KYUNGPOOK Math. J. 56(2016), 727-735
http://dx.doi.org/10.5666/KMJ.2016.56.3.727
pISSN 1225-6951 eISSN 0454-8124

Strongly Prime Ideals and Primal Ideals in Posets

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Abstract. In this paper, we study and establish some interesting results of ideals in a poset. It is shown that for a nonzero ideal I of a poset P, there are at most two strongly prime ideals of P that are minimal over I. Also, we study the notion of primal ideals in a poset and the relationship among the primal ideals and strongly prime ideals is considered.

1. Introduction

Throughout this paper $(P, \leq)$ denotes a poset with smallest element 0. For basic terminology and notation for posets, we refer [9] and [6]. For $M \subseteq P$, let $L(M) = \{x \in P : x \leq m \text{ for all } m \in M\}$ denote the lower cone of M in P and dually, let $U(M) = \{x \in P : m \leq x \text{ for all } m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$, we shall write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, x_2, ..., x_n\}$ is finite, then we use the notation $L(x_1, x_2, ..., x_n)$ instead of $L(\{x_1, x_2, ..., x_n\})$(and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$. Following [10], a non-empty subset I of P is called a semi-ideal if $b \in I$ and $a \leq b$, then $a \in I$. A subset I of P is called an ideal if $a, b \in I$ implies $L(U((a, b)) \subseteq I$[9]. Following [8], for any subset X of P, $[X]$ is the smallest ideal of P containing X. If $X = \{b\}$, then $L(b)$ is called the principle ideal of P generated by b. A proper semi-ideal (ideal) I of P is called prime if $L(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$ [6]. An ideal I of a poset P is called semi-prime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c))) \subseteq I$[9]. Following [3], an ideal I of P is called strongly prime if $L(A^*, B^*) \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for different proper ideals A, B of P, where $A^* = A\backslash\{0\}$. A

* Corresponding Author.
Received July 23, 2015; revised March 21, 2016; accepted April 7, 2016.
2010 Mathematics Subject Classification: 06A06.
Key words and phrases: Poset, ideals, strongly prime ideal, primal ideal, minimal prime ideal.

non-empty subset M of P is called an m-system if for any $x_1, x_2 \in M$, there exists $t \in L(x_1, x_2)$ such that $t \in M$. Following [1], a non-empty subset M of P is called a strongly m-system if $A \cap M \neq \emptyset$ and $B \cap M \neq \emptyset$ imply $L(A^*, B^*) \cap M \neq \emptyset$ for different proper ideals A, B of P. It is clear that an ideal I of P is strongly prime if and only if P/I is a strongly m- system of P. Also every strongly m-system of P is an m-system. Following [3], an ideal I of P is called strongly semi-prime if $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$ together imply $L(A^*, U(B^*, C^*)) \subseteq I$ for any different ideals A, B and C of P. For any semi-ideal I of P and a subset A of P, we define $< A, I >= \{z \in P : L(a, z) \subseteq I \text{ for all } a \in A\} = \bigcap_{a \in A} < a, I >$[3]. If $A = \{x\}$, then we write $< x, I >$ instead of $< \{x\}, I >$. For any ideal I of P, a strongly prime ideal Q of P is said to be a minimal strongly prime ideal of I if $I \subseteq Q$ and there exists no strongly prime ideal R of P such that $I \subset R \subset Q$. The set of all strongly prime ideals of P is denoted by $Sspec(P)$ and the set of minimal strongly prime ideals of P is denoted by $Smin(P)$. For any ideal I of P, $P(I)$ and $SP(I)$ denotes the intersection of all prime semi-ideals and strongly prime ideals of P containing I. It is clear from Theorem 6 of [6] and Example 1.1 of [2] that $P(I) = I$ and $SP(I) \neq I$ for any ideal I of P. Following [1], let I be a semi-ideal of P. Then I is said to have (*) condition if whenever $L(A, B) \subseteq I$, we have $A \subseteq < B, I >$ for any subsets A and B of P.

2. Main Results

Theorem 2.1. *Let M be a nonempty strongly m-system of P and J be an ideal of P with $J \cap M = \emptyset$. Then J is contained in a strongly prime ideal I of P with $I \cap M = \emptyset$.*

Proof. Let $S = \{K : K \text{ is an ideal of } P \text{ with } K \cap M = \emptyset\}$. Then $S \neq \emptyset$ and by Zorn's lemma, there exists a maximal element $I \in S$ with $I \cap M = \emptyset$. Let A and B be ideals of P with $L(A^*, B^*) \subseteq I$ and suppose that $A \nsubseteq I$ and $B \nsubseteq I$. Then there exists $x \in A \backslash I$ and $y \in B \backslash I$ such that $I \subset I \cup \{x\} \subseteq [I \cup \{x\}]$ and $I \subset I \cup \{y\} \subseteq [I \cup \{y\}]$, which imply $[I \cup \{x\}] \cap M \neq \emptyset$ and $[I \cup \{y\}] \cap M \neq \emptyset$. Since M is strongly m-system, we have $L([I \cup \{x\}]^*, [I \cup \{y\}]^*) \cap M \neq \emptyset$. But $L([I \cup \{x\}]^*, [I \cup \{y\}]^*) \subseteq L([I \cup \{x\}]^*) \subseteq L(I^*) \subseteq I$, which implies $I \cap M \neq \emptyset$, a contradiction. □

Theorem 2.2. *Let I and J be ideals of P with $\{0\} \neq J \subseteq I$. Then the following are equivalent.*

(i) *I is a minimal strongly prime ideal of J.*

(ii) *For each $x \in I$, there exists $t \in U(x)$ and $y \in P \backslash I$ such that $L(L(t)^*, L(y)^*) \subseteq J$.*

(iii) *If I has (*) condition, then for any $x \in I$, we have $< x, J > \nsubseteq I$.*

Proof. (**i**) ⇒ (**ii**) Let I be a minimal strongly prime ideal of J. Suppose that there exists $x \in I$ such that $L(L(t_i)^*, L(y_j)^*) \nsubseteq J$ for all $t_i \in U(x)$ and $y_j \in P \backslash I$. Let

$M = \{a_{ij} : a_{ij} \in L(L(t_i)^*, L(y_j)^*)\backslash J$ for $t_i \in U(x)$ and $y_j \in P\backslash I\}$. Then $M \neq \emptyset$. For any ideals A, B of P, let $A \cap M \neq \emptyset$ and $B \cap M \neq \emptyset$. Then there exists $a \in A$ and $b \in B$ such that $a, b \in M$. Let $t \in L(A^*, B^*)$. Then $t \in L(a, b)$. Since $a, b \in M$, we have $a \in L(L(t_i)^*, L(y_j)^*)\backslash J$ and $b \in L(L(t_k)^*, L(y_l)^*)\backslash J$ for some $t_i, t_k \in U(x)$ and $y_j, y_l \in P\backslash I$, which imply $t \in L(L(t_i)^*, L(y_j)^*)$ with $t \notin J$. Indeed, if $t \in J$, then $a \in L(L(t_i)^*) \subseteq L(t) \subseteq J$, a contradiction. So M is a strongly m-system of P. Since $M \cap J = \emptyset$ and by Theorem , there exists a strongly prime ideal I_1 of P containing J with $I_1 \cap M = \emptyset$. If $x \in I_1$, then $L(L(x)^*, L(y_i)^*) \subseteq I_1$ for every $y_i \in P\backslash I$. But there exists $q \in L(L(t_i)^*, L(y_j)^*)\backslash J$ with $q \in M$, which implies $q \in L(L(t_i)^*, L(y_j)^*) \subseteq L(L(x)^*, L(y_i)^*) \subseteq I_1$ and $I_1 \cap M \neq \emptyset$, a contradiction. So $x \notin I_1$. Let $i_1 \in I_1$ and suppose $i_1 \notin I$. Then $i_1 \in P\backslash I$ and $L(L(x)^*, L(i_1)^*) \subseteq I_1$. But $L(L(t_i)^*, L(i_1)^*) \nsubseteq J$, which implies $I_1 \cap M \neq \emptyset$, a contradiction. Thus $I_1 \subset I$, which is again a contradiction to the minimality of I.

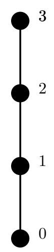
(**ii**) $\Rightarrow$ (**i**) Let I_1 be a strongly prime ideal of P with $J \subseteq I_1 \subseteq I$. Let $x \in I$. Then there exists $y \in P\backslash I$ and $t \in U(x)$ such that $L(L(t)^*, L(y)^*) \subseteq J \subseteq I_1$. Since $y \notin I_1$, we have $L(t) \subseteq I_1$, which implies $x \leq t \in I_1$. Thus $I \subseteq I_1$ and hence I is a minimal strongly prime ideal of J.

(**i**) $\Rightarrow$ (**iii**) Let $x \in I$. Then by (ii), there exists $y \notin I$ and $t \in U(x)$ such that $L(L(t)^*, L(y)^*) \subseteq J$. Since J satisfies (*) condition, we have $y \in L(y)^* \subseteq < L(t)^*, J > \subseteq < x, J >$, which implies $< x, J > \nsubseteq I$.

(**iii**) $\Rightarrow$ (**i**) Let Q be a strongly prime ideal of P such that $J \subseteq Q \subset I$ and $x \in I\backslash Q$. Then $< x, J > \nsubseteq I$. So there exists $y \in < x, J > \backslash I$ such that $L(L(x)^*, L(y)^*) \subseteq L(x, y) \subseteq J \subseteq Q$. Since $L(x) \nsubseteq Q$, we have $y \in Q$, a contradiction. □

The following example shows that the condition $J \neq \{0\}$ is not superficial in Theorem 2.2.

Example 2.3. Consider $P = \{0, 1, 2, 3\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I = \{0, 1\}$ is a minimal strongly prime ideal of $J = \{0\}$. But for $1 \in I\backslash J$, there is no $y \in P\backslash I$ and $t \in U(1)$ such that $L(L(t)^*, L(y)^*) \subseteq J$. □

Following [6], a semi-ideal I of P is called n-prime if for pairwise distinct elements $x_1, x_2, x_3, ..., x_n \in P$, if $L(x_1, x_2, x_3, ..., x_n) \subseteq I$, then at least $(n-1)$ of n subsets $L(x_2, x_3, x_4, ..., x_n), L(x_1, x_3, x_4, ..., x_n), ..., L(x_1, x_2, x_3, ..., x_{n-1})$ is a subset of I. One can define in an obvious way the concept of n-strongly prime ideal of P for $n \geq 2$. As a special case, if I is strongly prime, then I is a 2-strongly prime ideal. It might be hard to extend the results for $n > 2$. It is proved by the following theorem that n-primeness can not be generalized for $n > 2$.

Theorem 2.4. *Let I be an ideal of P. Then I has the following property that for $n > 2$, if pairwise distinct ideals $A_1, A_2, ..., A_n$ of P with $L(A_1^*, A_2^*, ..., A_n^*) \subseteq I$, then at least $(n-1)$ of n subsets $L(A_2^*, A_3^*, ..., A_n^*), L(A_1^*, A_3^*, ..., A_n^*), ..., L(A_1^*, A_2^*, ..., A_{n-1}^*)$ are subsets of I.*

Proof. Let $A_1, A_2, ..., A_n$ be distinct ideals of P with $L(A_1^*, A_2^*, ..., A_n^*) \subseteq I$. Suppose that $L(A_1^*, A_2^*, ..., A_{n-1}^*) \nsubseteq I$. Then we now prove $L(A_2^*, A_3^*, ..., A_n^*), L(A_1^*, A_3^*, ..., A_n^*), ..., L(A_1^*, A_2^*, ..., A_{n-2}^*, A_n^*)$ are subsets of I. Since $L(A_1^*, A_2^*, ..., A_{n-1}^*) \nsubseteq I$, then there exists $t \in L(A_1^*, A_2^*, ..., A_{n-1}^*) \backslash I$ such that $L(t) \subseteq L(A_1^*, A_2^*, ..., A_{n-1}^*)$, which implies $L(t, A_n^*) \subseteq I$. So for each $j \in \{1, 2, 3, ..., n-1\}$, we have $L(\bigcup_{i=1, i\neq j}^{n-1} A_i^*, A_n^*) \subseteq L(t, A_n^*) \subseteq I$. □

In the generalization of n-primeness in posets, by the above theorem, we get that the cases $n = 2$ and $n \geq 3$ are substantially different. Hence a 2-strongly prime ideal only exists in P. If $n > 2$, then every ideal of P is n-strongly prime.

Lemma 2.5. *Let I be a semi-prime ideal of P with (*) condition. Then $< C^*, I >$ is a strongly prime ideal of P for any ideal C of P.*

Proof. Let A, B and C be ideals of P with $L(A^*, B^*) \subseteq < C^*, I >$. Then $L(A^*, B^*, C^*) \subseteq I$. By Theorem 2.4, we have $A \subseteq < C^*, I >$ or $B \subseteq < C^*, I >$. □

Theorem 2.6. *Let $J \neq \{0\}$ be an ideal of P. Then there are at most two strongly prime ideals of P that are minimal over J.*

Proof. Suppose that I_1, I_2 and I_3 are three pairwise distinct strongly prime ideals of P that are minimal over J. Then there exist $x_1 \in I_1 \backslash I_2$ and $x_2 \in I_2 \backslash I_1$. Since $x_1 \in I_1$ and by Theorem 2.2, there exist $c_2 \notin I_1$ and $t_1 \in U(x_1)$ such that $L(L(t_1)^*, L(c_2)^*) \subseteq J$. Also for $x_2 \in I_2$, there exists $c_1 \notin I_2$ and $t_2 \in U(x_2)$ such that $L(L(t_2)^*, L(c_1)^*) \subseteq J$. Then $L(L(t_2)^*, L(c_1)^*) \subseteq < x_1, J >$, which implies $L(L(t_2)^*, L(c_1)^*, L(x_1)^*) \subseteq J$. Suppose $L(L(x_1)^*, L(c_1)^*) \subseteq J$. Then $L(L(x_1)^*, L(c_1)^*) \subseteq I_2$. Since I_2 is a strongly prime ideal of P, we have $c_1 \in L(c_1) \subseteq I_2$ or $x_1 \in L(x_1) \subseteq I_2$, a contradiction. By Theorem 2.4, we have $L(L(t_2)^*, L(c_1)^*) \subseteq J$ and $L(L(t_2)^*, L(x_1)^*) \subseteq J$. Clearly $I_1 \nsubseteq I_2 \cup I_3, I_2 \nsubseteq I_1 \cup I_3$ and $I_3 \nsubseteq I_1 \cup I_2$. Indeed, if $I_1 \subseteq I_2 \cup I_3$, then $I_1^* \subseteq I_2^* \cup I_3^*$, which implies $L(I_2^*, I_3^*) \subseteq I_1$. Since I_1 is strongly prime, we have $I_2 \subseteq I_1$ or $I_3 \subseteq I_1$, a contradiction. So we can choose $y_1 \in I_1 \backslash (I_2 \cup I_3); y_2 \in I_2 \backslash (I_1 \cup I_3); y_3 \in I_3 \backslash (I_2 \cup I_1)$. By the above argument, we have $L(L(t_2)^*, L(y_1)^*) \subseteq J \subseteq I_3$ for some $t_2 \in U(y_2)$. Since I_3 is strongly prime, we have $y_1 \in I_3$ or $y_2 \leq t_2 \in I_3$, a contradiction. So there are at most two strongly prime ideal that are minimal over J. □

Theorem 2.7. *Let $J \neq \{0\}$ be an ideal of P. Then at least one of the following statement must hold.*

(i) *$SP(J) = I_1$ is a strongly prime ideal of P.*

(ii) *$SP(J) = I_1 \cap I_2$, where I_1 and I_2 are the distinct strongly prime ideal of P that are minimal over J. If J satisfies (*) condition, then $L(I_1^*, I_2^*) \subseteq J$.*

Proof. By Theorem 2.6, we have $SP(J) = I_1$ is a strongly prime ideal of P or $SP(J) = I_1 \cap I_2$, where I_1 and I_2 are the distinct strongly prime ideals of P that are minimal over J. Since I_1 and I_2 are distinct, there exists $x_1 \in I_1 \backslash I_2$ and $x_2 \in I_2 \backslash I_1$. By Theorem 2.2, there exist $c_2 \notin I_1$ and $t_1 \in U(x)$ such that $L(L(t_1)^*, L(c_2)^*) \subseteq J$ and there exist $c_1 \notin I_2$ and $t_2 \in U(x)$ such that $L(L(t_2)^*, L(c_1)^*) \subseteq J$. Then $L(L(t_2)^*, L(c_1)^*) \subseteq < x_1, J >$, which implies $L(L(t_2)^*, L(c_1)^*, L(x_1)^*) \subseteq J$. Suppose $L(L(x_1)^*, L(c_1)^*) \subseteq J$. Then $L(L(x_1)^*, L(c_1)^*) \subseteq I_2$, so $c_1 \in L(c_1) \subseteq I_2$ or $x_1 \in L(x_1) \subseteq I_2$, a contradiction. By Theorem 2.4, we have $L(L(t_2)^*, L(c_1)^*) \subseteq J$ and $L(L(t_2)^*, L(x_1)^*) \subseteq J$, which imply $x_1 \in L(x_1)^* \subseteq < L(t_2)^*, J > \subseteq < x_2, J >$. Thus $L(x_1, x_2) \subseteq J$ and hence $L(I_1^*, I_2^*) \subseteq J$. □

Theorem 2.8. *Let $J \neq \{0\}$ be a semi-prime ideal of P such that $J \neq SP(J) = I$ is a strongly prime ideal of P and J satisfies (*) condition. Then for each $x \in I \backslash J$ and $I \subseteq < x, J >$, we have $< x, J >$ is a strongly prime of P. Furthermore $< y, J > \subseteq < x, J >$ or $< x, J > \subseteq < y, J >$ for every $x, y \in I \backslash J$.*

Proof. Let $x \in I \backslash J$ and $I \subset < x, J >$ with $L(A^*, B^*) \subseteq < x, J >$ for different proper ideals A, B of P. If $L(A^*, B^*) \subseteq I$, then $A \subseteq < x, J >$ or $B \subseteq < x, J >$. Let $L(A^*, B^*) \nsubseteq I$. Since $L(A^*, B^*) \subseteq < x, J >$, we have $L(A^*, B^*, L(x)^*) \subseteq J$. Then by Theorem 2.4, we have $L(A^*, L(x)^*) \subseteq J$ and $L(B^*, L(x)^*) \subseteq J$, which imply $A \subseteq < x, J >$ and $B \subseteq < x, J >$. So $< x, J >$ is a strongly prime ideal of P.

Let $x, y \in I \backslash J$. If $< x, J > \nsubseteq < y, J >$, then there exists $z \in < x, J >$ with $z \notin I$. Let $w \in < y, J >$. If $w \in I/J$, then $< y, J > \subseteq < x, J >$. Suppose $w \notin I$. Then $L(L(z)^*, L(w)^*) \nsubseteq I$. Since $L(L(z)^*, L(x)^*) \subseteq J$, we have $L(L(z)^*, L(x)^*, L(w)^*) \subseteq J$. By Theorem 2.4, we have $L(L(z)^*, L(x)^*) \subseteq J$ and $L(L(x)^*, L(w)^*) \subseteq J$, which imply $w \in < x, J >$. □

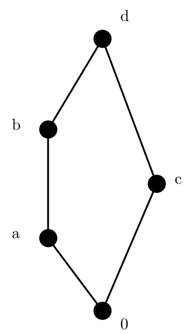
Theorem 2.9. *Let $J \neq \{0\}$ be a semi-prime ideal of P such that $J \neq SP(J) = I_1 \cap I_2$, where I_1, I_2 are distinct strongly prime ideals of P that are minimal over J and J satisfies (*) condition. Then for each $x \in SP(J) \backslash J$ and $SP(J) \subseteq < x, J >$, we have $< x, J >$ is a strongly prime of P containing I_1 and I_2. Furthermore either $< y, J > \subseteq < x, J >$ or $< x, J > \subseteq < y, J >$ for every $x, y \in SP(J) \backslash J$.*

Proof. Let $x \in (I_1 \cap I_2) \backslash J$. Then by Theorem 2.7, $L(I_1^*, I_2^*) \subseteq J$, which implies $I_1 \subseteq < x, J >$ and $I_2 \subseteq < x, J >$. Let $L(A^*, B^*) \subseteq < x, J >$ for some different ideals A, B of P. If $A, B \subseteq I_1$ or $A, B \subseteq I_2$, then $I_1 \subseteq < x, J >$ or $I_2 \subseteq < x, J >$. If $A, B \nsubseteq I_1$ or $A, B \nsubseteq I_2$, then $L(A^*, B^*) \nsubseteq J$. Since $L(A^*, B^*) \subseteq < x, J >$, we have $L(A^*, B^*, L(x)^*) \subseteq J$. By Theorem 2.4, we have $L(A^*, L(x)^*) \subseteq J$ and $L(B^*, L(x)^*) \subseteq J$. Hence $A \subseteq < x, J >$ and $B \subseteq < x, J >$. It follows from Theorem

2.8 that either $< y, J > \subseteq < x, J >$ or $< x, J > \subseteq < y, J >$ for every $x, y \in SP(J) \backslash J$.
/,□

Following [4] and [5], for an ideal I of P, an element $x \in P$ is called *prime* to I if $< x, I > = I$. A proper ideal I of P is called *primal* if the set $S(I) = \{x \in P : x$ is not prime to $I\}$ is an ideal of P and it is called *the adjoint* set of I. It is clear that for any ideal I of P, $I \subseteq S(I)$ and $S(I)$ is a semi-ideal of P, but $S(I)$ is not necessarily to be an ideal of P as shows in the following example.

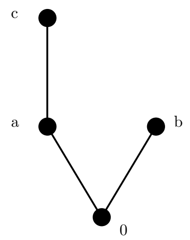
Example 2.10. Consider $P = \{0, a, b, c, d\}$ and define the partial relation $\leq$ on P as follows



Then $(P, \leq)$ is a poset and $I = \{0\}$ is an ideal of P. Here $S(I) = \{0, a, b, c\}$ is a semi-ideal of P, but not an ideal of P as $L(U(b, c)) = \{0, a, b, c, d\} \nsubseteq S(I)$. □

If I is a proper ideal of P, then $I = S(I)$ ([9], Theorem 20), so every proper prime ideal of P is primal. But Example 2.11 shows that there exists a primal ideal of P which is not necessarily to be prime.

Example 2.11. Consider $P = \{0, a, b, c\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset with a primal ideal $I = \{0\}$. Here I is not a prime ideal of P. □

Following [7], a proper ideal I of P is called irreducible if for any ideals J and K of P, $I = J \cap K$ implies $J = I$ or $K = I$.

Lemma 2.12. *Every prime ideal of a poset P is irreducible.*

Proof. Let I be a prime ideal of P such that $I = J \cap K$ for some ideals J and K of P. Then $I \subseteq J$ and $I \subseteq K$. Suppose $I \neq J$ and $I \neq K$. Then there exist $x, y \in P$ such that $x \in J \backslash I$ and $y \in K \backslash I$, which imply $L(x, y) \subseteq J \cap K \subseteq I$, so either $x \in I$ or $y \in I$, a contradiction. □

In Example 2.10, $I = \{0, a\}$ is an irreducible ideal of P, but not primal. So the converse of Lemma is not true in general. But we have the following.

Theorem 2.13. *Every irreducible semi-prime ideal of P is primal.*

Proof. Let I be an irreducible semi-prime ideal of P and $x_1, x_2 \in S(I)$. Then $I \subset < x_1, I > \cap < x_2, I >$ and there exists $a \in (< x_1, I > \cap < x_2, I >) \backslash I$ such that $L(a, x_1) \subseteq I$ and $L(a, x_2) \subseteq I$. Since I is semi-prime, we have $L(a, U(x_1, x_2)) \subseteq I$. So for any $t \in L(U(x_1, x_2))$, we have $L(a, t) \subseteq I$. □

For any ideal I of P, if $S(I)$ is a strongly prime ideal of P, then I is called a *S-primal* ideal of P. Following [2], for an ideal I and a strongly prime ideal Q of P, we define $I_Q = \{x \in P : L(s, x) \subseteq I \text{ for some } s \in P \backslash Q\} = \bigcup_{s \in P \backslash Q} < s, I >$.

Lemma 2.14 *Let I be a Q-primal ideal of P with (*) condition. Then $< x, I >_Q = < x, I_Q >$.*

Proof. Let $y \in < x, I >_Q$. Then there exists $c \notin Q$ such that $L(y, c) \subseteq < x, I >$, which implies $L(x, t) \subseteq I$ for all $t \in L(y, c)$. Since $t \in L(y)$, we have $L(L(y), x) \subseteq L(t, x) \subseteq I \subseteq I_Q$. Then $y \in < x, I_Q >$. Let $a \in < x, I_Q >$. Then $L(x, a) \subseteq I_Q$, which implies $c \notin Q$ such that $L(t, c) \subseteq I$ for all $t \in L(x, a)$. Since $c \notin Q$, we have $t \in < c, I >= I$. Hence $a \in < x, I >_Q$. □

Theorem 2.15. *Let I be a Q_1-primal ideal of P. Then*

(i) *If Q_2 is a strongly prime ideal of P containing Q_1, then $I_{Q_2} = I$.*

(ii) *If Q_2 is a strongly prime ideal of P not containing Q_1, then $I_{Q_2} \supset I$*

(iii) *If I_{Q_2} is a Q_2-primal ideal for some strongly prime ideal Q_2 containing I and I satisfies (*) condition, then $Q_2 \subseteq Q_1$.*

Proof. (i) Let $x \in I_{Q_2}$. Then there exists $c \notin Q_2$ such that $L(x, c) \subseteq I$, which implies $c \notin Q_1$ with $L(x, c) \subseteq I$. Since I is Q_1-primal and $c \notin Q_1$, we have c is prime to I. So $x \in < c, I >= I$.
(ii) Let $x \in Q_1 \backslash Q_2$. Then x is not prime to I and for some $y \notin I$, we have $L(x, y) \subseteq I$. Since $x \notin Q_2$, we have $y \in I_{Q_2}$. But $y \notin I$. Hence $I \subset I_{Q_2}$.
(iii) Suppose Q_2 is not a subset of Q_1. Then there exists $q \in Q_2 \backslash Q_1$ such that $L(q, x) \subseteq I_{Q_2}$ for some $x \in P \backslash I_{Q_2}$, which implies $c \in P \backslash Q_2$ such that $L(t, c) \subseteq I$ for all $t \in L(q, x)$. Since $t \in L(x)$, we have $L(L(x), c) \subseteq L(t, c) \subseteq I$. However $c \notin Q_2$, so that $x \in I_{Q_2}$, a contradiction. □

We now present more properties of primal ideals in a poset by using the following definition. For an ideal I of a poset P, the notation $C(I) = \{x \in P \backslash I : L(x, y) \subseteq I$ for some $y \in P \backslash I\}$. If I is strongly prime, then $C(I) = \emptyset$.

Lemma 2.16. *Let I be a proper ideal of P. Then the following hold.*

(i) *$I \subseteq S$, where S is the adjoint set of I.*

(ii) *$C(I) = S \backslash I$.*

Proof. (i) Let $x \in I$. Then $L(x, y) \subseteq I$ with $y \notin I$, which implies x is not prime to I. So $x \in S$.
(ii) Let $r \in C(I)$. Then $r \notin I$ and $L(x, r) \subseteq I$ for some $x \notin I$, which imply $r \in S$. Since $r \notin I$, we have $C(I) \subseteq S \backslash I$. Conversely, let $a \in S \backslash I$. Then there exists $y \notin I$ such that $L(a, y) \subseteq I$. So $a \in C(I)$. Hence $C(I) = S \backslash I$. □

Theorem 2.17. *Let I and J be proper ideals of P with $I \subseteq J$. Then I is a J-primal ideal of P if and only if $C(I) = J \backslash I$.*

Proof. Let I be a J-primal ideal of P. Then by Lemma , we have $C(I) = S \backslash I = J \backslash I$. Let $C(I) = J \backslash I$. Then it is enough to prove that J is exactly the set of elements that are not prime to I. Let $c \in J$. If $c \in I$, then $< c, I >= P \neq I$. So c is not prime to I. If $c \in J \backslash I = C(I)$, then there exists $z \notin I$ such that $L(z, c) \subseteq I$. It gives c is not prime to I. Suppose $x \notin J$ and x is not prime to I. Then there exists $t \notin I$ such that $L(x, t) \subseteq I$, which implies $x \in C(I) = J \backslash I$, a contradiction. We now prove that J is strongly prime. It is enough to prove that $S(J) = J$. Clearly $J \subseteq S(J)$. Let $t \in S(J)$. If $t \in I$, then $t \in S(J)$. If $t \notin I$, then there exists $s \notin I$ such that $L(s, t) \subseteq I$, which implies $s \in C(I) = J \backslash I \subseteq J$. Hence I is a J-primal ideal of P. □

Corollary 2.18. *Let I be an ideal of P. Then I is a primal ideal of P if and only if $C(I) \cup I$ is an ideal (prime ideal) of P.*

Corollary 2.19. *Let I and J be Q-primal ideals of P. Then $C(I) = C(J)$ if and only if $I = J$.*

Lemma 2.20. *Let I be an ideal of P with (*) condition. Then I is strongly semi prime of P if and only if $< t, I >$ is a strongly semi-prime ideal of P for any $t \in P$.*

Proof. It follows directly from Theorem 2.7 of [3]. □

Theorem 2.21. *Let $J \neq \{0\}$ be a strongly semi-prime ideal of P such that $J \neq SP(J) \subseteq < x, J >$ for $x \in SP(J) \backslash J$ and J satisfies (*) condition. Then J is a Q-primal ideal of P, where* $Q = \bigcup_{x \in SP(J) \backslash J} < x, J >$.

Proof It is clear that $J \subseteq Q$. We now prove that all elements of Q are not prime to J. Let $a, b \in P \backslash J$ such that $L(a, b) \subseteq J$. It is enough to prove $a, b \in < t, J >$ for some $t \in SP(J) \backslash J$. By Theorem 2.7, we have $SP(J) = I$ is a strongly prime ideal of P or $SP(J) = I_1 \cap I_2$, where I_1 and I_2 are the only distinct strongly prime ideals of P that are minimal over J.

If $SP(J) = I$ is a strongly prime ideal of P, then either $a \in I \backslash J$ or $b \in I \backslash J$. By Theorem 2.8, we have $< a, J > \subseteq < b, J >$ or $< b, J > \subseteq < a, J >$, which implies

$a, b \in < a, J >$ or $a, b \in < b, J >$. Then $D = \{< t, J >: t \in SP(J)\backslash J\}$ is a linearly ordered set of ideals and by Lemma , they are strongly semi-prime ideals. Following Theorem 2.8 of [3] and by Zorn's lemma, there exists a strongly prime ideal $Q = \bigcup_{x \in SP(J)\backslash J} < x, J >$ of P.

If $SP(J) = I_1 \cap I_2$, where I_1 and I_2 are the only distinct strongly prime ideals of P that are minimal over J, then either $a \in SP(J)\backslash I$ or $a \in I_1\backslash I_2$ and $b \in I_2\backslash I_1$. If $a \in SP(J)\backslash J$, then $a, b \in < a, J >$. Suppose that $a \in I_1\backslash I_2$ and $b \in I_2\backslash I_1$. Since $J \neq SP(J)$, there exists $d \in SP(J)\backslash J$. By Theorem 2.9, $I_1^* \subseteq < d, J >$ and $I_2^* \subseteq < d, J >$, which imply $a, b \in < d, J >$. Then $D = \{< d, J >: d \in SP(J)\backslash J\}$ is a linearly ordered set of strongly semi-prime ideals. By Zorn's lemma and Theorem 2.8 of [3], there exists a strongly prime ideal $Q = \bigcup_{x \in SP(J)\backslash J} < x, J >$ of P. □

Acknowledgments. The authors express their sincere thanks to the referee for his/her valuable comments and suggestions which improve the paper a lot.

References

[1] J. Catherine Grace John and B. Elavarasan, *Primeness of extension of semi-ideals in posets*, Applied Mathematical Sciences, **164(8)**(2014), 8227-8232.

[2] J. Catherine Grace John and B. Elavarasan, *z^J-ideals and strongly prime ideals in posets*, submitted.

[3] J. Catherine Grace John and B. Elavarasan, *Strongly prime and strongly semiprime ideals in Posets*, Global Journal of Pure and Applied Mathematics, **11(5)**(2015), 2965-2970.

[4] L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc., **1**(1950), 1 - 6.

[5] L. Fuchs, W. Heninzer and B. Olberding, *Commutative ideal theory without finiteness conditions: Primal ideal*, Trans. Amer. Math. Soc., **357(7)**(2004), 2771-2798.

[6] R. Halaš, *On extension of ideals in posets*, Discrete Mathematics, **308**(2008), 4972-4977.

[7] G. H. Jiang and L. S. Xu, *Maximal ideals relative to a filter on posets and some applications*, Int. J. Contemp. Math. Sciences, **3(9)**(2008), 401-410.

[8] Jan Paseka, *Note on distributive posets*, Math. Appl., **1**(2012), 197-206.

[9] V. S. Kharat and K. A. Mokbel, *Primeness and semiprimeness in posets*, Math. Bohem., **134(1)**(2009), 19-30.

[10] P. V. Venkatanarasimhan, *Semi ideals in posets*, Math. Ann., **185(4)**(1970), 338-348.