KYUNGPOOK Math. J. 56(2016), 683-714
http://dx.doi.org/10.5666/KMJ.2016.56.3.683
pISSN 1225-6951 eISSN 0454-8124

When the Comaximal Graph of a Lattice is Toroidal

Mojgan Afkhami
Department of Mathematics, University of Neyshabur, P. O. Box 91136-899, Neyshabur, Iran
e-mail : mojgan.afkhami@yahoo.com

Khadijeh Ahmad Javaheri and Kazem Khashyarmanesh*
Department of Pure Mathematics, Internatinal Compus of Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran
e-mail : javaheri1158kh@yahoo.com *and* khashyar@ipm.ir

Abstract. In this paper we investigate the toroidality of the comaximal graph of a finite lattice.

1. Introduction

The comaximal graph of a commutative ring R was first defined in [9]. Also, in [6] and [10], the authors studied several properties of the comaximal graph. Recently, in [1], the comaximal graph of a lattice was defined and studied.

The comaximal graph of a lattice $L = (L, \wedge, \vee)$, denoted by $\Gamma(L)$, is an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $a \vee b = 1$. In this paper, we study the finite lattices L with toroidal comaximal graphs.

First we recall some definitions and notation on lattices and graphs.

Recall that a *lattice* is an algebra $L = (L, \wedge, \vee)$ satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a,\ a \vee a = a$,
2. $a \wedge b = b \wedge a,\ a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

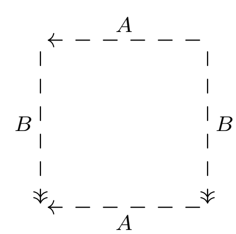
* Corresponding Author.
Received September 17, 2013; accepted April 11, 2014.
2010 Mathematics Subject Classification: 05C10, 06A07, 06B10.
Key words and phrases: Lattice, Comaximal graph, Toroidal graph.

Note that in every lattice the equality $a \wedge b = a$ always implies that $a \vee b = b$. Also, by [7, Theorem 2.1], one can define an order $\leqslant$ on L as follows: For any $a, b \in L$, we set $a \leqslant b$ if and only if $a \wedge b = a$. Then $(L, \leqslant)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let L be an ordered set such that, for every pair $a, b \in L$, g.l.b.(a, b), l.u.b.$(a, b) \in L$. For each a and b in L, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Then $(L, \wedge, \vee)$ is a lattice. A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in L$.

Clearly, every finite lattice is bounded. Recall that in a partially ordered set $(P, \leqslant)$, we say that a covers b or b is covered by a, in notation $b \prec a$, if and only if $b < a$ and there is no element p in P such that $b < p < a$. An element a in L is called a *co-atom* if $a \prec 1$. We denote the sets of all co-atoms in a lattice L by $C(L)$. Also, for an element $a \in L$, we set $[a]^l = \{b \in L \mid b \leq a\}$.

In a graph G, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. For a positive integer r, an *r-partite graph* is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph G is said to be *contracted to* a graph H if there exists a sequence of elementary contractions which transforms G into H, where an *elementary contraction* consists of deletion of a vertex or an edge or the identification of two adjacent vertices. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [2, p.153]).

By a surface we mean a connected compact 2-dimensional manifold without boundary, that is a topological space such that each point has a neiborhood homeomorphic to the open disc. It is well-known that every oriented compact surface is homeomorphic to a sphere with g handles. This number g is a called *the genus* of the surface. The torus can be though of as a sphere with one handle. This means that the genus of torus is 1.



The canonical representation of a torus

A graph G is embeddable in a surface S if the vertices of G are assigned to distinct points in S such that every edge of G is a simple arc in S connecting the two vertices which are joined in G. If G can not be embedded in S, then G has at least two edges intersecting at a point which is not a vertex of G. We say a graph G is *irreducible* for a surface S if G does not embed in S, but any proper subgraph of G embeds in S. A *toroidal graph* is a graph that can be embedded in a torus. Note that the genus of a planar graph is zero. So the planar graph is not considered as a toroidal graph. Also, a complete graph K_n is toroidal if $n = 5$, 6 or 7, and the only toroidal complete bipartite graphs are $K_{4,4}$ and $K_{3,n}$, with $n = 3, 4, 5, 6$ (see [3] or [8]).

2. Toroidal Ccomaximal Graph of a Lattice

In this paper, we assume that L is a finite lattice. The comaximal graph of a lattice L, denoted by $\Gamma(L)$, is an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $a \vee b = 1$ (see [1]). We denote the induced subgraph of $\Gamma(L)$ with vertex set $L \setminus (J(L) \cup \{1\})$, by $\Gamma_2(L)$, where $J(L)$ is the set $\bigcap_{m \in C(L)} [m]^l$. It is easy to see that the vertex 1 is adjacent to all vertices, also the vertices in $J(L)$ are isolated vertices in the induced subgraph with vertex set $L \setminus \{1\}$.

In this paper, we explore the toroidality of the graph $\Gamma_2(L)$. Clearly, by [1], if $\Gamma_2(L)$ is planar, then $|C(L)| \leq 4$. As $|C(L)| = 1$, then the graph $\Gamma_2(L)$ is an empty graph. Note that when $|C(L)| = 2$, we have that $\Gamma_2(L)$ is a complete bipartite graph. So $\Gamma_2(L)$ is planar if and only if either $|[m_1]^l \setminus [m_2]^l| \leq 2$ or $|[m_2]^l \setminus [m_1]^l| \leq 2$, where $C(L) = \{m_1, m_2\}$. Also, one can easily see that $\Gamma_2(L)$ is toroidal if and only if either $|[m_1]^l \setminus [m_2]^l| = |[m_2]^l \setminus [m_1]^l| = 4$ or $|[m_1]^l \setminus [m_2]^l| = 3$ and $|[m_2]^l \setminus [m_1]^l| \in \{3, 4, 5, 6\}$, where $C(L) = \{m_1, m_2\}$. We begin this section by the following lemma.

Lemma 2.1. *If $\Gamma_2(L)$ is toroidal, then the size of $C(L)$ is at most seven.*

Proof. Assume to the contrary that $|C(L)| \geq 8$. Then the induced subgraph of $\Gamma_2(L)$ with vertex set $C(L)$ is isomorphic to K_8, which is a contradiction. $\square$

By Lemma 2.1., it is sufficient to study the toroidality of the graph $\Gamma_2(L)$ in the cases that $C(L)$ has $3, 4, 5, 6$ or 7 elements. In this paper, we discuss on the case that $|C(L)| = 4$. First we begin by the following notation.

Notation. Suppose that $|C(L)| = n$, where $n > 1$. To simplify notation, for $1 \leq i \leq n$, we denote the set $[m_i]^l$, where $m_i \in C(L)$, by $\mathfrak{m}_i$. We set $S_t := \mathfrak{m}_t \setminus \bigcup_{i \notin \{t\}} \mathfrak{m}_i$, where $1 \leq i, t \leq n$. Also, $S_{t_1 t_2 \ldots t_k} := (\mathfrak{m}_{t_1} \cap \mathfrak{m}_{t_2} \cap \cdots \cap \mathfrak{m}_{t_k}) \setminus \bigcup_{i \notin \{t_1, t_2, \ldots, t_k\}} \mathfrak{m}_i$, where $1 \leq t_1 < t_2 < \cdots < t_k \leq n$. Note that each element in S_i is adjacent to all elements of S_j, for $1 \leq i \neq j \leq n$, and also it is adjacent to all elements of $S_{t_1 t_2 \ldots t_k}$, where $t_1, \ldots, t_k \notin \{i\}$.

Now, suppose that $|\bigcup_{t=1}^4 S_t| \geq 10$. Then it is easy to find a subgraph isomorphic to $K_{3,7}$ in the contraction of $\Gamma_2(L)$, and so it is not toroidal. Hence we have the following lemma.

Lemma 2.2. *If* $\Gamma_2(L)$ *is toroidal, then* $|\bigcup_{t=1}^{4} S_t| \leq 9$.

In this section, we study the toroidality of the graph $\Gamma_2(L)$, whenever $5 \leq |\bigcup_{t=1}^{4} S_t| \leq 9$.

Lemma 2.3. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 5$, $|S_1| = 2$ *and* $\Gamma_2(L)$ *is a toroidal graph. Then one of the following conditions holds:*

(i) $|S_{1i_1}| = 3$, *for some unique* $i_1 \in \{2,3,4\}$ *and* $|S_{i_2i_3}| = 1$, $S_{i_1i_2i_3} = S_{i_1i_2} = S_{i_1i_3} = \varnothing$, *for* $i_2, i_3 \notin \{1, i_1\}$.

(ii) $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{2,3,4\}$ *and* $|S_{i_2i_3}| = 1$, *for all* $i_2, i_3 \notin \{1\}$, *and also* $S_{1i_4} = \varnothing$, *for all* $i_4 \notin \{1, i_1\}$.

(iii) $|S_{1i_1}| = 2$, *for some unique* $i_1 \notin \{1\}$ *and* $|S_{i_1i_2}| = |S_{i_2i_3}| = 1$, $|S_{1i_2}| \geq 0$, $S_{1i_3} = S_{i_1i_3} = \varnothing$, *for some unique* $i_2, i_3 \in \{2,3,4\} \setminus \{i_1\}$.

(iv) $|S_{1i_1}| = |S_{i_1i_2}| = |S_{i_1i_3}| = 1$, $|S_{i_2i_3}| = 2$, *and* $S_{1i_2} = S_{1i_3} = \varnothing$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$.

(v) $|S_{1i_1}| = |S_{i_2i_3}| = 1$, $|S_{i_1i_2}| = |S_{i_1i_3}| = 2$, *and* $S_{1i_2} = S_{1i_3} = \varnothing$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$.

(vi) $|S_{1i_1}| = |S_{1i_2}| = 1$, $S_{1i_3} = \varnothing$ *and* $|S_{i_1i_2}| = |S_{i_1i_3}| = |S_{i_2i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$.

(vii) $|S_{i_1i_2}| = 2$ *and* $S_{1i_1} = \varnothing$, *for all* $i_1, i_2 \notin \{1\}$.

(viii) $|S_{i_1i_2}| = 3$, $|S_{i_2i_3}| = 2$, $|S_{i_1i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$ *and* $S_{1i_1} = \varnothing$, *for all* $i_1 \notin \{1\}$.

(ix) $|S_{i_1i_2}| = 4$, $|S_{i_1i_3}| = |S_{i_2i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$ *and* $S_{1i_1} = \varnothing$, *for all* $i_1 \notin \{1\}$.

(x) $|S_{1i_1}| = |S_{i_2i_3}| = 1$ *and* $S_{i_1i_2i_3} = \varnothing$, *for all* $i_1, i_2, i_3 \notin \{1\}$.

(xi) $S_{i_1i_2} = \varnothing$, $|S_{1i_1}| \geq 0$, $|S_{i_1i_2i_3}| \geq 0$, *for all* $i_1, i_2 \notin \{1\}$ *and for some unique* $i_3 \in \{2,3,4\}$.

Proof. By our hypothesis, $\Gamma_2(L)$ is toroidal. If S_{234}, S_{23}, S_{24} and S_{34} are empty, then $\Gamma_2(L)$ is planar, which is not toroidal. We know that, if the size of one of the sets S_{23}, S_{24} or S_{34} is at least five, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$, which is impossible. So the size of all of the above sets is at most four. We have the following situations.

(i) We assume that $|S_{14}| = 4$ and $|S_{23}| = 1$. Then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. So it is not toroidal. Also, if $|S_{14}| = 3$, $|S_{23}| = 1$ and S_{234} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5] (see Figure i). So $\Gamma_2(L)$ is not toroidal. In

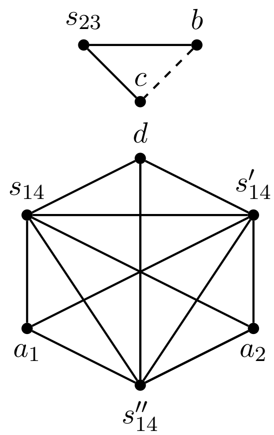


Figure 1

Figure 1, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14}, s'_{14}, s''_{14} \in S_{14}$ and $s_{23} \in S_{23}$.

Moreover, if $|S_{14}| = 3$, $|S_{23}| = |S_{34}| = 1$, then $\Gamma_2(L)$ contains $E6, 2$, one of the listed graphs in [11] (see Figure 2). Hence it is not toroidal. In Figure 2, $a_1, a_2 \in S_1, b \in S_2, c \in S_3, d \in S_4, s_{14}, s'_{14}, s''_{14} \in S_{14}$, $s_{23} \in S_{23}$ and $s_{34} \in S_{34}$. In addition, if $|S_{14}| = |S_{23}| = 2$, then the contraction of $\Gamma_2(L)$ contains a copy

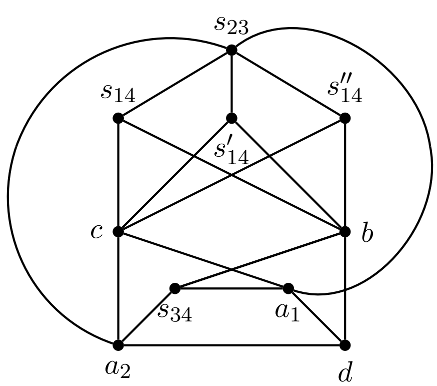


Figure 2

of $K_{4,5}$. So it is not toroidal. Thus, we may assume that $|S_{14}| = 3$, $|S_{23}| = 1$, $S_{24} = S_{34} = S_{234} = \varnothing$. In this situation, the complement of $\Gamma_2(L)$ contains $C415$, one of the listed graphs in [5] (see Figure 3). So it is toroidal. In Figure 3, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14}, s'_{14}, s''_{14} \in S_{14}$ and $s_{23} \in S_{23}$.

(ii) If $|S_{14}| = |S_{34}| = 2$ and $|S_{23}| = 1$, then the graph $\Gamma_2(L)$ contains $E6, 2$,

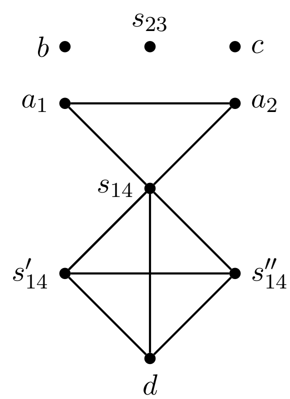


Figure 3

one of the listed graphs in [11]. So it is not toroidal. Also, if $|S_{14}| = 2$ and $|S_{13}| = |S_{23}| = |S_{24}| = 1$, then the graph $\Gamma_2(L)$ contains G_3, one of the listed graphs in [11] (see Figure 4). Hence it is not toroidal. In Figure ii, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{14}, s'_{14} \in S_{14}$, $s_{23} \in S_{23}$ and $s_{24} \in S_{24}$.

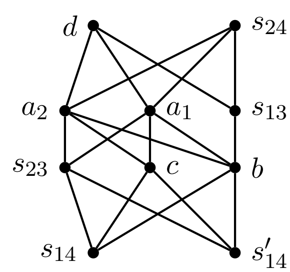


Figure 4

We may assume that $|S_{14}| = 2$, $|S_{23}| = |S_{24}| = |S_{34}| = 1$ and $S_{12} = S_{13} = \varnothing$. Then the graph $\Gamma_2(L)$, which is pictured in Figure 5, is toroidal. In Figure 5, we have $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14}, s'_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$ and $s_{34} \in S_{34}$.

(iii) In view of the previous situations, we may assume that $|S_{14}| = 2$, $|S_{23}| = |S_{34}| = 1$, $|S_{13}| \geq 0$ and $S_{12} = S_{24} = \varnothing$. In this case, the graph $\Gamma_2(L)$, in Figure 6, is toroidal. In Figure 6, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{14}, s'_{14} \in S_{14}$, $s_{23} \in S_{23}$ and $s_{34} \in S_{34}$.

(iv) If $|S_{23}| = 3$ and $|S_{14}| = 1$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$, and so it is not toroidal. On the other hand, if $|S_{23}| = 2$ and $|S_{24}| = |S_{13}| = |S_{14}| = 1$, then the graph $\Gamma_2(L)$ contains G_3, one of the listed graphs in [11]. Hence it is not toroidal. Also, if $|S_{23}| = |S_{34}| = 2$ and $|S_{14}| = 1$, then

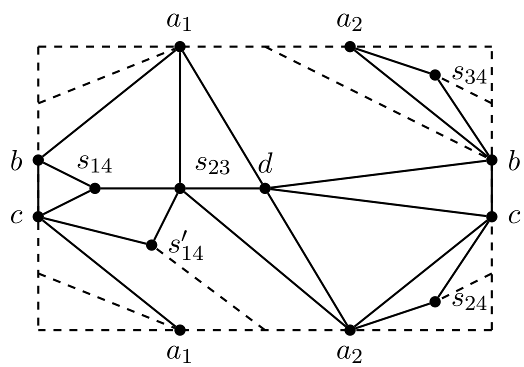


Figure 5

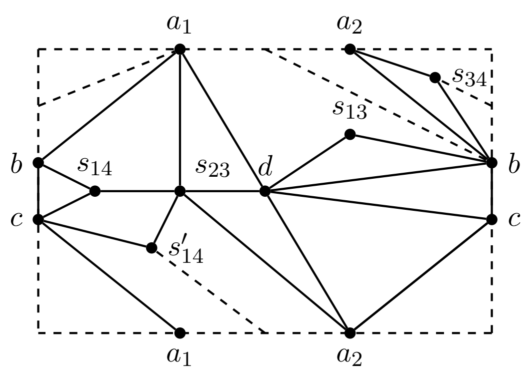


Figure 6

$\Gamma_2(L)$ contains G_3, one of the listed graphs in [11]. Hence it is not toroidal. Therefore we may assume that $|S_{23}| = 2$, $|S_{14}| = |S_{24}| = |S_{34}| = 1$ and $S_{12} = S_{13} = \varnothing$. Then the graph $\Gamma_2(L)$ is toroidal, which is pictured in Figure 7. In Figure 7, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{23}, s'_{23} \in S_{23}$, $s_{24} \in S_{24}$ and $s_{34} \in S_{34}$.

(v) In view of the previous situations, we may assume that $|S_{34}| = |S_{24}| = 2$, $|S_{23}| = |S_{14}| = 1$ and $S_{12} = S_{13} = \varnothing$. In this case, the graph $\Gamma_2(L)$, which is pictured in Figure 8, is toroidal. In Figure 8, we have $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24}, s'_{24} \in S_{24}$ and $s_{34}, s'_{34} \in S_{34}$.

(vi) If $|S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = 1$ and $|S_{34}| = 2$, then the complement of the contraction of $\Gamma_2(L)$ is contained in $V6.5$, one of the listed graphs in [5] (see Figure 9). So it is not toroidal. In Figure 9, we have $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{23} \in S_{23}$, $s_{14} \in S_{14}$, $s_{24} \in S_{24}$ and $s_{34}, s'_{34} \in S_{34}$. If $|S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = |S_{34}| = 1$ and $|S_{234}| \geq 0$, then $S_{12} = \varnothing$.

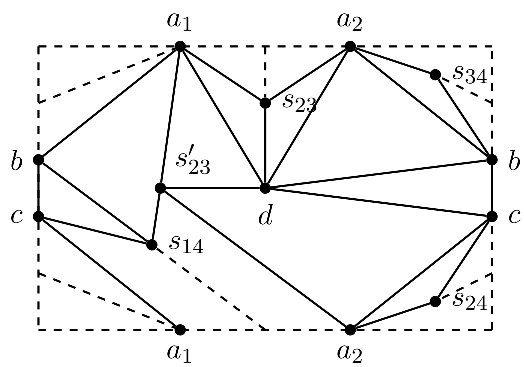


Figure 7

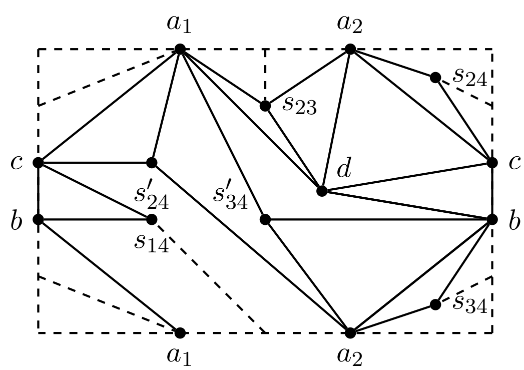


Figure 8

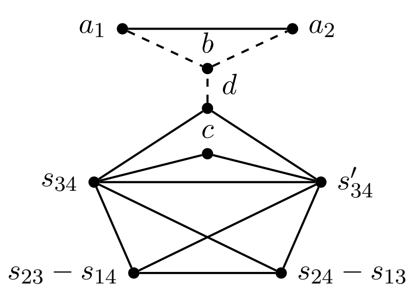


Figure 9

In this situation, the graph $\Gamma_2(L)$, in Figure 10, is toroidal. In Figure 10, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, $s_{34} \in S_{34}$ and $s_{234} \in S_{234}$.

(vii) In view of the previous situations, we may assume that $|S_{23}| = |S_{24}| = |S_{34}| =$

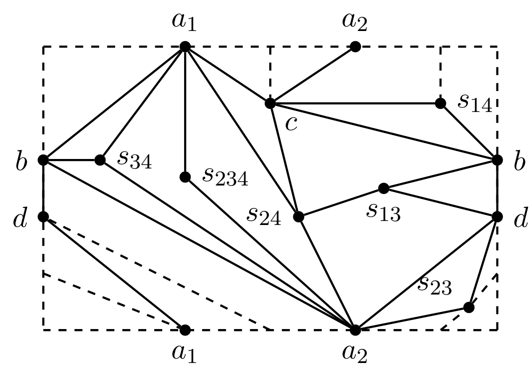


Figure 10

2 and $S_{12} = S_{13} = S_{14} = \varnothing$. In this case, the graph $\Gamma_2(L)$, in Figure 11, is toroidal. In Figure 11, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23}, s'_{23} \in S_{23}$, $s_{24}, s'_{24} \in S_{24}$ and $s_{34}, s'_{34} \in S_{34}$.

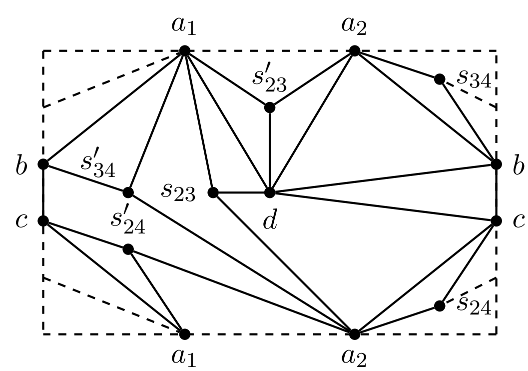


Figure 11

(viii) If $|S_{34}| = 3$ and $|S_{23}| = |S_{14}| = 1$, then the graph $\Gamma_2(L)$ contains a subgraph isomorphic to $E6, 2$, one of the listed graphs in [11]. So it is not toroidal. Therefore we may assume that $|S_{34}| = 3$, $|S_{24}| = 2$, $|S_{23}| = 1$ and $S_{12} = S_{13} = S_{14} = \varnothing$. Then the graph $\Gamma_2(L)$ is pictured in Figure 12, is toroidal. In Figure 12, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23} \in S_{23}$, $s_{24}, s'_{24} \in S_{24}$ and $s_{34}, s'_{34}, s''_{34} \in S_{34}$.

(ix) In view of the previous situations, we may assume that $|S_{34}| = 4$, $|S_{23}| = |S_{24}| = 1$ and $S_{12} = S_{13} = S_{14} = \varnothing$. In this case, the graph $\Gamma_2(L)$, in Figure 13, is toroidal. In Figure 13, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$ and $s_1, s_2, s_3, s_4 \in S_{34}$.

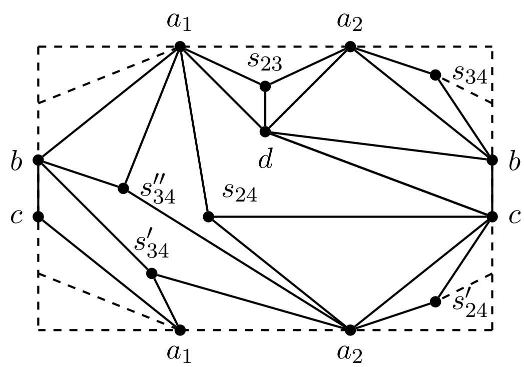


Figure 12

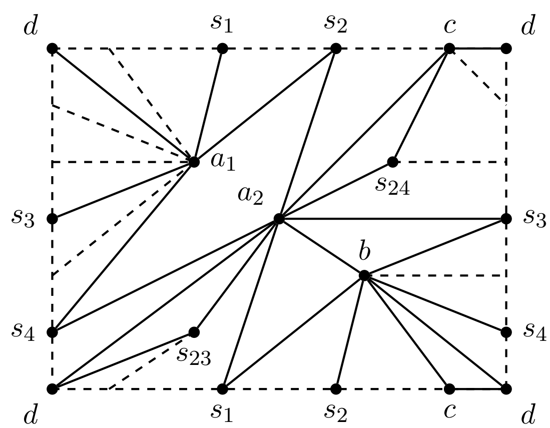


Figure 13

(x) If $|S_{12}| = |S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = |S_{34}| = |S_{234}| = 1$, then the complement of the contraction of $\Gamma_2(L)$ is contained in $Y7.4$, one of the listed graphs in [5] (see Figure 14). Thus it is not toroidal. In Figure 14, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, $s_{34} \in S_{34}$ and $s_{234} \in S_{234}$.
We may assume that $|S_{12}| = |S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = |S_{34}| = 1$ and $S_{234} = \varnothing$. Then the graph $\Gamma_2(L)$, in Figure 15, is toroidal. In Figure 15, $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$,$s_{24} \in S_{24}$ and $s_{234} \in S_{234}$.

(xi) When $S_{23} = S_{24} = S_{34} = \varnothing$, $|S_{234}| \geq 0$, $|S_{12}| \geq 0$, $|S_{13}| \geq 0$ and $|S_{14}| \geq 0$, the graph $\Gamma_2(L)$ is isomorphic to a subdivision of K_5, and so it is toroidal. □

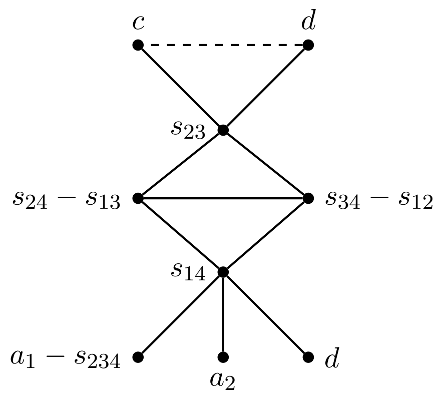


Figure 14

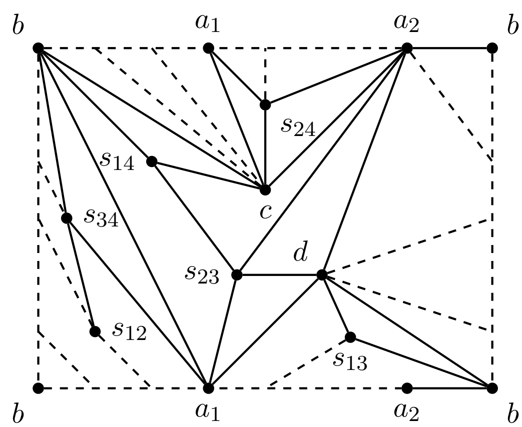


Figure 15

Remark 2.4. Note that if the size of the set $\bigcup_{t=1}^{4} S_t$ is five, $|S_1| = 2$ and one of the following cases holds, then it is a question that whether $\Gamma_2(L)$ is toroidal or not.

Case 1. $|S_{34}| = 3$ and $|S_{23}| = |S_{24}| = 2$.

Case 2. $|S_{24}| = |S_{34}| = 3$.

Case 3. $|S_{24}| = 2$ and $|S_{34}| = 4$.

Now, the next theorem follows immediately from Lemma 2.3 and Remark 2.4.

Theorem 2.5. *Suppose that $|\bigcup_{t=1}^{4} S_t| = 5$ and $|S_1| = 2$, and also the cases which are mentioned in Remark 2.4. do not hold. Then $\Gamma_2(L)$ is toroidal if and only if one of the following statements holds:*

(i) *$|S_{1i_1}| = 3$, for some unique $i_1 \in \{2,3,4\}$ and $|S_{i_2i_3}| = 1$, $S_{i_1i_2i_3} = S_{i_1i_2} = S_{i_1i_3} = \varnothing$, for $i_2, i_3 \notin \{1, i_1\}$,*

(ii) $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{2,3,4\}$ *and* $|S_{i_2i_3}| = 1$, *for all* $i_2, i_3 \notin \{1\}$, *and also* $S_{1i_4} = \varnothing$, *for all* $i_4 \notin \{1, i_1\}$,

(iii) $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{2,3,4\}$, *and* $|S_{i_1i_2}| = |S_{i_2i_3}| = 1$, $|S_{1i_2}| \geq 0$, $S_{1i_3} = S_{i_1i_3} = \varnothing$, *for some unique* $i_2, i_3 \notin \{1, i_1\}$,

(iv) $|S_{1i_1}| = |S_{i_1i_2}| = |S_{i_1i_3}| = 1$, $|S_{i_2i_3}| = 2$, *and* $S_{1i_2} = S_{1i_3} = \varnothing$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$,

(v) $|S_{1i_1}| = |S_{i_2i_3}| = 1$, $|S_{i_1i_2}| = |S_{i_1i_3}| = 2$, *and* $S_{1i_2} = S_{1i_3} = \varnothing$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$,

(vi) $|S_{1i_1}| = |S_{1i_2}| = 1$, $S_{1i_3} = \varnothing$ *and* $|S_{i_1i_2}| = |S_{i_1i_3}| = |S_{i_2i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$,

(vii) $|S_{i_1i_2}| = 2$, *and* $S_{1i_1} = \varnothing$, *for all* $i_1, i_2 \notin \{1\}$.

(viii) $|S_{i_1i_2}| = 3$, $|S_{i_2i_3}| = 2$, $|S_{i_1i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$ *and* $S_{1i_1} = \varnothing$, *for all* $i_1 \notin \{1\}$,

(ix) $|S_{i_1i_2}| = 4$, $|S_{i_1i_3}| = |S_{i_2i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$, *and* $S_{1i_1} = \varnothing$, *for all* $i_1 \notin \{1\}$,

(x) $|S_{1i_1}| = |S_{i_2i_3}| = 1$ *and* $S_{i_1i_2i_3} = \varnothing$, *for all* $i_1, i_2, i_3 \notin \{1\}$.

(xi) $S_{i_1i_2} = \varnothing$, $|S_{1i_1}| \geq 0$ *and* $|S_{i_1i_2i_3}| \geq 0$, *for all* $i_1, i_2 \notin \{1\}$ *and for some unique* $i_3 \in \{2,3,4\}$.

Suppose that $|\bigcup_{t=1}^{4} S_t| = 6$. Then either one of the sets S_t's, $1 \leq t \leq 4$, say S_1, has three elements or two of the S_t's, $1 \leq t \leq 4$, say S_1 and S_2, have two elements, exactly.

In the first case, if $|S_{234}| \geq 3$, then the complement of $\Gamma_2(L)$ is isomorphic to $U6.6b$, one of the listed graphs in [5]. So it is not toroidal.

And if $|S_{234}| = 2$ and $|S_{23}| \geq 1$, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5]. Thus it is not toroidal.

So we may assume that $|S_{234}| = 2$ and $S_{23} = S_{24} = S_{34} = \varnothing$. Then $\Gamma_2(L)$ contains a subgraph isomorphic to $K_8 \setminus (K_3 \cup K_2)$, which is toroidal (cf. [5, p.55]).

Now, suppose that $|S_{234}| = |S_{23}| = |S_{24}| = 1$. Then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5]. Thus it is not toroidal.

In addition, if $|S_{234}| = 1$ and $|S_{23}| = 2$, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5]. So it is not toroidal.

Also, if $|S_{234}| = |S_{23}| = 1$ and $|S_{14}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomrphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

Therefore we may assume that $|S_{234}| = |S_{23}| = |S_{14}| = 1$ and $S_{24} = S_{34} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [5], which is toroidal (see Figure 16). In Figure 16, we have the vertices $a_1, a_2, a_3 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$ and $s_{234} \in S_{234}$.

When $|S_{23}| = 3$, one can easily find a copy of $K_{4,5}$ in the structure of the contraction of $\Gamma_2(L)$. Hence it is not toroidal.

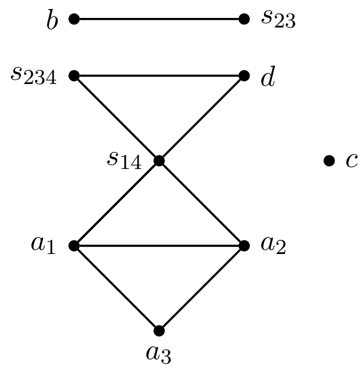


Figure 16

Also, if $|S_{14}| = 1$ and $|S_{23}| = 2$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5] (see Figure 17). Thus it is not toroidal. In Figure 17, $a_1, a_2, a_3 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$ and $s_{23}, s'_{23} \in S_{23}$.

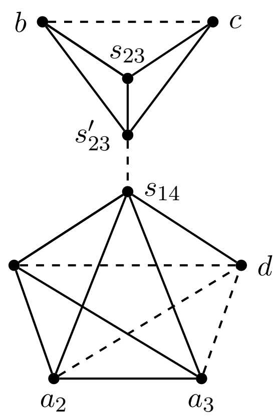


Figure 17

In addition, if $|S_{23}| = 2$ and $|S_{24}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5]. So it is not toroidal.

Hence we assume that $|S_{23}| = 2$ and $S_{14} = S_{24} = S_{34} = S_{234} = \varnothing$. Then $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3.K_2)$, ($K_3.K_2$ is the union of K_3 with K_2 such that intersect in one vertex), which is toroidal (cf. [5, p.55]).

Now, suppose that $|S_{14}| = 2$ and $|S_{23}| = |S_{24}| = 1$. Then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

If $|S_{14}| = 3$ and $|S_{23}| = 1$, then one can find a copy of $K_{3,7}$ in the contraction of $\Gamma_2(L)$. So it is not toroidal.

Also, if $|S_{14}| = |S_{23}| = 2$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. Hence it is not toroidal.

So we may assume that $|S_{14}| = 2$, $|S_{23}| = 1$ and $S_{234} = S_{24} = S_{34} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [5]. Thus it is toroidal. To do this, in Figure 18, consider the vertices $a_1, a_2 \in S_1$, $b \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14}, s'_{14} \in S_{14}$ and $s_{23} \in S_{23}$.

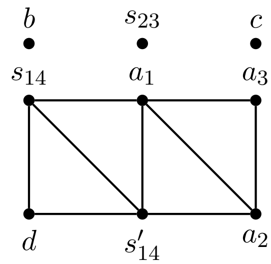


Figure 18

If $|S_{23}| = |S_{24}| = |S_{34}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5]. So it is not toroidal.

Also, if $|S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

Therefore we may assume that $|S_{12}| = |S_{23}| = |S_{34}| = 1$ and $S_{234} = S_{14} = S_{24} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [5]. So it is toroidal.

In the second case, when $|S_1| = |S_2| = 2$, if S_{34} has at least three elements, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. Hence it is not toroidal.

Also, if $|S_{34}| = 2$ and $|S_{12}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. So it is not toroidal.

Moreover, if $|S_{34}| = |S_{24}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

In addition, if $|S_{34}| = 2$ and $|S_{14}| = |S_{23}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

We may assume that $|S_{34}| = 2$, $|S_{23}| = |S_{24}| = 1$, $|S_{134} \geq 0|$, $|S_{234}| \geq 0$ and $S_{12} = S_{13} = S_{14} = \varnothing$. Then $\Gamma_2(L)$ is toroidal, since in Figure 19, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, $s_{34}, s'_{34} \in S_{34}$, $s_{134} \in S_{134}$ and $s_{234} \in S_{234}$.

Also, we may assume that $|S_{34}| = 2$, $|S_{13}| = |S_{23}| = 1$, $S_{12} = S_{14} = S_{24} = \varnothing$, $|S_{134}| \geq 0$ and $|S_{234}| \geq 0$. Then $\Gamma_2(L)$ is toroidal, since in Figure 20, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{23} \in S_{23}$, $s_{34}, s'_{34} \in S_{34}$, $s_{134} \in S_{134}$ and $s_{234} \in S_{234}$.

Now, suppose that $|S_{134}| = |S_{14}| = |S_{23}| = |S_{24}| = |S_{34}| = 1$. Then the complement of the contraction of $\Gamma_2(L)$ is contained in $W6.6a$, one of the listed graphs in [5] (see Figure 21). So it is not toroidal. In Figure 21, we have $a_1, a_2 \in S_1$,

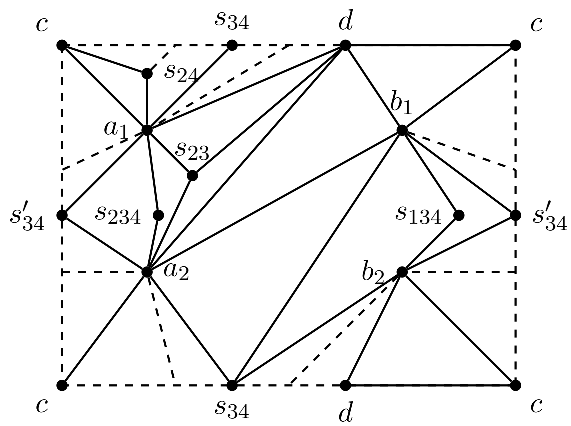


Figure 19

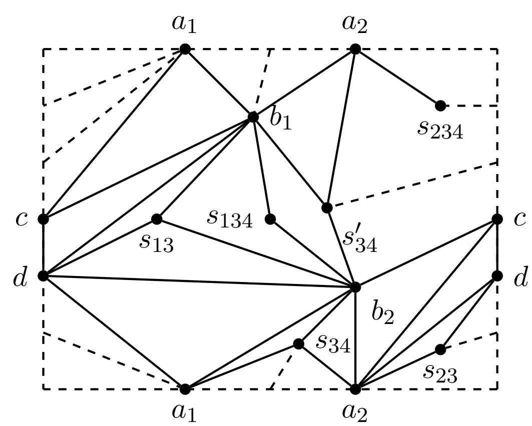


Figure 20

$b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, $s_{34} \in S_{34}$ and $s_{134} \in S_{134}$.

Now, if $|S_{13}| = |S_{14}| = |S_{23}| = |S_{24}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

In addition, if $|S_{12}| = |S_{14}| = |S_{23}| = |S_{34}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

Whenever $|S_{13}| = |S_{23}| = |S_{24}| = 1$ and $S_{34} = S_{14} = \varnothing$, the complement of $\Gamma_2(L)$ contains a subgraph isomorphic to $C416$, one of the listed graphs in [5] (see Fugure 22). Thus it is toroidal. In Figure 22, $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$, $s_{23} \in S_{23}$ and $s_{24} \in S_{24}$.

So we may assume that $|S_{14}| = |S_{23}| = |S_{24}| = |S_{34}| = 1$, $|S_{234}| \geq 0$ and

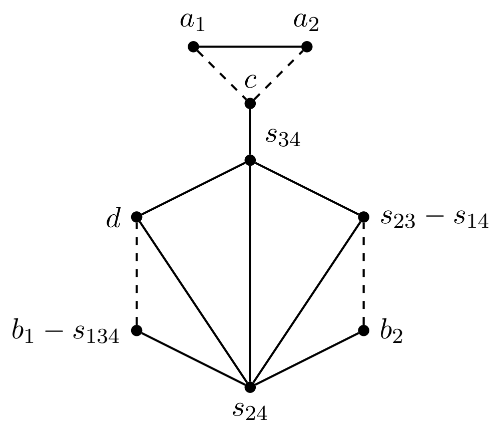


Figure 21

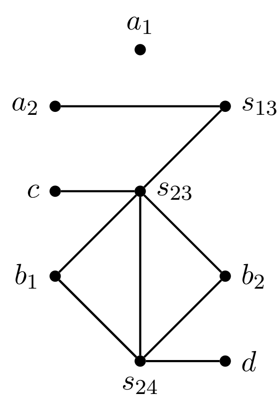


Figure 22

$S_{12} = S_{13} = S_{134} = \varnothing$. Then $\Gamma_2(L)$ is toroidal, since in Figure 23, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, $s_{34} \in S_{34}$ and $s_{234} \in S_{234}$.

Consider $|S_{24}| = 3$ and $|S_{34}| = 1$. Since $\Gamma_2(L)$ contains $E6, 2$, one of the listed graphs in [11], it is not toroidal.

When $|S_{24}| = 2$ and $|S_{12}| = |S_{34}| = 1$, $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

Now, we may assume that $|S_{23}| = |S_{24}| = 2$, $|S_{34}| = 1$, $|S_{234}| \geq 0$ and $S_{12} = S_{13} = S_{14} = S_{134} = \varnothing$. Then $\Gamma_2(L)$ is toroidal, since in Figure 24, we have $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23}, s'_{23} \in S_{23}$, $s_{24}, s'_{24} \in S_{24}$, $s_{34} \in S_{34}$ and $s_{234} \in S_{234}$.

If $|S_{14}| = 1$ and $|S_{23}| = 2$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$,

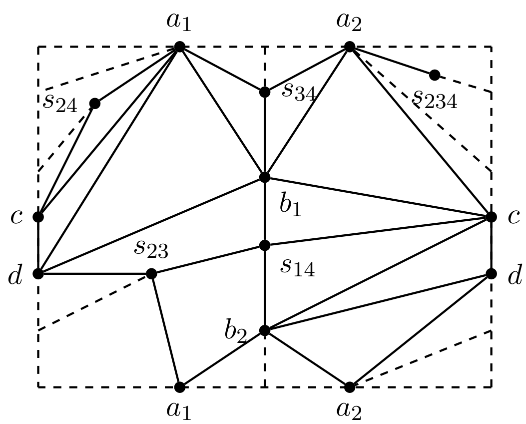


Figure 23

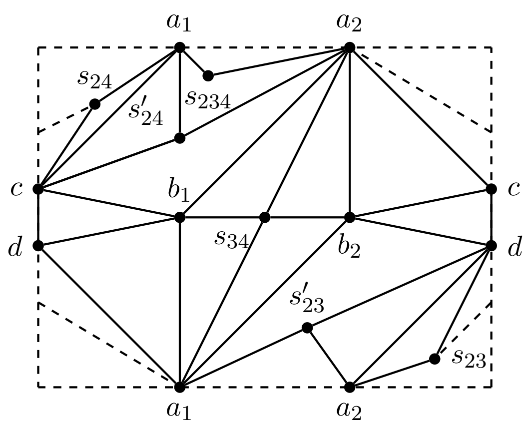


Figure 24

one of the listed graphs in [5]. So it is not toroidal.

We may assume that $|S_{23}| = |S_{24}| = 2$, $|S_{12}| \geq 0$, $|S_{134}| \geq 0$, $|S_{234}| \geq 0$ and $S_{13} = S_{14} = S_{34} = \varnothing$. Then $\Gamma_2(L)$ is toroidal. To do this, in Figure 25, consider the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{23}, s'_{23} \in S_{23}$, $s_{24}, s'_{24} \in S_{24}$, $s_{134} \in S_{134}$ and $s_{234} \in S_{234}$.

As $|S_{24}| \geq 4$, the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. Hence it is not toroidal.

If $|S_{24}| = 3$ and $|S_{14}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to $E6, 2$, one of the listed graphs in [11]. So it is not toroidal.

If $|S_{24}| = 3$ and $|S_{23}| = |S_{134}| = 1$, then the complement of the contraction of $\Gamma_2(L)$ contains $U6.6b$, one of the listed graphs in [5]. So it is not toroidal.

Now, consider $|S_{24}| = 3$, $|S_{23}| = 1$, $S_{13} = S_{14} = S_{34} = S_{134} = \varnothing$, $|S_{12}| \geq 0$

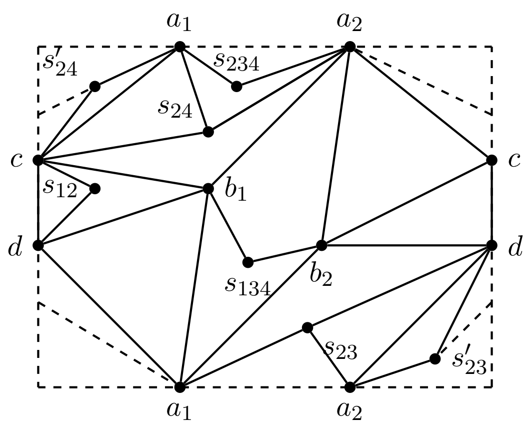


Figure 25

and $|S_{234}| \geq 0$. Then the graph $\Gamma_2(L)$, in Figure 26, is toroidal. In Figure 26, $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{23} \in S_{23}$, $s_{24}, s'_{24}, s''_{24} \in S_{24}$ and $s_{234} \in S_{234}$.

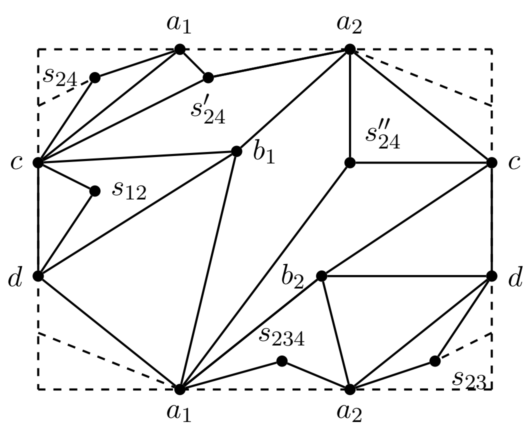


Figure 26

If $|S_{12}| = 2$ and $|S_{23}| = |S_{34}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

In addition, if $|S_{12}| = 2$ and $|S_{34}| = |S_{234}| = 1$, then the complement of the contraction of $\Gamma_2(L)$ contains $V6.5$, one of the listed graphs in [5]. So it is not toroidal.

If $|S_{12}| = 3$ and $|S_{34}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to $E6, 2$, one of the listed graphs in [11]. So it is not toroidal.

Now, we may assume that $|S_{12}| = 2$, $|S_{34}| = 1$ and $S_{134} = S_{234} = S_{13} = S_{14} =$

$S_{23} = S_{24} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C610$, one of the listed graphs in [5]. So it is toroidal, since in Figure 27, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12}, s'_{12} \in S_{12}$ and $s_{34} \in S_{34}$.

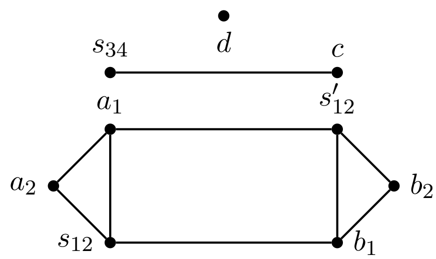


Figure 27

If $|S_{24}| = 2$ and $|S_{14}| = |S_{23}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11]. Hence it is not toroidal.

If $|S_{14}| = |S_{24}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomorphic to $E6, 2$, one of the listed graphs in [11]. So it is not toroidal.

Suppose that $|S_{24}| = 2$, $|S_{14}| = |S_{34}| = 1$, $S_{12} = S_{13} = S_{23} = \varnothing$, $|S_{134}| \geq 0$ and $|S_{234}| \geq 0$. Then the graph $\Gamma_2(L)$, in Figure 28, is toroidal. In Figure 28, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$, $s_{24}, s'_{24} \in S_{24}$, $s_{34} \in S_{34}$, $s_{134} \in S_{134}$ and $s_{234} \in S_{234}$.

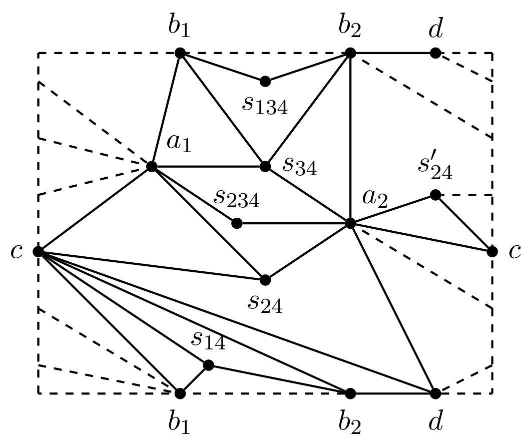


Figure 28

Moreover, if $|S_{12}| = |S_{13}| = |S_{23}| = |S_{34}| = 1$, $S_{14} = S_{24} = \varnothing$, $|S_{134}| \geq 0$ and $|S_{234}| \geq 0$, then the graph $\Gamma_2(L)$ is toroidal, which is pictured in Figure 29. In Figure 29, $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$, $s_{23} \in S_{23}$, $s_{34} \in S_{34}$, $s_{134} \in S_{134}$ and $s_{234} \in S_{234}$.

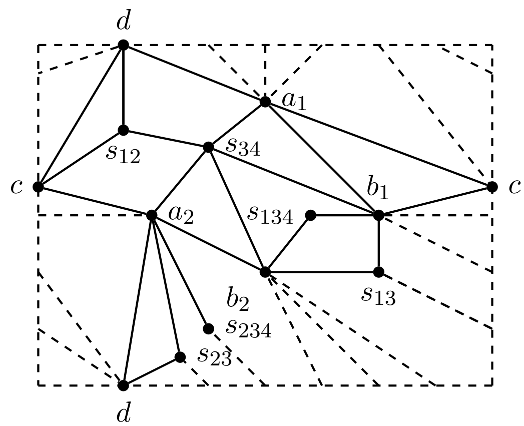


Figure 29

Also, if $|S_{12}| = |S_{23}| = |S_{24}| = |S_{34}| = 1$, $S_{13} = S_{14} = \varnothing$, $|S_{134}| \geq 0$ and $|S_{234}| \geq 0$, then the graph $\Gamma_2(L)$ is toroidal, which is pictured in Figure 30. In Figure 30, $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$, and $s_{34} \in S_{34}$.

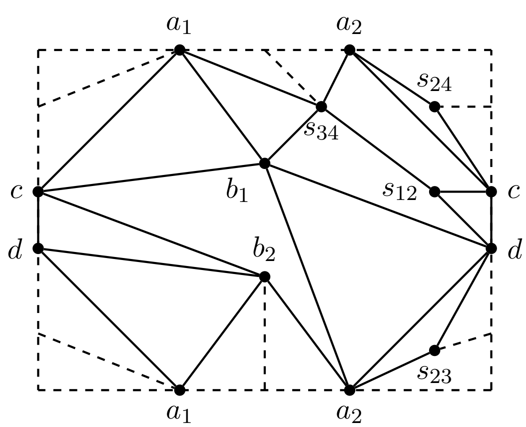


Figure 30

Now, from the above discussion, we state some necessary and sufficient conditions for the toroidality of $\Gamma_2(L)$ in the next two theorems.

Theorem 2.6. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 6$ *and* $|S_1| = 3$. *Then* $\Gamma_2(L)$ *is toroidal if and only if one of the following statements holds:*

(i) *If* $|S_{i_1 i_2 i_3}| = 2$, *then* $S_{i_1 i_2} = \varnothing$, *for all* $i_1, i_2, i_3 \notin \{1\}$,

(ii) *If* $|S_{1i_1}| = 1$, *for some unique* $i_1 \in \{2,3,4\}$, *then* $|S_{i_2i_3}| = 1$, *for* $\{i_2, i_3\} = \{2,3,4\} \setminus \{i_1\}$ *and* $S_{i_1i_4} = \varnothing$, *for all* $i_4 \in \{i_2, i_3\}$, *and also* $|S_{i_1i_2i_3}| = 1$,

(iii) *If* $|S_{i_1i_2}| = 2$, *for some unique* $i_1, i_2 \in \{2,3,4\}$, *then* $S_{i_3i_4} = \varnothing$, *for all* $i_3 \notin \{1, i_1, i_2\}$, $i_4 \in \{1, i_1, i_2\}$ *and* $S_{i_1i_2i_3} = \varnothing$,

(iv) *If* $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{2,3,4\}$, *then* $|S_{i_2i_3}| = 1$, *for* $\{i_2, i_3\} = \{2,3,4\} \setminus \{i_1\}$ *and* $S_{i_1i_4} = \varnothing$, *for all* $i_4 \in \{i_2, i_3\}$, *and also* $S_{i_1i_2i_3} = \varnothing$,

(v) *If* $|S_{1i_1}| = 1$, *for some unique* $i_1 \in \{2,3,4\}$, *then* $|S_{i_2i_3}| = 1$, *for* $\{i_2, i_3\} = \{2,3,4\} \setminus \{i_1\}$ *and* $|S_{i_1i_4}| = 1$, *for some unique* $i_4 \in \{i_2, i_3\}$, *and also* $S_{1i_5} = S_{i_1i_5} = \varnothing$, *for some unique* $i_5 \in \{i_2, i_3\} \setminus \{i_4\}$ *and* $S_{i_1i_2i_3} = \varnothing$.

Remark 2.7. Note that if $\bigcup_{t=1}^{4} S_t$ has six elements, $|S_1| = |S_2| = 2$ and one of the following cases holds, then it is a question that whether $\Gamma_2(L)$ is toroidal or not.

Case 1. $|S_{23}| = 2$ and $|S_{24}| = 3$.

Case 2. $|S_{23}| = |S_{24}| = 2$ and $|S_{34}| = |S_{134}| = 1$.

Theorem 2.8. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 6$ *and* $|S_1| = |S_2| = 2$, *and also the cases which are mentioned in remark 2.7. do not hold. Then* $\Gamma_2(L)$ *is toroidal if and only if one of the following statements holds:*

(i) $S_{12} = \varnothing$ *and* $|S_{i_1i_2}| = 2$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_1i_3}| = |S_{i_2i_3}| = 1$, *for some unique* $i_3 \in \{1,2\}$, *then* $S_{i_1i_4} = S_{i_2i_4} = \varnothing$, *for* $i_4 \in \{1,2\} \setminus \{i_3\}$,

(ii) $S_{12} = \varnothing$ *and* $|S_{i_1i_2}| = 2$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{1i_3}| = |S_{2i_3}| = 1$, *for some unique* $i_3 \in \{3,4\}$, *then* $S_{1i_4} = S_{2i_4} = \varnothing$, *for* $i_4 \notin \{1,2,i_3\}$,

(iii) $S_{i_1i_2} = \varnothing$, *for* $i_1, i_2 \notin \{1,2\}$ *and* $|S_{i_3i_4}| = 1$, *for some unique* $i_3 \in \{1,2\}$ *and for all* $i_4 \in \{i_1, i_2\}$. *Also, if* $|S_{i_4i_5}| = 1$, *for some unique* $i_4 \in \{i_1, i_2\}$ *and for* $i_5 \in \{1,2\} \setminus \{i_3\}$, *then* $S_{i_5i_6} = \varnothing$, *for* $i_6 \in \{i_1, i_2\} \setminus \{i_4\}$,

(iv) $S_{12} = \varnothing$, $|S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$ *and* $|S_{i_3i_4}| = 1$, *for some unique* $i_3 \in \{1,2\}$, *for all* $i_4 \in \{i_1, i_2\}$. *Also, if* $|S_{i_4i_5}| = 1$, *for some unique* $i_4 \in \{i_1, i_2\}$, *for* $i_5 \in \{1,2\} \setminus \{i_3\}$, *then* $S_{i_5i_6} = \varnothing$, *for* $i_6 \in \{i_1, i_2\} \setminus \{i_4\}$ *and* $S_{i_1i_2i_5} = \varnothing$,

(v) $S_{12} = \varnothing$ *and* $|S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 2$, *for some unique* $i_3 \in \{1,2\}$ *and for all* $i_4 \in \{i_1, i_2\}$, *then* $S_{i_4i_5} = \varnothing$, *for* $i_5 \in \{1,2\} \setminus \{i_3\}$ *and* $S_{i_1i_2i_5} = \varnothing$,

(vi) $S_{i_1i_2} = \varnothing$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 2$, *for some unique* $i_3 \in \{1,2\}$ *and for all* $i_4 \in \{i_1, i_2\}$, *then* $S_{i_4i_5} = \varnothing$, *for all* $i_4 \in \{i_1, i_2\}$ *and for* $i_5 \in \{1,2\} \setminus \{i_3\}$,

(vii) $S_{i_1i_2} = \varnothing$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 3$, *for some unique* $i_3 \in \{1,2\}$ *and for some unique* $i_4 \in \{i_1, i_2\}$, *then* $|S_{i_3i_5}| = 1$, *for* $i_5 \in \{i_1, i_2\} \setminus \{i_4\}$ *and* $S_{i_1i_6} = S_{i_2i_6} = S_{i_1i_2i_6} = \varnothing$, *for* $i_6 \in \{1,2\} \setminus \{i_3\}$,

(viii) $|S_{12}| = 2$. *Also, if* $|S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$, *then* $S_{i_1i_3} = S_{i_2i_3} = S_{i_1i_2i_3} = \varnothing$, *for all* $i_3 \in \{1,2\}$,

(ix) $S_{12} = \varnothing$ *and* $|S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 2$, *for some unique* $i_3 \in \{1,2\}$, *for some unique* $i_4 \in \{i_1, i_2\}$ *and* $|S_{i_4i_5}| = 1$, *for* $i_5 \in \{1,2\} \setminus \{i_3\}$, *then* $S_{1i_6} = S_{2i_6} = \varnothing$, *for* $i_6 \in \{i_1, i_2\} \setminus \{i_4\}$,

(x) $|S_{12}| = |S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 1$, *for all* $i_3 \in \{1,2\}$ *and for some unique* $i_4 \in \{i_1, i_2\}$, *then* $S_{i_3i_5} = \varnothing$, *for all* $i_3 \in \{1,2\}$ *and for some unique* $i_5 \in \{i_1, i_2\} \setminus \{i_4\}$,

(xi) $|S_{12}| = |S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1,2\}$. *Also, if* $|S_{i_3i_4}| = 1$, *for some unique* $i_3 \in \{1,2\}$ *and for all* $i_4 \in \{i_1, i_2\}$, *then* $S_{i_4i_5} = \varnothing$, *for* $i_5 \in \{1,2\} \setminus \{i_3\}$.

Lemma 2.9. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 7$ *and* $|S_1| = 4$. *If one of the following conditions holds, then* $\Gamma_2(L)$ *is not a toroidal graph.*

(i) $|S_{i_1i_2}| = |S_{i_1i_3}| = 1$, *for* $i_1, i_2, i_3 \notin \{1\}$.

(ii) $|S_{1i_1}| = |S_{i_2i_3}| = 1$, *for some unique* $i_2, i_3 \notin \{1, i_1\}$.

(iii) $|S_{i_1i_2i_3}| = |S_{i_1i_2}| = 1$, *for some unique* $i_1, i_2, i_3 \notin \{1\}$.

(iv) $|S_{i_1i_2i_3}| = 2$, *for* $i_1, i_2, i_3 \notin \{1\}$.

(v) $|S_{i_1i_2}| = 2$, *for some unique* $i_1, i_2 \notin \{1\}$.

Proof.

(i) If $|S_{234}| \geq 2$, then the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{4,5}$.

(ii) If S_{23}, S_{24} or S_{34} has at least two elements, then one can find a copy of $K_{4,5}$ in the structure of the contraction of $\Gamma_2(L)$.

(iii) If $|S_{234}| = 1$ and S_{23}, S_{24} or S_{34} has one element, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(iv) If $|S_{23}| = |S_{24}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(v) If $|S_{12}| = |S_{34}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

In all of the above cases, $\Gamma_2(L)$ is not a toroidal graph. □

Now, we may assume that S_{234} has at most one element and $S_{23} = S_{24} = S_{34} = \varnothing$. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$. Hence it is a toroidal graph (cf. [5, p.55]). In addition, we assume that S_{34} has exactly one element. Then $S_{234} = S_{12} = S_{23} = S_{24} = \varnothing$. So $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, which is toroidal (cf. [5, p.55]).

As a consequence of the above discussion and Lemma 2.9., one can easily check that the toroidality of the graph $\Gamma_2(L)$, when $|\bigcup_{t=1}^{4} S_t| = 7$ and $|S_1| = 4$.

Theorem 2.10. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 7$ *and* $|S_1| = 4$. *Then* $\Gamma_2(L)$ *is a toroidal graph if and only if one of the following conditions is satisfied:*

(i) *If* $|S_{i_1i_2i_3}| = 1$, *then* $S_{i_1i_2} = \varnothing$, *for all* $i_1, i_2, i_3 \notin \{1\}$.

(ii) *If* $|S_{i_1i_2}| = 1$, *for some unique* $i_1, i_2 \notin \{1\}$, *then* $S_{i_1i_2i_3} = \varnothing$, *for* $i_3 \notin \{1\}$ *and* $S_{1i_4} = S_{i_1i_4} = S_{i_2i_4} = \varnothing$, *for* $i_4 \notin \{i_1, i_2\}$.

Lemma 2.11. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 7$, $|S_1| = 3$ *and* $|S_2| = 2$. *If one of the following conditions holds, then the graph* $\Gamma_2(L)$ *is not toroidal.*

(i) $|S_{2i_1i_2}| \geq 2$, *for* $i_1, i_2 \notin \{1\}$.

(ii) $|S_{1i_1}| \geq 3$, *for some unique* $i_1 \notin \{2\}$.

(iii) $|S_{2i_1}| \geq 2$, *for some unique* $i_1 \notin \{1\}$.

(iv) $|S_{i_1i_2}| \geq 2$, *for* $i_1, i_2 \notin \{1, 2\}$.

(v) $|S_{2i_1i_2}| = |S_{i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$.

(vi) $|S_{2i_1i_2}| = |S_{2i_3}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$ *and for some unique* $i_3 \in \{i_1, i_2\}$.

(vii) $|S_{2i_1i_2}| = 1$ *and* $|S_{1i_3}| = 2$, *for* $i_1, i_2 \notin \{1, 2\}$ *and for some unique* $i_3 \in \{i_1, i_2\}$.

(viii) $|S_{1i_1i_2}| = |S_{1i_1}| = |S_{2i_1}| = |S_{i_1i_2}| = 1$, *for some unique* $i_1, i_2 \in \{3, 4\}$.

(ix) $|S_{i_1i_2}| = 1$ *and* $|S_{1i_1}| = 2$, *for some unique* $i_1, i_2 \in \{3, 4\}$.

(x) $|S_{i_1i_2}| = 1$ *and* $|S_{12}| = 2$, *for* $i_1, i_2 \notin \{1, 2\}$.

(xi) $|S_{2i_1}| = 1$ *and* $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{3, 4\}$.

(xii) $|S_{i_1i_2}| = |S_{2i_3}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$ *and for some unique* $i_3 \in \{i_1, i_2\}$.

(xiii) $|S_{2i_1}| = |S_{2i_2}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$.

(xiv) $|S_{1i_1}| = |S_{2i_2}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$.

(xv) $|S_{12}| = |S_{1i_1}| = |S_{i_1i_2}| = 1$, *for some unique* $i_1, i_2 \in \{3, 4\}$.

(xvi) $|S_{12}| = |S_{i_1i_2}| = |S_{1i_1i_2}| = 1$, *for* $i_1, i_2 \notin \{1, 2\}$.

Proof.

(i) If $|S_{234}| \geq 2$, then the complement of $\Gamma_2(L)$ is contained in $V6.5$, one of the listed graphs in [5].

(ii) If $|S_{13}| \geq 3$ or $|S_{14}| \geq 3$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$.

(iii) If $|S_{23}| \geq 2$ or $|S_{24}| \geq 2$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$.

(iv) If $|S_{34}| \geq 2$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$.

(v) If $|S_{234}| = |S_{34}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [5].

(vi) If $|S_{234}| = |S_{24}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(vii) If $|S_{234}| = 1$ and $|S_{14}| = 2$, then $\Gamma_2(L)$ contains a subgroph isomorphic to G_3, one of the listed graphs in [11].

(viii) If $|S_{134}| = |S_{13}| = |S_{23}| = |S_{34}| = 1$, then the complement of the contraction of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(ix) If $|S_{34}| = 1$ and $|S_{14}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11].

(x) If $|S_{34}| = 1$ and $|S_{12}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11].

(xi) If $|S_{24}| = 1$ and $|S_{14}| = 2$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11].

(xii) If $|S_{24}| = |S_{34}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(xiii) If $|S_{23}| = |S_{24}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $W^*7.5$, one of the listed graphs in [5], and so it is not toroidal (see Figure 31). Since in Figure 31, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{23} \in S_{23}$ and $s_{24} \in S_{24}$.

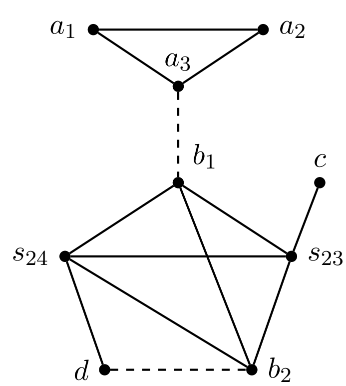


Figure 31

(xiv) If $|S_{13}| = |S_{24}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5].

(xv) If $|S_{12}| = |S_{13}| = |S_{34}| = 1$, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [11].

(xvi) If $|S_{12}| = |S_{34}| = |S_{134}| = 1$, then $\Gamma_2(L)$ is contained in $S5.6$, one of the listed graphs in [5] (see Figure 32). In Figure 32, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{12} \in S_{12}$ and $s_{34} \in S_{34}$. □

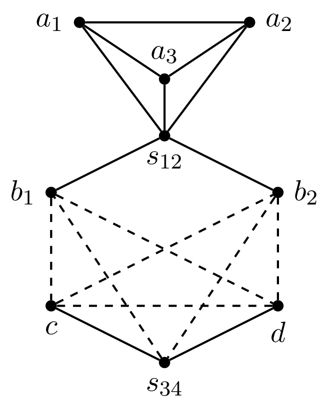


Figure 32

Moreover, we assume that S_{34} and S_{13} are singleton sets and $S_{12} = S_{23} = S_{24} = S_{234} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C420$, one of the listed graphs in [5] (see Figure 33). In Figure 33, $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$ and $s_{34} \in S_{34}$. So it is toroidal.

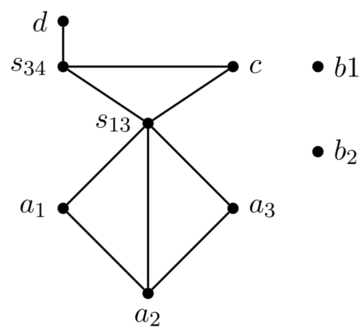


Figure 33

Also, if S_{12} and S_{34} are singleton sets and $S_{13} = S_{14} = S_{23} = S_{24} = S_{134} = S_{234} = \varnothing$, then the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [5]. So it is toroidal. Now, consider S_{13} and S_{14} have exactly one element and $S_{23} = S_{24} = S_{34} = S_{234} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C517$, one of the listed graphs in [5] (see Figure 34). In Figure 34, we have $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{13} \in S_{13}$ and $s_{14} \in S_{14}$. So it is toroidal.

In the case that S_{14} and S_{24} have exactly one element and $S_{13} = S_{23} = S_{34} = S_{234} = \varnothing$, the complement of $\Gamma_2(L)$ contains $C403$, one of the listed graphs in [5] (see Figure 35). In Figure 35, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{14} \in S_{14}$ and $s_{24} \in S_{24}$. Therefore it is toroidal.

Now, if S_{234} and S_{14} have exactly one element and $S_{13} = S_{23} = S_{24} = S_{34} = \varnothing$,

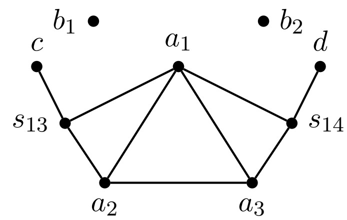


Figure 34

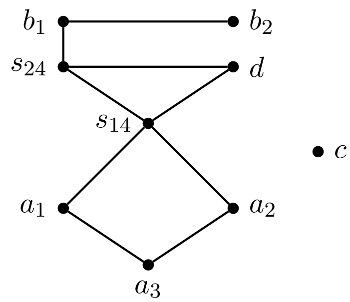


Figure 35

then the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [5]. So it is toroidal.

Finally, if S_{14} has two elements and $S_{13} = S_{23} = S_{24} = S_{34} = S_{234} = \varnothing$, then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [5]. So it is toroidal.

Remark 2.12. Note that if the size of the set $\bigcup_{t=1}^{4} S_t$ is seven, $|S_1| = 3$, $|S_2| = 2$ and one of the following cases holds, then it is a question that whether $\Gamma_2(L)$ is toroidal or not.

Case 1. $|S_{13}| = |S_{14}| = |S_{234}| = 1$.
Case 2. $|S_{13}| = 1$ and $|S_{14}| = 2$.
Case 3. $|S_{13}| = |S_{14}| = |S_{34}| = 1$

Now, the next theorem follows immediately from Lemma 2.11 and Remark 2.12.

Theorem 2.13. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 7$, $|S_1| = 3$, $|S_2| = 2$, *and also the cases which are mentioned in Remark 2.12. do not hold. Then the graph* $\Gamma_2(L)$ *is toroidal if and only if one of the following conditions holds:*

(i) $|S_{i_1 i_2}| = |S_{1 i_1}| = 1$ *and* $S_{2 i_1 i_2} = S_{12} = S_{1 i_2} = S_{2 i_1} = S_{2 i_2} = \varnothing$, *for some unique* $i_1, i_2 \in \{3, 4\}$,

(ii) $|S_{i_1 i_2}| = |S_{12}| = 1$ *and* $S_{1 i_1 i_2} = S_{2 i_1 i_2} = S_{1 i_1} = S_{1 i_2} = S_{2 i_1} = S_{2 i_2} = \varnothing$, *for* $i_1, i_2 \notin \{1, 2\}$,

(iii) $|S_{1i_1}| = |S_{1i_2}| = 1$ *and* $S_{2i_1i_2} = S_{2i_1} = S_{2i_2} = S_{i_1i_2} = \varnothing$, *for* $i_1, i_2 \notin \{1,2\}$,

(iv) $|S_{1i_1}| = |S_{2i_1}| = 1$ *and* $S_{2i_1i_2} = S_{1i_2} = S_{2i_2} = S_{i_1i_2} = \varnothing$, *for some unique* $i_1, i_2 \in \{3,4\}$,

(v) $|S_{2i_1i_2}| = |S_{1i_1}| = 1$ *and* $S_{1i_2} = S_{2i_1} = S_{2i_2} = S_{i_1i_2} = \varnothing$, *for some unique* $i_1, i_2 \in \{3,4\}$,

(vi) $|S_{1i_1}| = 2$ *and* $S_{2i_1i_2} = S_{1i_2} = S_{2i_1} = S_{2i_2} = S_{i_1i_2} = \varnothing$, *for some unique* $i_1, i_2 \in \{3,4\}$.

Lemma 2.14. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 7$ *and* $|S_1| = 1$. *If one of the following conditions holds, then the graph* $\Gamma_2(L)$ *is not toroidal.*

(1) $|S_{i_1i_2}| = 3$, *for some unique* $i_1, i_2 \in \{2,3,4\}$.

(2) $|S_{1i_1}| = 2$, *for some unique* $i_1 \in \{2,3,4\}$.

(3) $|S_{1i_1}| = |S_{i_2i_3}| = 1$, *for some unique* $i_1, i_2, i_3 \in \{2,3,4\}$.

(4) $|S_{1i_1}| = 1$ *and* $|S_{i_1i_2}| = 2$, *for some unique* $i_1, i_2 \in \{2,3,4\}$.

(5) $|S_{i_1i_2}| = 2$ *and* $|S_{i_1i_3}| = 1$, *for* $i_1, i_2, i_3 \notin \{1\}$.

(6) $|S_{i_1i_2}| = 2$ *and* $|S_{1i_1i_3}| = 1$, *for* $i_1, i_2, i_3 \notin \{1\}$.

(7) $|S_{i_1i_2}| = 2$ *and* $|S_{1i_2i_3}| = 1$, *for* $i_1, i_2, i_3 \notin \{1\}$.

(8) $|S_{1i_1}| = |S_{1i_2}| = |S_{i_1i_2}| = 1$, *for some unique* $i_1, i_2 \in \{2,3,4\}$.

(9) $|S_{1i_1}| = 1$, *for all* $i_1 \notin \{1\}$ *and* $|S_{1i_1i_2}| = 1$, *for some unique* $i_1, i_2 \in \{2,3,4\}$.

(10) $|S_{1i_1}| = |S_{i_1i_2}| = |S_{i_1i_3}| = |S_{1i_2i_3}| = 1$, *for some unique* $i_2, i_3 \in \{2,3,4\}$.

Proof. In (1) and (2), the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{3,7}$ and $K_{4,5}$, respectively. In (3), the complement of $\Gamma_2(L)$ is contained in $S5.5$. In (4) and (5), we can find a copy of G_3 in the structure of $\Gamma_2(L)$. In (6) and (7), the complement of $\Gamma_2(L)$ is contained in $U6.6b$. In (8), the complement of the contraction of $\Gamma_2(L)$ is contained in $Z^*8.3$, one of the listed graphs in [5] (see Figure 36). In Figure 36, we have $a \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $d_1, d_2 \in S_4$, $s_{13} \in S_{13}$, $s_{14} \in S_{14}$ and $s_{34} \in S_{34}$.

In (9), the complement of the contraction of $\Gamma_2(L)$ is contained in $Z^*8.3$, one of the listed graphs in [5]. In (10), the complement of the contraction of $\Gamma_2(L)$ is contained in $W6.6a$, one of the listed graphs in [5]. In the all of the above cases, $\Gamma_2(L)$ is not toroidal. □

In the sequel, we assume that S_{12}, S_{13} and S_{14} are singelton sets and $S_{23} = S_{24} = S_{34} = S_{123} = S_{124} = S_{134} = \varnothing$. Then the graph $\Gamma_2(L)$ is toroidal, which is pictured in Figure 37. In Figure 37, $a \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $d_1, d_2 \in S_4$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$ and $s_{14} \in S_{14}$.
Also, consider S_{14}, S_{24} and S_{34} are singleton sets and $S_{12} = S_{13} = S_{23} = S_{123} = \varnothing$.

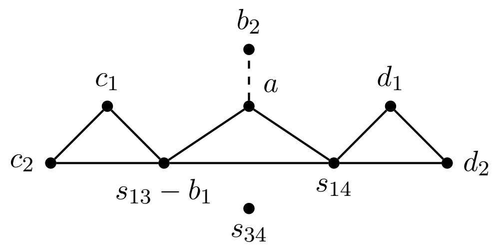


Figure 36

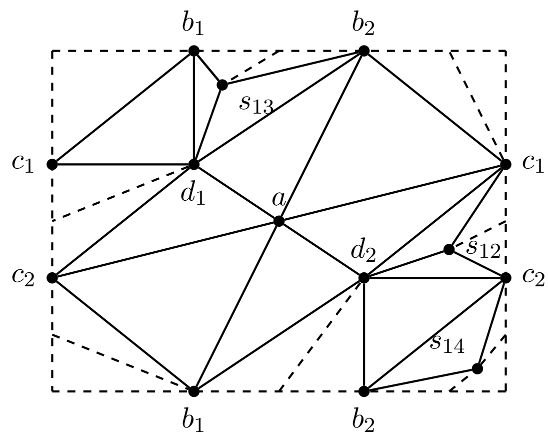


Figure 37

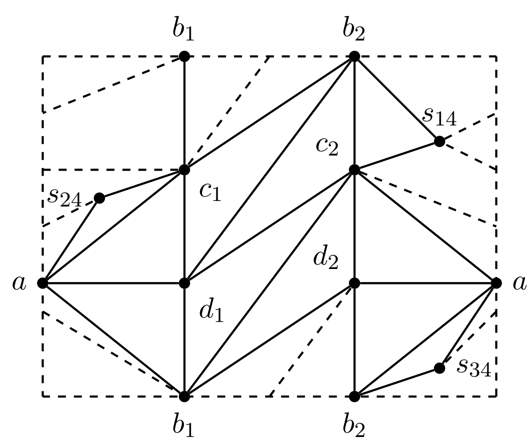


Figure 38

Then $\Gamma_2(L)$ is toroidal (see Figure 38). In Figure 38, we have the vertices $a \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $d_1, d_2 \in S_4$, $s_{14} \in S_{14}$, $s_{24} \in S_{24}$ and $s_{34} \in S_{34}$. We observe that if S_{23} has two elements and $S_{12} = S_{13} = S_{14} = S_{24} = S_{34} = S_{124} = S_{134} = \varnothing$, then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [5]. So it is a toroidal graph. Finally, if S_{23}, S_{24} and S_{34} are singleton sets and $S_{12} = S_{13} = S_{14} = \varnothing$, then the graph $\Gamma_2(L)$ is pictured in Figure 39, which is toroidal. In Figure 39, we have the vertices $a \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $d_1, d_2 \in S_4$, $s_{23} \in S_{23}$, $s_{24} \in S_{24}$ and $s_{34} \in S_{34}$. $\square$

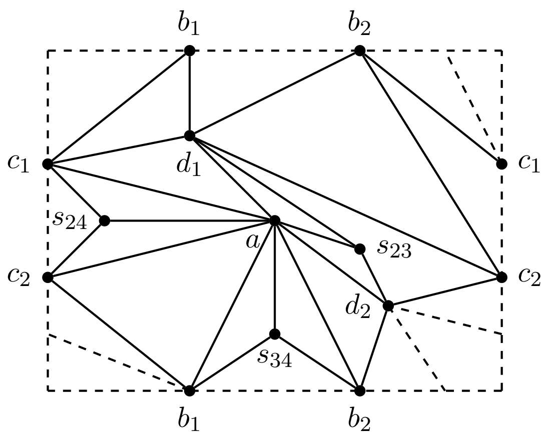


Figure 39

Theorem 2.15. *Suppose that $|\bigcup_{t=1}^{4} S_t| = 7$ and $|S_1| = 1$. Then $\Gamma_2(L)$ is toroidal if and only if one of the following conditions holds:*

(i) *If $|S_{1i_1}| = 1$, for all $i_1 \notin \{1\}$, then $S_{i_1 i_2} = S_{1 i_1 i_2} = \varnothing$, for all $i_1, i_2 \notin \{1\}$,*

(ii) *If $|S_{1i_1}| = |S_{i_1 i_2}| = |S_{i_1 i_3}| = 1$, for some unique $i_1, i_2, i_3 \in \{2, 3, 4\}$, then $S_{1i_2} = S_{1i_3} = S_{1 i_2 i_3} = \varnothing$,*

(iii) *If $|S_{i_1 i_2}| = 2$, for some unique $i_1, i_2 \in \{2, 3, 4\}$, then $S_{i_1 i_3} = S_{i_2 i_3} = S_{1 i_1 i_3} = S_{1 i_2 i_3} = \varnothing$, for $i_3 \notin \{1, i_1, i_2\}$ and $S_{1i_1} = \varnothing$, for all $i_1 \notin \{1\}$,*

(iv) *If $|S_{i_1 i_2}| = 1$, for all $i_1, i_2 \notin \{1\}$, then $S_{1i_1} = \varnothing$, for all $i_1 \notin \{1\}$.*

Theorem 2.16. *Suppose that $|\bigcup_{t=1}^{4} S_t| = 8$. Then the graph $\Gamma_2(L)$ is toroidal if and only if one of the following conditions holds:*

(i) *There is S_i with $|S_i| = 5$, for $1 \leq i \leq 4$ and $S_{i_1 i_2 i_3} = S_{i_1 i_2} = \varnothing$, for all $i_1, i_2, i_3 \notin \{i\}$,*

(ii) *There are unique S_i, S_j with $|S_i| = 4$ and $|S_j| = 2$, for $1 \leq i, j \leq 4$ and $S_{i_1 i_2 i_3} = S_{i_1 i_2} = \varnothing$, for all $i_1, i_2, i_3 \notin \{i\}$, $S_{ii_1} = \varnothing$, for $i_1 \notin \{i, j\}$,*

(iii) *There are unique* S_i *and* S_j *with* $|S_i| = |S_j| = 3$*, for* $1 \leq i, j \leq 4$ *and* $S_{1i_1i_2} = S_{ji_1i_2} = S_{ii_1} = S_{ji_1} = S_{i_1i_2} = \varnothing$*, for all* $i_1, i_2 \notin \{i, j\}$*,*

(iv) *There are unique* S_i *and* S_j *with* $|S_i| = 3$ *and* $|S_j| = 1$*, for* $1 \leq i, j \leq 4$ *and* $|S_{ii_1}| = 1$*, for some unique* $i_1 \in \{1, 2, 3, 4\} \setminus \{i, j\}$*. Also,* $S_{ij} = S_{ii_2} = S_{i_3i_4} = S_{ii_2j} = S_{i_1i_2j} = \varnothing$*, for* $i_2 \notin \{i, j, i_1\}$ *and for all* $i_3, i_4 \notin \{i\}$*,*

(v) *For all* $1 \leq i \leq 4$*,* $|S_i| = 2$ *and* $S_{i_1i_2i_3} = S_{i_1i_2} = \varnothing$*, for* $1 \leq i_1, i_2, i_3 \leq 4$*.*

Proof. If one of the above conditions holds, then one can easily check that $\Gamma_2(L)$ is toroidal.

Conversely, let $\Gamma_2(L)$ be a toroidal graph. Then we can consider the following cases:

(i) Assume that there is a unique S_i, say S_1, such that $|S_1| = 5$. If S_{234} has at least one element, then we can find a copy of $K_{4,5}$ in the contraction of $\Gamma_2(L)$, which is impossible. Also, if S_{23}, S_{24} or S_{34} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. Thus it is not toroidal, which is again impossible. Hence for toroidality of $\Gamma_2(L)$, we assume that $S_{234} = S_{23} = S_{24} = S_{34} = \varnothing$. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$ (cf. [5, p.55]).

(ii) Assume that there are unique S_i and S_j, say S_1 and S_2, such that $|S_1| = 4$ and $|S_2| = 2$. If S_{234}, S_{23} or S_{24} has at least one element, then we can find a copy of $K_{4,5}$ in the contraction of $\Gamma_2(L)$. So it is not toroidal, which is impossible. Also, if S_{13} or S_{14} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. Thus it is not toroidal, which is impossible. In addition, if S_{34} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $W7.7d$, one of the listed graphs in [5] (see Figure 40). In Figure 40, $a_1, a_2, a_3, a_4 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $d \in S_4$, $s_{34} \in S_{34}$. Thus it is not toroidal, which is impossible.

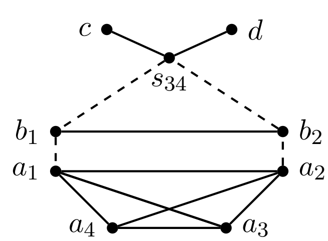


Figure 40

Therefore we assume that all of the above sets are empty. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, which is toroidal (cf. [5, p.55]).

(iii) Assume that there are unique S_i and S_j, say S_1 and S_2, such that $|S_1| = |S_2| = 3$. If S_{134} or S_{234} has at least one element, then the complement of

$\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. Thus it is not toroidal, which is impossible. Moreover if S_{13}, S_{14}, S_{23} or S_{24} has at least one element, then the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{4,5}$. It means that $\Gamma_2(L)$ is not toroidal, a contradiction. Also, if S_{34} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. Thus it is not toroidal, a contradiction. Therefore, for toroidality of $\Gamma_2(L)$, we assume that all of the above sets are empty. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$ (cf. [5, p.55]).

(iv) Assume that there are unique S_i and S_j, say S_1 and S_2, such that $|S_1| = 3$ and $|S_2| = 1$. If S_{13} or S_{14} has at least two elements, then we can find a copy of $K_{3,7}$ in the contraction of $\Gamma_2(L)$, which is impossible. Also, if S_{12}, S_{23}, S_{24} or S_{34} has at least one element, then we can find a copy of $K_{4,5}$ in the contraction of $\Gamma_2(L)$, a contradiction. In addition, if S_{234} has at least one element, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [5]. Thus it is not toroidal, a contradiction. In the case that $|S_{13}| = |S_{14}| = 1$, the graph $\Gamma_2(L)$ contains G_3, one of the listed graphs in [11]. Hence it is not toroidal, which is contradiction. Finally, if $|S_{13}| = |S_{124}| = 1$ or $|S_{14}| = |S_{123}| = 1$, then the complement of $\Gamma_2(L)$ is contained in $S5.6$, one of the listed graphs in [5]. Thus it is not toroidal, a contradiction. So, for the toroidality of $\Gamma_2(L)$, we assume that either S_{13} has at most one element by condition $S_{12} = S_{14} = S_{23} = S_{24} = S_{34} = S_{124} = S_{234} = \varnothing$, or S_{14} has at most one element by considering $S_{12} = S_{13} = S_{23} = S_{24} = S_{34} = S_{123} = S_{234} = \varnothing$. In these cases, the complement of $\Gamma_2(L)$ contains a subgraph isomorphic to $C415$, one of the listed graphs in [5], which is toroidal.

(v) Assume that S_1, S_2, S_3 and S_4 have two elements. As S_{123}, S_{124}, S_{134} or S_{234} has at least one element, then $\Gamma_2(L)$ contains $K_8 \setminus (K_{12} \cup 2K_2)$. So it is not toroidal (see [4]). Also, if S_{12}, S_{13}, S_{14}, S_{23}, S_{24} or S_{34} has at least one element, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. The contradiction is clear. Hence we assume that all of the above sets are empty. In this situation, $\Gamma_2(L)$ is isomorphic to a complete 4-partite graph with all parts of size two, which is toroidal. □

We end this paper with the following theorem.

Theorem 2.17. *Suppose that* $|\bigcup_{t=1}^{4} S_t| = 9$ *. Then the graph* $\Gamma_2(L)$ *is toroidal if and only if there exists* S_i *with* $|S_i| = 6$*, for* $1 \leq i \leq 4$ *and* $S_{i_1 i_2 i_3} = S_{i_1 i_2} = \varnothing$*, for all* $i_1, i_2, i_3 \notin \{i\}$*.*

Proof. Let $\Gamma_2(L)$ be a toroidal graph and for all $1 \leq i \leq 4$, $|S_i| \neq 6$. Then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. Hence it is not toroidal, which is impossible. Therefore we assume that there exists a unique S_i with $|S_i| = 6$, for $1 \leq i \leq 4$. Now, if $|S_{234}| \geq 1$, then we can see a subgraph isomorphic to $K_{4,6}$ in the contraction of $\Gamma_2(L)$, which is not toroidal. When S_{23}, S_{24} or S_{34} has at least one element, the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. It means that $\Gamma_2(L)$ is not toroidal, which is impossible. Now, assume that all of the sets S_{23}, S_{24}, S_{34} and S_{234} are empty. Then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [5], which is toroidal.

The converse statement is clear. □

Acknowledgments. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

References

[1] M. Afkhami and K. Khashyarmanesh, *The comaximal graph of a lattice*, Bull. Malays. Math. Sci. Soc., **37**(2014), 261–269.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.

[3] A. Bouchet, *Orientable and nonorientable genus of the complete bipartite graph*, J. Combin. Theory Ser. B **24**(1978), 24–33.

[4] F. Harary, P. C. Kainen, A. J. Schwenk and A. T. White, *A maximal toroidal graph which is not a triangulation*, Math.Scand. **33**(1973), 108–112.

[5] A. L. Hlavacek, *9-vertex irreducible graphs for the torus*, PhD. thesis, The university of Ohio State, 1997.

[6] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319**(2008), 1801-1808.

[7] J. B. Nation, *Notes on Lattice Theory*, Cambridge studies in advanced mathematics, Vol. **60**, Cambridge University Press, Cambridge, 1998.

[8] G. Ringel, *Map Color Theorem*, Springer-Verlag, New York/Heidelberg, 1974.

[9] P. K. Sharma, S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra **176**(1995), 124–127.

[10] H. J. Wang, *Graphs associated to co-maximal ideals of commutative rings*, J. Algebra **320**(2008), 2917–2933.

[11] D. Zeps, *Förbidden minors for projective plane are free-toroidal or non-toroidal*, IUUK-CE-ITI series (2009).