$. By Theorem 3.5 the vertex v is adjacent to at least two vertices of S , say x and y . If $\rho^o(G) = 1$, then v is the private common neighbour of the pair (x, y) as S is an open packing set of $G - v$ so that $pd(v) \geq 2$. Suppose $\rho^o(G) = 2$. Then $|S| \geq 3$. Now, choose a vertex $z \notin \{x, y\}$ in S such that it is not adjacent with one of the vertices x and y , say x . As $\text{diam}(G) = 2$, we have $d(x, z) = 2$ and also the vertices x and z have no common neighbour in $G - v$. Therefore, v is the only common neighbour of x and z in G . That is, v is the private common neighbour of the pair (x, z) . Certainly, v is also the private common neighbour of the pairs (x, y) and (y, z) in G as S is an open packing set of $G - v$ and hence $pd(v) \geq 3$.

Conversely, suppose $\rho^o(G) = 1$ and $pd(v) \geq 2$. Then there exist two vertices u and w such that v is the private common neighbour of the pair (u, w) so that $\{u, w\}$ forms an open packing set of $G - v$ and therefore $\rho^o(G - v) \geq 2$. Hence $v \in V^+(G)$. Now, suppose $\rho^o(G) = 2$ and $pd(v) \geq 3$. Then the private common neighbour set

B of v forms an open packing set of $G - v$ and $|B| \geq 3$ so that $\rho^o(G - v) \geq 3$. Thus $v \in V^+(G)$. \square

Lemma 4.2. *Let G be a graph such that $\rho^o(G) = 2$. Then $v \in V^-(G)$ if and only if v lies in every ρ^o -set of G and $pd(v) = 0$.*

Proof. Suppose $v \in V^-(G)$. By Theorem 3.2, the vertex v lies in every ρ^o -set of G . Further, as the private common neighbour set B of the vertex v forms an open packing set of $G - v$, it follows that $|B| = 0$; for otherwise $\rho^o(G - v) \geq |B| \geq 2$, a contradiction to the assumption that $v \in V^-$. Thus $pd(v) = 0$.

Conversely, suppose the vertex v lies in every ρ^o -set of G and $pd(v) = 0$. Let S be a ρ^o -set of $G - v$. As $pd(v) = 0$, the set S is an open packing set of G and consequently $|S| \leq 2$. Certainly, $|S|$ cannot be 2; if not S becomes a ρ^o -set of G not containing the vertex v , a contradiction to the assumption. \square

Lemma 4.3. *Let G be a graph with $diam(G) = 2$ and $\rho^o(G) = 2$. Then a vertex $v \in V^0(G)$ if and only if one of the following holds.*

- (i) $pd(v) = 2$
- (ii) $pd(v) = 0$ with the property that $G - v$ contains an edge not lying on any triangle of $G - v$.

Proof. If v is a vertex with $pd(v) = 2$, then $\rho^o(G - v) \geq 2$ as the private common neighbour set of v is always an open packing set of $G - v$. However, Lemma 4.1 says that the vertex v cannot be in $V^+(G)$ so that $v \in V^0(G)$. On the other hand, let v be a vertex as in (ii) and let xy be an edge in $G - v$ not lying on any triangle of $G - v$. Then $\{x, y\}$ becomes an open packing set of $G - v$ and so again by Lemma 4.1 the vertex v belongs to $V^0(G)$. Conversely, if v lies in $V^0(G)$ such that $pd(v) \neq 2$, then $pd(v) = 0$ in view of Lemma 4.1. Consider a ρ^o -set $\{x, y\}$ of $G - v$. If x and y are adjacent, then the edge xy will serve the purpose and in fact it is true; for otherwise they both must be adjacent to the vertex v being $diam(G) = 2$, which is however not possible because $pd(v) = 0$. \square

Lemma 4.4. *Let G be a graph with $\rho^o(G) = 2$. Then $V(G) = V^-(G)$ if and only if every component of G is either K_1 or K_2 .*

Proof. Suppose $V(G) = V^-(G)$. Then by Theorem 3.2, the vertex set $V(G)$ becomes a ρ^o -set of G and hence the fact that each component of G is either K_1 or K_2 follows immediately from the definition of open packing. Converse is obvious. \square

Now, Theorem A immediately follows from Lemma 4.1. Note that, as any vertex of a graph G with $\rho^o(G) = 1$ is either in V^+ or in V^0 , Theorem A can also be stated as “for a graph G of diameter 2 with $\rho^o(G) = 1$, $V(G) = V^+(G)$ if and only if $pd(v) \geq 2$ for all $v \in V(G)$ ”. Further, Theorem B is an immediate consequence of the remaining lemmas. Of course, Theorem A and B do not provide the structural characterization of those graphs of diameter two for which whole vertex set equals one of the sets V^+ , V^- and V^0 ; however as you can see below,

they are helpful in determining the complete structure of such graphs when they admit a major vertex. By a half support in a graph G , we mean a vertex that is adjacent to a vertex of degree 2 in G .

Proposition 4.5. *Let G be graph on $n \geq 4$ vertices with $\Delta(G) = n - 1$ and $\delta(G) = 1$. Then*

- (a) *If there are more than one pendant vertices in G , then $V^+ = \{u\}$ and $V^0 = V - \{u\}$, where u is the unique vertex with $\deg u = n - 1$.*
- (b) *If there exists exactly one pendant vertex, say x , then (i) $V^- = \{x\}$ (ii) $V^+ = \phi$ or $\{u\}$ with $V^+ = \{u\}$ if and only if $pd(u) \geq 3$.*

Proof. Since $\delta(G) = 1$, the graph G contains exactly one major vertex, say u . Suppose there are more than one pendant vertices in G . Then $pd(u) \geq 3$ and consequently Lemma 4.1 implies that $u \in V^+$. Further, as the vertex u together with one of its pendant neighbours forms an open packing set of $G - w$ for each $w \in N(u)$ we have $w \in V^0 \cup V^+$. But $w \notin V^+$ as $pd(w) \leq 2$. Thus $V^+ = \{u\}$ and $V^0 = V - \{u\}$.

Now, if x is the only pendant neighbour of u , then the set $\{x, u\}$ is the unique ρ^o -set of G . Therefore Lemma 4.2 implies that $w \notin V^-$, for all $w \in N(u) - \{x\}$. Also, $pd(x) = 0$ and $pd(w) \leq 2$, for all $w \in N(u) - \{x\}$. Therefore $x \in V^-$ by Lemma 4.2 and $w \notin V^+$ by Lemma 4.1. Hence $V^- = \{x\}$ and $N(u) - \{x\} \subseteq V^0$ and of course $V^+ = \{u\}$ or ϕ . The rest follows again by Lemma 4.1. \square

Proposition 4.6. *Let G be graph on $n \geq 4$ vertices with $\Delta(G) = n - 1$ and $\delta(G) \geq 2$. Then $V^-(G) = \phi$ and further*

- (a) *If G contains more than two major vertices, then $V = V^0$.*
- (b) *If G contains exactly two major vertices, say u and v , then $V^+ = \phi$ or $\{u, v\}$ with $V^+ = \{u, v\}$ only when $\delta(G) = 2$.*
- (c) *Suppose G contains exactly one major vertex, say u . Then (i) when $\delta(G) = 2$, $V^+ = S$, where S is the set of all half supports of G . (ii) when $\delta(G) \geq 3$, $V^+ = \phi$ or $\{u\}$ with $V^+ = \{u\}$ if and only if $pd(u) \geq 2$.*

Proof. Certainly, $V^-(G) = \phi$ as $\rho^o(G) = 1$ when $\delta(G) \geq 2$. Consider a non-major vertex w of G that is also a half support. Let w' be a vertex of degree 2 that is adjacent to the vertex w . Then w is the private common neighbour of the vertex w' and a major vertex. Therefore $pd(w) \geq 2$ and so by Lemma 4.1 we have $w \in V^+(G)$. On the other hand, if a non-major vertex v is not a half support, then every neighbour of v has degree at least 3. As a result, the private common neighbour set of v is empty so that $pd(v) = 0$ and so again by Lemma 4.1 that $v \notin V^+(G)$. So, the conclusion we draw is that a non-major vertex belongs to $V^+(G)$ only if it is a half support in G . We need to discuss what are the major vertices that belong to $V^+(G)$.

Now, suppose G has more than two major vertices. Then there is no half support and so no non-major vertex belongs to $V^+(G)$. Further, the major vertices also not in $V^+(G)$ as they are of private common degree zero and thus $V^+(G) = \phi$. Suppose G has exactly two major vertices, say u and v . Then no non-major vertex is a half support in G so that they are not in $V^+(G)$. However, if there is a vertex of degree 2, then both u and v are of private common degree 2 so that they will be in $V^+(G)$; and conversely. Thus $V^+(G) = \{u, v\}$ or ϕ with $V^+(G) = \{u, v\}$ only when $\delta(G) = 2$.

In case, there is exactly one major vertex, say u . Certainly, $pd(u) \geq 2$ when $\delta(G) = 2$ so that $u \in V^+(G)$. Therefore, when $\delta(G) = 2$, $V^+(G)$ is the set of all half supports. When $\delta(G) \geq 3$, no vertex is a half support so that no non-major vertex belongs to $V^+(G)$. However, the vertex u may belong to $V^+(G)$ and it is possible only when $pd(u) \geq 2$. \square

Now, the above two results constitute the following.

Theorem 4.7. *If G is a graph of order $n \geq 2$ with $\Delta(G) = n - 1$ and $\delta(G) = 1$, then it is not possible that $V(G) = V^+(G)$. Further, (i) $V(G) = V^-(G)$ if and only if G is K_2 (ii) $V(G) = V^0(G)$ if and only if G is P_3 . \square*

Theorem 4.8. *Let G be a graph of order $n \geq 4$ such that $\Delta(G) = n - 1$ and $\delta(G) \geq 2$. Then (i) $V(G) = V^+(G)$ if and only if G is a Fan graph (ii) $V(G) = V^0(G)$ if and only if either G has more than two major vertices or, $\delta(G) \geq 3$ and the major vertex is of private common degree zero in the case when G has exactly one major vertex.*

5. Open Problems

The study of the effect of the removal of a vertex or an edge on any graph theoretic parameter has interesting applications in the context of network. This type of study has been carried out in the case of domination number. In this paper, a similar study has been initiated with respect to the open packing number for a graph G . We conclude the paper by listing some interesting problems and directions for further research that we encounter during the course of our investigation.

1. Obtain a necessary and sufficient condition for the vertices of a graph G that belonging to the sets $V^+(G)$ and $V^0(G)$.
2. Determination of the sets V^+ , V^- and V^0 in the case of trees and regular graphs seems to be a little challenging problems.
3. Obtain good bounds for $|V^-(G)|$, $|V^0(G)|$ and $|V^+(G)|$ of a graph G .
4. Similar study can be initiated for edge removal.
5. By analogy with the bondage number for domination, one can define *the open packing bondage number* $ob(G)$ of a graph G to be the minimum number of edges whose removal results in a graph H such that $\rho^o(H) > \rho^o(G)$. This

parameter is well defined as the value of ρ^o for the totally disconnected graph is its order. Now, one can initiate a study on this parameter.

References

- [1] G. Chartrand and Lesniak, *Graphs and Digraphs*, Fourth edition, CRC press, Boca Raton, 2005.
- [2] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in Graphs*, Marcel Dekker, Newyork(1998).
- [3] M. A. Henning, P. J. Slater, *Open packing in graphs*, JCMCC, **29**(1999), 3–16.
- [4] M. A. Henning, *Upper bounds on the lower open packing number of a tree*, Quaestiones Mathematicae, **21**(1998), 235–245.
- [5] I. Sahul Hamid, S. Saravanakumar, *Packing parameters in graphs*, Discussiones Mathematicae Graph theory, **35**(2015), 5–16.