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On Some Modular Equations in the Spirit of Ramanujan

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Abstract. In this paper, we establish some new P-Q type modular equations, by using the modular equations given by Srinivasa Ramanujan.

1. Introduction

In Chapter 16 of his second notebook [9], S. Ramanujan developed, theory of theta-function and his theta-function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Note that, if we set $a = q^{2iz}$, $b = q^{-2iz}$, where z is complex and $Im(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [16, p. 464]. The three most important special cases of $f(a,b)$ [4, p, 36] are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2{}_\infty (q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty,$$

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where we employ the customary notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

We now define a modular equation as given by Ramanujan. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2Sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{(n!)^2}k^{2n} = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.1}$$

where $0<k<1$. The series representation in (1.1) is found by expanding the integrand in a binomial series and integrating termwise and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \qquad |z|<1,$$

with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

where a, b and c are complex numbers such that c is not a nonpositive integer. The number k is called the modulus of K and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with moduli k, k' l and l' respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.2}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.2). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We often say that β has degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}.$$

Ramanujan [4, p. 122-124] recorded several formulae for φ, ψ, f and χ at different arguments of α q and $z := {}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)$ by using

$$\varphi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \qquad q = exp(-\pi K'/K).$$

Ramanujan's modular equations involve quotients of function $f(-q)$ at certain arguments. For example [5, p. 206], let

$$P := \frac{f(-q)}{q^{1/6}f(-q^5)} \qquad \text{and} \qquad Q := \frac{f(-q^2)}{q^{1/3}f(-q^{10})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{1.3}$$

These modular equations are also called Schläfli-type. Since the publication of [5], several authors, including N. D. Baruah [2], [3] M. S. M. Naikia [7], [8] K. R. Vasuki [12], [13] and K. R. Vasuki and B. R. Srivatsa Kumar [14] have found additional modular equations of the type (1.3). Recently C. Adiga, et. al. [1] have established several modular relations for the Rogers-Ramanujan type functions of order eleven which analogous to Ramanuja's forty identities for Rogers-Ramanujan functions and also they established certain interesting partition-theoritic interpretation of some of the modular relations and H. M. Srivastava and M. P. Chaudhary [11] established a set of four new results which depicit the interrelationships between q-product identities, coninued fraction identities and combinatorial partition identities.

On page 366 of his 'Lost' notebook [10], Ramanujan has recorded a continued fraction

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+\ldots} \qquad |q|<1,$$

and claimed that there are many results of $G(q)$ which are analogous to the famous Roger's-Ramanujan continued fraction. Motivated by Ramanujan's claim H. H. Chan [6], N. D. Baruah [2], K. R. Vasuki and B. R. Srivatsa Kumar [15] have established new identities providing the relations between $G(q)$ and seven continued fractions $G(-q), G(q^2), G(q^3), G(q^5),\ G(q^7), G(q^{11})$ and $G(q^{13})$. We conclude this introduction by recalling certain results on $G(q)$ stated by Ramanujan [4] and H. H. Chan [6].

$$G(-q) := q^{1/3}\frac{\chi(q)}{\chi(q^3)} \tag{1.4}$$

where $\chi(q)$ is defined as $\chi(q) = (-q;q^2)_\infty$.

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0 \tag{1.5}$$

and

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0. \tag{1.6}$$

For a proof of (1.5) and (1.6), see [6].
Motivated by the above works in this paper, we establish some new P-Q type modular equations, by employing Ramanujan's modular equations.

2. Main Results

Theorem 2.1. *If*

$$X := q^{1/3}\frac{\chi(q)\chi(q^6)}{\chi(q^3)\chi(q^2)} \qquad \text{and} \qquad Y := q^{2/3}\frac{\chi(q^2)\chi(q^{12})}{\chi(q^6)\chi(q^4)}$$

then

$$2X^2-22Y^4X^3-2Y+4Y^2X-18X^2Y^3+17Y^9X^2-10Y^8X+17Y^{10}X^3+Y^{11}X$$
$$+34Y^5X+328Y^7X^3-160Y^6X^2-30Y^7X^6-30Y^6X^5+12Y^5X^4-371Y^8X^4+328Y^9X^5$$
$$-10Y^{11}X^4-160Y^{10}X^6+34Y^{11}X^7-22Y^9X^8+12Y^8X^7+4Y^{11}X^{10}-18Y^{10}X^9-2Y^{12}X^{11}$$
$$+10Y^2X^4+20Y^4X^6+20Y^6X^8+10X^{10}Y^8+2Y^{10}X^{12}=0.$$

Proof. From (1.4) and the definition of X and Y, it can be seen that

$$B-AX=0 \qquad \text{and} \qquad C-BY=0. \tag{2.1}$$

where $A=G(-q)$, $B=G(-q^2)$ and $C=G(-q^4)$. On changing q to q^2 in (1.5), we have

$$G(q^2)+G(-q^2)+2G^2(-q^2)G^2(q^2)=0 \tag{2.2}$$

and also change q to $-q$ in (1.6), we have

$$G^2(-q)+2G^2(q^2)G(-q)-G(q^2)=0. \tag{2.3}$$

Eliminating $G(q^2)$ between (2.2) and (2.3) using Maple,

$$2(AB)^4-4(AB)^3+3(AB)^2+AB+A^3+B^3=0. \tag{2.4}$$

Now on using first identity of (2.1) in (2.4), we obtain

$$2B^6-4B^4X+3B^2X^2+X^3+BX+BX^4=0. \tag{2.5}$$

On replacing q to q^2 in (2.4) we see that

$$2(BC)^4-4(BC)^3+3(BC)^2+BC+B^3+C^3=0.$$

Using second identity of (2.1) in the above, it is easy ton see that

$$2B^6Y^4-4B^4Y^3+3B^2Y^2+Y+B+BY^3=0.$$

Finally, on eliminating B between (2.5) and the above, using Maple we obtain

$$P(X,Y)Q(X,Y)=0,$$

where

$$P(X,Y)=X-16Y^4X^2-6XY^3-6Y^5X^3-2Y^5+Y^5X^6+10Y^3X^4+10Y^2X^3 +5YX^2+5Y^4X^5-2Y^6X$$

and

$$Q(X,Y)=-2Y+2X^2-22Y^4X^3+4Y^2X-18X^2Y^3+17Y^9X^2-10Y^8X+17Y^{10}X^3 +Y^{11}X+328Y^7X^3+34Y^5X-160Y^6X^2-30Y^7X^6-30Y^6X^5+12Y^5X^4-18Y^{10}X^9 -371Y^8X^4+328Y^9X^5-10Y^{11}X^4-160Y^{10}X^6+34Y^{11}X^7-22Y^9X^8+12Y^8X^7 +4Y^{11}X^{10}-2Y^{12}X^{11}+2Y^{10}X^{12}+10Y^2X^4+20Y^6X^8+20Y^4X^6+10X^{10}Y^8.$$

By examining the behavour of the first factor near $q=0$, it can be seen that there is a neighbourhood about the origin, where $P(X,Y)\neq 0$ and $Q(X,Y)=0$ in this neighbourhood. Hence by the identity theorem, we have $Q(X,Y)=0$. □

Theorem 2.2. *If*

$$X:=q^{1/6}\frac{\chi^2(q^3)}{\chi(q)\chi(q^9)} \qquad \textit{and} \qquad Y:=q^{1/3}\frac{\chi^2(q^6)}{\chi(q^2)\chi(q^{18})}$$

then

$$\left(\frac{X}{Y}\right)^3+\left(\frac{Y}{X}\right)^3+\left\{(XY)^{5/2}+\frac{1}{(XY)^{5/2}}+11\left((XY)^{1/2}+\frac{1}{(XY)^{1/2}}\right)\right\}\left[\left(\frac{X}{Y}\right)^{3/2}+\left(\frac{Y}{X}\right)^{3/2}\right]$$

(2.6)

$$=(XY)^3+\frac{1}{(XY)^3}-11\left((XY)^2+\frac{1}{(XY)^2}\right)+44\left(XY+\frac{1}{XY}\right)+8\left(X^3+\frac{1}{X^3}\right)+8\left(Y^3+\frac{1}{Y^3}\right)-86.$$

Proof. From Entry 12(v)of Chapter 17 [4, p. 124], we have

$$X=\left\{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right\}^{1/24}. \tag{2.7}$$

where β and γ be of the third and ninth degrees respectively, with respect to α. Let

$$B:=q^{1/3}\frac{\chi^2(-q^6)}{\chi(-q^2)\chi(-q^{18})}.$$

Then from Entry 12(vii) of Chapter 17 [4, p. 124], we have

$$B = \left\{ \frac{\alpha^2\gamma^2(1-\beta)^2}{\beta^4(1-\alpha)(1-\gamma)} \right\}^{1/24}. \tag{2.8}$$

By (2.7) and (2.8), we deduce that

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/8} = XB \qquad \text{and} \qquad \left\{ \frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2} \right\}^{1/8} = \frac{X^2}{B}. \tag{2.9}$$

From Entry 3 (xii) and (xiii) of Chapter 20 [4, p. 352-358], we have

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = -\frac{3m}{m'} \tag{2.10}$$

and

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} = \frac{m'}{m}. \tag{2.11}$$

where $m = z_1/z_3$ and $m' = z_3/z_9$. Thus (2.9), (2.10) and (2.11) yields

$$M(X^2B^4 + X^4 - B^2X^6) - B^2 = 0 \qquad \text{and} \qquad X^4 + X^2B^4 - B^2 + 3MX^6B^2 = 0.$$

where $M = m/m'$. Which implies

$$X^6B^6 - 6B^4X^4 - B^8X^2 + B^6 - X^6 + B^2X^2 + X^8B^2 = 0. \tag{2.12}$$

Let

$$A := q^{1/6}\frac{\chi^2(-q^3)}{\chi(-q)\chi(-q^9)}$$

Then, from Entry 12(vi) of Chapter 17 [4, p. 124], we have

$$A = \left\{ \frac{\alpha\gamma(1-\beta)^4}{\beta^2(1-\alpha)^2(1-\gamma)^2} \right\}^{1/24}. \tag{2.13}$$

From (2.7) and (2.13), we obtain

$$\left\{ \frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2} \right\}^{1/8} = \frac{X}{A} \qquad \text{and} \qquad \left(\frac{\alpha\gamma}{\beta^2}\right)^{1/8} = AX^2.$$

Using the above in (2.10) and (2.11), we deduce

$$(X^4A^4 + X^2 - X^6A^2)M - A^2 = 0 \qquad \text{and} \qquad X^2 + X^4A^4 - A^2 + 3MX^6A^2 = 0.$$

From the above two identities, we obtain

$$X^8A^6 - 6X^4A^4 - A^8X^6 + A^6X^2 - X^2 + A^2 + X^6A^2 = 0.$$

Changing q to q^2 in the above, we have

$$Y^8B^6 - 6Y^4B^4 - B^8Y^6 + B^6Y^2 - Y^2 + B^2 + Y^6B^2 = 0. \tag{2.14}$$

Now on eliminating B, between (2.12) and (2.14), using Maple we obtain

$$C(X,Y)D(X,Y) = 0.$$

where

$$C(X,Y) = X^4Y + X^3 + XY + 6Y^2X^2 + Y^3X^3 + Y^3 + XY^4$$

and

$$D(X,Y) = X^8Y^5 - Y^7X^7 - 8Y^4X^7 + X^7Y + 11X^6Y^6 + 11Y^3X^6 + X^5Y^8 - 44Y^5X^5 + 11X^5Y^2 - 8X^4Y^7 + 86Y^4X^4 - 8X^4Y + 11Y^6X^3 - 44Y^3X^3 + X^3 + 11Y^5X^2 + 11Y^2X^2 + XY^7 - 8XY^4 - XY + Y^3.$$

By examining the behavour of $C(X,Y)$ near $q = 0$, it can be seen that there is a neighbourhood about the origin, where this factor is not zero. Then the second factor $D(X,Y) = 0$ in this neighbourhood. Hence by the identity theorem, we have

$$D(X,Y) = 0.$$

On dividing the above throughout by $(XY)^4$, we obtain the result. □

Theorem 2.3. *If*

$$X := q^{1/3}\frac{\chi(q^3)\chi(q^5)}{\chi(q)\chi(q^{15})} \qquad \text{and} \qquad Y := q^{2/3}\frac{\chi(q^6)\chi(q^{10})}{\chi(q^2)\chi(q^{30})}$$

then

$$\left(\frac{X}{Y}\right)^3 + \left(\frac{Y}{X}\right)^3 + \left[(XY)^{5/2} + \frac{1}{(XY)^{5/2}} + (XY)^{1/2} + \frac{1}{(XY)^{1/2}}\right]\left(\left(\frac{X}{Y}\right)^{3/2} + \left(\frac{Y}{X}\right)^{3/2}\right)$$

$$= (XY)^3 + \frac{1}{(XY)^3} - 5\left((XY)^2 + \frac{1}{(XY)^2}\right) + 10\left(XY + \frac{1}{XY}\right) + 4\left(X^3 + \frac{1}{X^3} + Y^3 + \frac{1}{Y^3}\right) - 20. \tag{2.15}$$

Proof. Let

$$B := q^{2/3}\frac{\chi(-q^6)\chi(-q^{10})}{\chi(-q^2)\chi(-q^{30})}.$$

By Entry 12(v) and (vii) of Chapter 17 [4, p. 124], we have

$$X = \left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/24} \qquad \text{and} \qquad B = \left\{\frac{\alpha^2\delta^2(1-\beta)(1-\gamma)}{\beta^2\gamma^2(1-\alpha)(1-\delta)}\right\}^{1/24}, \tag{2.16}$$

where α, β, γ and δ are of the first, third, fifth and fifteenth degrees respectively. From (2.16), we deduce that

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} = XB, \qquad \left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/8} = \frac{X^2}{B}. \tag{2.17}$$

From Entry 11(viii) and (ix) of Chapter 20 [4, p. 383-397], we have

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}} \tag{2.18}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.19}$$

Employing (2.17) in (2.18) and (2.19), we obtain

$$M(XB^2 + X^2 - X^3B) - B = 0 \quad \text{and} \quad X^2 + B^2X - B + MBX^3 = 0,$$

where $M = \sqrt{m/m'}$. Which implies

$$4X^2B^2 + X^3 - X^4B + XB^4 - X^3B^3 - B^3 - BX = 0. \tag{2.20}$$

Let

$$A := q^{1/3}\frac{\chi(-q^3)\chi(-q^5)}{\chi(-q)\chi(-q^{15})}.$$

Then, by employing Entry 12(vi) of Chapter 17 [4, p. 124] and (2.16) we deduce that

$$\left\{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right\}^{1/8} = \frac{X}{A} \quad \text{and} \quad \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} = AX^2.$$

Using these in (2.18) and (2.19), upon simplifying the resulting identities, and then replacing q by q^2, we obtain

$$4B^2Y^2 + Y - BY^3 + B^4Y^3 - B^3Y^4 - B^3Y - B = 0. \tag{2.21}$$

Eliminating B from (2.20) and (2.21), using Maple we obtain

$$C(X,Y)D(X,Y) = 0.$$

where

$$C(X,Y) = X^4Y + X^3 + XY + 4Y^2X^2 + Y^3X^3 + Y^3 + XY^4$$

and

$$D(X,Y) = X^8Y^5 - X^7Y^7 - 4X^7Y^4 + X^7Y + 5X^6Y^6 + X^6Y^3 + X^5Y^2 + X^5Y^8$$

$$-10Y^5X^5-4X^4Y^7+20X^4Y^4-4X^4Y+X^3Y^6-10Y^3X^3+X^3+X^2Y^5$$
$$+5Y^2X^2+XY^7-4XY^4-XY+Y^3.$$

It is same as discussed in Theorem 2.2, that $C(X,Y)\neq 0$ near $q=0$ whereas $D(X,Y)=0$ in some neighbourhood $q=0$. Hence by identity theorem, we have

$$D(X,Y)=0.$$

Finally, on dividing the above throughout by $(XY)^4$, we obtain the result. □

Theorem 2.4. *If*

$$X:=q^{2/3}\frac{\chi(q)\chi(q^7)}{\chi(q^3)\chi(q^{21})}\qquad \textit{and}\qquad Y:=q^{4/3}\frac{\chi(q^2)\chi(q^{14})}{\chi(q^6)\chi(q^{42})}$$

then

$p_{12}+14p_{11}+229p_{10}+1328p_9+1635p_8-15550p_7-8529p_6-177572p_5-37641p_4+764070p_3+2368728p_2+4125694p_1-2(2q_{23}+24q_{21}+158q_{19}+586q_{17}+663q_{15}+13509q_{13}+43169q_{11}+36801q_9-14490q_7-612613q_5-1259739q_3-1742545q_1)r_3-2(2q_{21}-6q_{19}+51q_{17}+208q_{15}-111q_{13}-2275q_{11}-8880q_9-22598q_7-43267q_5+65339q_3-79989q_1)r_9-2(q_{15}+2q_{13}+4q_{11}+20q_9+78q_7+88q_5+38q_3+155q_1)r_{15}+(6p_{11}+60p_{10}+162p_9-560p_8-5129p_7-11254p_6+10488p_5+126726p_4+406080p_3+828738p_2+1238441p_1+1410116)s_3+(p_{10}-10p_9+11p_8+60p_7+218p_6+896p_5+2022p_4+3816p_3+7277p_2+111558p_1+13838)s_6+(p_5-2p_4-3p_3+8p_2+2p_1-12)s_9+4907562=0.$

where

(2.22)
$$p_n=(XY)^n+\frac{1}{(XY)^n},\quad q_n=(XY)^{n/2}+\frac{1}{(XY)^{n/2}},$$
$$r_n=\left(\frac{X}{Y}\right)^{n/2}+\left(\frac{Y}{X}\right)^{n/2},\quad s_n=\left(\frac{X}{Y}\right)^{n}+\left(\frac{Y}{X}\right)^{n}.$$

Proof. Let

$$B:=q^{4/3}\frac{\chi(-q^2)\chi(-q^{14})}{\chi(-q^6)\chi(-q^{42})}.$$

Then from Entry 12 (v) and (vii) of Chapter 17 [4, p. 124], we have

(2.23)
$$X=\left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/24}\qquad\text{and}\qquad B=\left\{\frac{\beta^2\delta^2(1-\alpha)(1-\gamma)}{\alpha^2\gamma^2(1-\beta)(1-\delta)}\right\}^{1/24}.$$

where α,β,γ and δ are of the degrees 1, 3, 7 and 21 respectively. From (2.23), we deduce that

(2.24)
$$XB=\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8},\qquad \frac{X^2}{B}=\left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8}.$$

From Entry 13 of Chapter 20 [4, p. 400-403], we have

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}+\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$

$$-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1+\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\}=mm' \tag{2.25}$$

and

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}$$

$$-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\}=\frac{9}{mm'}. \tag{2.26}$$

Employing (2.24) in (2.25) and (2.26), we obtain

$$X^2B^4+X^4+B^2X^6-2BX^3(B+XB^2+X^2)-B^2M=0$$

and

$$(X^2B^4+X^4+B^2-2BX(BX^2+X+B^2))M-9B^2X^6=0.$$

where $M=mm'$, which implies

$$X^6+B^6+6B^4X^4+X^8B^2-2X^7B+X^2B^8+X^6B^6-2X^4B^7$$

$$+B^2X^2-2B^4X-2B^4X^7-2BX^4-2B^7X=0. \tag{2.27}$$

Let

$$A:=q^{2/3}\frac{\chi(-q)\chi(-q^7)}{\chi(-q^3)\chi(-q^{21})}$$

From Entry 12 (vi) of Chapter 17 [4, p. 124] and (2.23), we deduce that

$$\frac{X}{A}=\left\{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right\}^{1/8}\qquad\text{and}\qquad AX^2=\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}.$$

Employing these in (2.25) and (2.26) up on simplifying, the resulting identities and then replacing q by q^2, we obtain

$$B^2Y^6+B^2-2BY+B^8Y^6+B^6Y^8-2B^7Y^7+B^6Y^2-2B^4Y$$

$$-2B^4Y^7-2B^7Y^4-2BY^4+Y^2+6B^4Y^4=0. \tag{2.28}$$

On eliminating B between (2.27) and (2.28), using Maple we obtain

$$C(X,Y)D(X,Y)E(X,Y)=0.$$

where

$$C(X,Y)=X^6Y^6-2X^4Y^7+X^2Y^8-2XY^7-2X^7Y-2X^4Y+Y^6+Y^2X^8$$
$$+X^2Y^2+6Y^4X^4-2Y^4X-2Y^4X^7+X^6,$$

$$D(X,Y)=Y^6+256X^6Y^6+38X^4Y^7+2X^2Y^8-2X^4Y+2Y^2X^8+X^2Y^2+29Y^4X^4$$
$$-2Y^4X+38Y^4X^7-16X^{14}Y^5+66X^{10}Y^7+14X^8Y^5-10X^5Y^{11}-20X^3Y^{12}-10X^{11}Y^5$$
$$+X^{14}Y^2+72X^7Y^7-20Y^{13}X^4-35X^4Y^{10}+66Y^9X^6-16X^5Y^{14}-35X^{10}Y^4+X^{10}Y^{16}$$
$$-2X^6Y^{15}-35X^6Y^{12}-4X^7Y^{13}+66X^7Y^{10}+2X^8Y^{14}+14X^8Y^{11}+466X^8Y^8+38X^9Y^{12}$$
$$+72X^9Y^9+256X^{10}Y^{10}+12X^{11}Y^{11}+14X^{11}Y^8-2X^{12}Y^{15}+29X^{12}Y^{12}+38X^{12}Y^9$$
$$-35X^{12}Y^6-14X^{10}Y^{13}-2X^{15}Y^6-2X^{13}Y^{13}-14X^{13}Y^{10}-4X^{13}Y^7+X^{14}Y^{14}+2X^{14}Y^8$$
$$-2X^{15}Y^{12}+X^{16}Y^{10}+14X^5Y^8-4X^3Y^9+Y^{14}X^2-2XY^{10}-2YX^{10}-16X^2Y^{11}+X^6$$
$$-2X^3Y^3-20X^{13}Y^4+66X^9Y^6-16Y^2X^{11}-14X^6Y^3-20Y^3X^{12}-4Y^3X^9+12X^5Y^5-14X^3Y^6.$$

and $E(X,Y)$ is as in (2.22).

As discussed in Theorem 2.2, by examining the behaviour of $C(X,Y)$ and $D(X,Y)$ near $q=0$, it can be seen that there is a neighbourhood about the origin, where these factors are not zero. Then the third factor $E(X,Y)=0$ in this neighbourhood. Hence by identity theorem, we have $E(X,Y)=0$. Finally, on dividing $E(X,Y)$ throughout by $(PQ)^{16}$ and then simplifying we have the result. □

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