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Note on Cellular Structure of Edge Colored Partition Algebras

A. Joseph Kennedy*
Department of Mathematics, Pondicherry University, Pondicherry, India
e-mail : kennedy.pondi@gmail.com

G. Muniasamy
Department of Mathematics, MIT, Anna University, Chennai, India
e-mail : gmuniasamy32@gmail.com

Abstract. In this paper, we study the cellular structure of the G-edge colored partition algebras, when G is a finite group. Further, we classified all the irreducible representations of these algebras using their cellular structure whenever G is a finite cyclic group. Also we prove that the $\mathbb{Z}/r\mathbb{Z}$-Edge colored partition algebras are quasi-hereditary over a field of characteristic zero which contains a primitive r^{th} root of unity.

1. Introduction

Cellular structure of algebras has been studied in the last few years, and a variety of algebras have been proved as cellular, which are like Ariki-Koike Hecke algebra, Brauer algebra, Partition algebra, etc. Cellular algebras, which were introduced by Graham and Lehrer in [5], were defined by the existence of a basis with some multiplicative properties. Later, König and Xi in [10], have given equivalent definition for cellular algebra in terms of cell ideals, but not in terms of basis. One of the main problem in the representation theory is to parameterize all irreducible modules for an algebra. But in cellular algebras, the structure provides a complete list of irreducible modules for the algebra over any field in a systematic way.

The partition algebras have been studied independently by Martin in [11] and Jones as generalizations of the Temperley-Lieb algebras and the Potts model in sta-

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tistical mechanics. In 1993, Jones considered the algebra as the centralizer algebra of the symmetric group S_n on $V^{\otimes k}$ (see [7]). In [14], Xi gave a sufficient condition for a given algebra to be cellular and proved that the partition algebras are cellular by using this condition.

In [2], Matthew Bloss introduced a G-edge colored partition algebra (or G-colored partition algebra) as the centralizer algebra of the wreath product $G \wr S_n$, where G is any finite group. This algebra has an important subalgebra called Ramified partition algebra (or Class partition algebra) which has been introduced by P.P Martin and A. Elgamal in [12] and by A.J Kennedy in [9] in connection with some physical problem in Statistical Mechanics and as the centralizer of $S_{|G|} \wr S_n$ respectively. Further, the G-edge colored partition algebra has been identified as subalgebra of the G-vertex colored partition algebra which was introduced and realized as the centralizer algebra of the subgroup $G \times S_n$ of $G \wr S_n$ in [13].

We are interested in studying the cellular structure and the representations of this algebras. In this paper, we decompose G-edge colored partition algebra as a direct sum of vector spaces $\bigoplus_{l=0}^{k} V_l \otimes_F V_l \otimes_F F[G \wr S_l]$. If G is a finite group and $F[G \wr S_l]$ are cellular for $0 \leq l \leq k$, we prove that the G-edge colored partition algebras are cellular by using cellular structure of $F[G \wr S_l]$.

The Ariki-Koike Hecke algrbras $\mathcal{H}_{\zeta,F}$ were introduced by Ariki and Koike in [1], as deformation of $\mathbb{Z}/r\mathbb{Z} \wr S_n$. This algebras have been proved to have a cellular basis by Graham and Lehrer in [5] also by Dipper, James and Mathas in [4].

Let F be a field with a primitive r^{th} root of unity. If $\zeta = 1$, then the algebra $\mathcal{H}_{\zeta,F}$ is isomorphic to $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$. By using a cellular structure of $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$, we have parameterized the index set of all irreducible representations of $\mathbb{Z}/r\mathbb{Z}$-edge colored partition algebra. Also we prove that the $\mathbb{Z}/r\mathbb{Z}$-edge colored partition algebras are quasi-hereditary if the characteristic of F is zero.

2. Cellular Algebra

The original definition of cellular algebra was introduced by Graham and Lehrer in [5]. Here, we restrict ourself to an arbitrary field instead of commutative ring in the following definition.

Definition 2.1([5]). An associative F-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following condition are satisfied.

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an F-basis $C^{\lambda}_{S,T}$ where (S, T) runs through all element of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an F-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{S,T}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $\sum_{U \in M(\lambda)} r_a(U, S)C^{\lambda}_{U,T} + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficient $r_a(U,S) \in F$ do not depend on T.

For each $\lambda \in I$, there is a cell module $W(\lambda)$ with F- basis $\{C_S | S \in M(\lambda)\}$, the action is given by $aC_S = \sum_{T\in M(\lambda)} r_a(T,S)C_T$, where $r_a(T,S)$ is in F as in the above definition(C3).

For a cell module $W(\lambda)$, we can associate a bilinear form $\Phi_\lambda : W(\lambda)\times W(\lambda) \to F$ by $C^\lambda_{S,S}C^\lambda_{T,T} \equiv \Phi(C_S, C_T)C^\lambda_{S,T}$ modulo the ideal generated by all basis elements $C^\mu_{U,V}$ with upper index $\mu < \lambda$. And the isomorphism class of simple modules is parameterized by the set $\{\lambda \in I | \Phi_\lambda \neq o\}$. Next we recall the equivalent definition of cellular algebra in terms of cell ideals which was introduced in [10] by Koing and Xi.

Definition 2.2([14]). Let A be an F-algebra. Assume that there is an involution i on A. A two sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over F and there is an isomorphism of A-module $\alpha : J \simeq \Delta \otimes_F i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_F i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x\otimes y \mapsto i(y)\otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_F i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is an F-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \oplus^j_{i=1} J'_j$ gives a chain of two sided ideals of A : $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j $(j = 1, 2, \cdots n)$ the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

Note that, the modules $\Delta(j)$ for $1 \leq j \leq n$, are called the *standard modules* of the cellular algebra. These modules are called the *cell modules* in the sense of Graham and Lehrer in [5]. And the above chain of ideals in A is called *cell chain* of A.

Lemma 2.3([14]). *Let A be an F-algebra with an involution i. Suppose there is a decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes_F V_j \otimes_F B_j \quad \textit{as direct sum of vector spaces} \tag{2.1}$$

where V_j is a vector space and B_j is a cellular algebra with respect to an involution σ_j and a cell chain $J^{(j)}_1 \subset \cdots \subset J^{(j)}_{s_j} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^t V_j \otimes_F V_j \otimes_F B_j$.

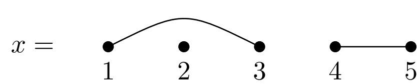
Assume that the restriction of i on $V_j \otimes_F V_j \otimes_F B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form $\phi_j : V_j \otimes_F V_j \to B_j$ such that $\sigma_j(\phi_j(w,v)) = \phi_j(v,w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes V_j \otimes B_j$ is governed ϕ_j modulo J_{j-1}, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have $(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y,u)c$ modulo the ideal J_{j-1}, and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in A for all l and j, then A is a cellular algebra.

In [14], Xi have given this Lemma 2.3 as a sufficient condition, especially for diagram algebras to be cellular. We are going to use this lemma to prove G-edge colored partition algebras are cellular.

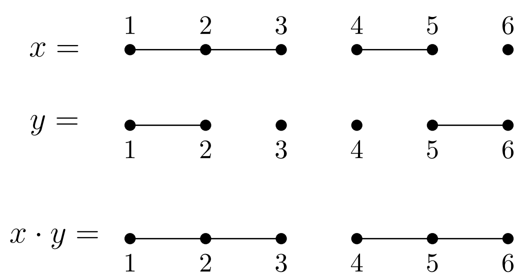
3. Edge Colored Partition Algebra

Let N be a finite set. A partition x on N is a collection $\{A_1, A_2, \cdots, A_n\}$ of pairwise disjoint non-empty subsets of N whose union is N. The sets $A_1, A_2, \cdots, A_n$ are called blocks of that partition. We say that a partition x is finer than a partition y if every block of x is contained in some block of y. In this case we write $x \leq y$.

Let k be a positive integer and denote $\mathbf{k} = \{1, 2, \cdots, k\}$ with usual order. Let x be a partition on $\mathbf{k}$. Then the partition x can be represented as diagram on $\mathbf{k}$ as follows, arrange vertices $1, 2, \cdots, k$ in a row, and then two vertices are connected by a path if and only if they are in a same block of x. For if $x = \{\{1,3\},\{2\},\{4,5\}\}$ is a partition of $\{1,2,3,4,5\}$ then



Let us denote $P_{\mathbf{k}}$ be the set of all such partition diagram on $\mathbf{k}$. Suppose x, y are two partitions on $\mathbf{k}$, we define $x \cdot y$ is the smallest partition z on $\mathbf{k}$ such that $x, y \leq z$. As diagrammatically,



Let $\mathbf{k}' = \{1', 2', \cdots, k'\}$. Suppose d is a partition on $\mathbf{k} \cup \mathbf{k}'$, then d can be represented as diagram on $\mathbf{k} \cup \mathbf{k}'$ as follows, arrange vertices $1, 2, \cdots, k$ in a row and vertices $1', 2', \cdots, k'$ in parallel row directly below. Then two vertices are connected by a path if and only if they are in a same block in d. Such a partition diagram is called

k**-partition** diagram on $\mathbf{k} \cup \mathbf{k}'$. Two partition diagrams are equivalent if and only if they determine the same partition on $\mathbf{k} \cup \mathbf{k}'$.

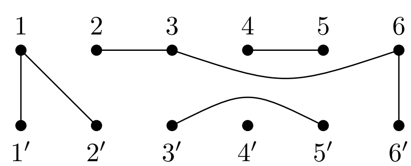
A standard k-partition diagram is a k-partition diagram whose blocks partition $\mathbf{k}$ into top blocks and partition $\mathbf{k}'$ into bottom blocks by restriction on $\mathbf{k}$ and $\mathbf{k}'$ respectively and if a top block connects to a bottom block (such blocks are called through block) then it connects with a single edge joining the leftmost vertex in each block. Such edges are called propagating edges and the number of propagating edges is called the propagating number of the diagram and its denoted by $pn(d)$.

The set of all k-partition diagram under this relation on $\mathbf{k} \cup \mathbf{k}'$ is denoted by $P_{\mathbf{k} \cup \mathbf{k}'}$.

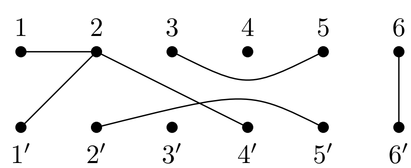
Definition 3.1([11, 8]). Let F be any field and $q \in F$. The *partition algebra* $P_{\mathbf{k} \cup \mathbf{k}'}(q)$ is F-algebra with basis $P_{\mathbf{k} \cup \mathbf{k}'}$ with the following multiplication on diagrams. Let d_1 and d_2 be diagram. To obtain the product $d_1 d_2$

- Place d_1 above d_2 so that the bottom row of d_1 coincide with the top row of d_2. We now have a diagram with a top, middle and bottom row.
- Count the number of connected components that lie entirely in the middle row. Let this number be n.
- Make a new k-partition diagram d_3 by eliminating that middle row of vertices, by keeping the top and bottom rows and maintaining the connection between them.
- We define $d_1 d_2 = q^n d_3$.

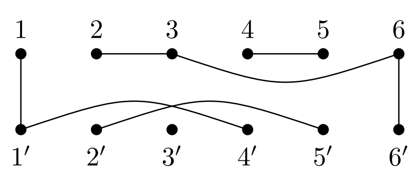
$d_1 =$



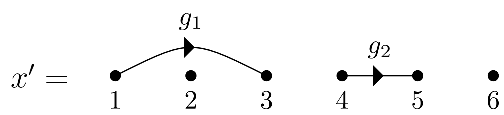
$d_2 =$



$d_1 d_2 = q^2$

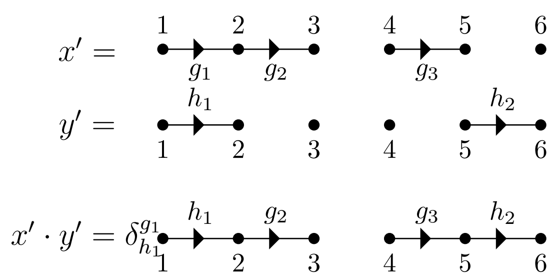


Let G be any group. We denote $P_{\mathbf{k}}(G)$ as the set of all elements of $P_{\mathbf{k}}$ whose edges are labeled by the elements of G, with orientation from left to right. For example, let $g_1, g_2 \in G$. Then the following diagram is an element of $P_{\mathbf{6}}(G)$.



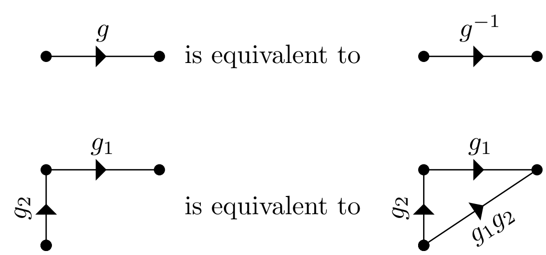
Let $x', y' \in P_{\mathbf{k}}(G)$ with underlying partition diagrams $x, y \in P_{\mathbf{k}}$ respectively, we define $x' \cdot y' \in P_{\mathbf{k}}(G)$ as follows,

- $x' \cdot y' = 0$ if and only if there exist an edge from some vertex i to j in x' and in y' with different colour.
- otherwise, $x' \cdot y'$ is the diagram whose underlying partition diagram is $x \cdot y \in P_k$ and with same labels.



where $\delta_{h_1}^{g_1}$ is a kroneker delta.

A (G, k)-partition diagram is a k-partition diagram with oriented edges, where each edge is colored(or labeled) by an element of the group G. When k is understood, we will call such diagrams as G diagrams. Two G-diagrams are equivalent if the underlying partitions are equivalent and the G-diagrams are equivalent up to vector addition, that is the following holds.

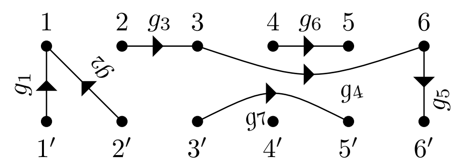


Thus when we speak of a G-diagram, we are really speaking of its equivalence class. The set of all such G-partition diagrams is denoted by $P_{\mathbf{k}\cup\mathbf{k}'}(G)$. If G is finite, then $|P_{\mathbf{k}\cup\mathbf{k}'}(G)| = \sum_{l=1}^{2k} |G|^{2k-l} S(2k,l)$, where $S(2k,l)$ is the Stirling number.

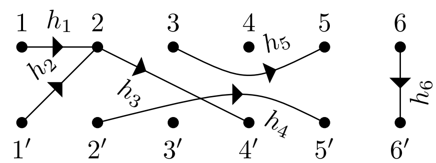
Definition 3.2([2]). The *edge colored partition algebra* $P_{\mathbf{k}\cup\mathbf{k}'}(q,G)$ is the F-algebra $F[P_{k\cup k'}(G)]$ with basis consisting of G-diagrams and the multiplication on G-diagrams is defined as follows:
Let d_1, d_2 be two G-diagrams

- Multiply the underlying partition diagram of d_1 and d_2. This will give the underlying partition diagram of the G-diagram d_1d_2.
- In carrying out the previous step, d_1 is placed above d_2. If during the concatenation, a bottom edge of d_1 coincide with a top edge of d_2 with the same orientation but with different label, then $d_1d_2 = 0$.
- Perform vector addition of the labels along imposed connection between d_1 and d_2. Start in d_1 and follow a path into d_2, performing vector addition as you go. When doing this, the labels on the edges in the diagram d_2 are multiplied on the right of the d_1 edge labels.
- For each connected components of edges entirely in the middle row, a factor of q appears in the product.

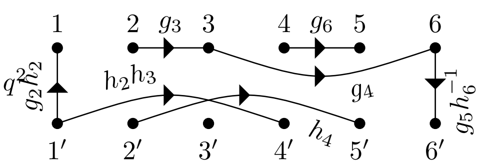
$\bar{d}_1 =$



$\bar{d}_2 =$



$\bar{d}_1\bar{d}_2 = \delta^{(h_1,h_5)}_{(g_1g_2^{-1},g_7)}\ q^2$



where

$$\delta^{(h_1,h_5)}_{(g_1g_2^{-1},g_7)} = \begin{cases} 1 & \text{if } h_1 = g_1g_2^{-1} \text{ and } h_5 = g_7 \\ 0 & \text{Otherwise} \end{cases}$$

Standard form of a G-diagram

- The underlying partition diagram is in standard form
- The orientation of edges are either from left to right or from top to bottom.

For each equivalence class we can choose a standard G-diagram as representative, so hereafter a G-diagram means that it is a standard G-diagram.

Let $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$, define $flip(d) \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$ as follows: Rotate the diagram from top to bottom and change the orientation and colour of the propagating edges by their inverse. Clearly, $flip(flip(d)) = d$ for all $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$.

Let $\eta : P_{\mathbf{k}\cup\mathbf{k}'}(q, G) \to P_{\mathbf{k}\cup\mathbf{k}'}(q, G)$ be the linear extension of the map $flip$ on $P_{\mathbf{k}\cup\mathbf{k}'}(G)$.

Lemma 3.3. *The map η is an anti-automorphism of $P_{\mathbf{k}\cup\mathbf{k}'}(q, G)$ with $\eta^2 = id$.*

Proof. Clearly, η is a linear. Since $flip(flip(d)) = d$, $\eta^2(d) = d$ for all $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$. From the definition of the multiplication on G-diagrams, $flip(d_1 d_2) = flip(d_2)flip(d_1)$ for every $d_1, d_2 \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$. Therefore, $\eta(d_1 d_2) = \eta(d_2)\eta(d_2)$ for all $d_1, d_2 \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$. □

4. Cellular Structure of $P_{\mathbf{k}\cup\mathbf{k}'}(q, G)$

Let us recall that $P_{\mathbf{k}}(G)$ be the set all partition diagrams on $\mathbf{k}$ with G-labeled edges. For $l \in \{0, 1, \cdots, k\}$, we define a vector space V_l, which has as a basis set

$$s_l = \{(x, S) \mid x \in P_{\mathbf{k}}(G), |x| \geq l \text{ and } S \text{ is a collection of any } l\text{-blocks of } x\}$$

Note that, the dimension of V_l is $\displaystyle\sum_{i=l}^{k} |G|^{k-l} S(k, l) \binom{i}{l}$. Let $(x, S) \in V_l$. We denote $[i]$ for the block of x with the left most vertex i.

We define an order on the blocks of x that $[i] < [j]$ if $i < j$, this gives an order on S. We denote $j_{[i]}$ for the j^{th} element of S with the left most vertex i. So, we can always write S as $\{1_{[i_1]}, 2_{[i_2]}, \cdots, l_{[i_l]}\}$. Let us denote $d_{\mathbf{k}}$ is the partition on $\mathbf{k}$ which is obtained from $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$ by deleting all elements in $\mathbf{k}'$ of d (i.e., by restricting on $\mathbf{k}$).

Definition 4.1. The *wreath product* of a group G with the symmetric group S_n is a group

$$G \wr S_n = \{(g_1, g_2, \cdots, g_n; \pi) \mid g_i \in G \text{ and } \pi \in S_n\}$$

under the multiplication

$$(g_1, g_2, \cdots, g_n; \pi_1)(h_1, h_2, \cdots, h_n; \pi_2) = (g_1 h_{\pi_1(1)}, g_2 h_{\pi_1(2)}, \cdots, g_n h_{\pi_1(n)}; \pi_1 \pi_2).$$

Lemma 4.2. *There is a bijection from $P_{\mathbf{k}\cup\mathbf{k}'}(G)$ to $\amalg_{l=0}^{k} s_l \times s_l \times G \wr S_l$*

Proof. Let $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$. Define $x := d_{\mathbf{k}} \in P_{\mathbf{k}}(G)$ and $y := d_{\mathbf{k}'} \in P_{\mathbf{k}}(G)$ (by identifying $\mathbf{k}'$ with $\mathbf{k}$ by sending j' to j). Let S_d be the set of all through blocks of d, then $|S_d| = pn(d) = l$ (say). Now consider $S_d = \{C^1, C^2, \cdots, C^l\}$. Let us define $S = \{C^1_{\mathbf{k}}, C^2_{\mathbf{k}}, \cdots, C^l_{\mathbf{k}}\}$ and $T = \{C^1_{\mathbf{k}'}, C^2_{\mathbf{k}'}, \cdots, C^l_{\mathbf{k}'}\}$, where $C^i_{\mathbf{k}}$ (*resp* $C^i_{\mathbf{k}'}$) are the blocks of x (*resp* y) which are obtained from $C^i \in S_d$ by deleting the numbers contained in $\mathbf{k}'$ (*resp* $\mathbf{k}$). Then we can rewrite $S = \{1_{[i_1]}, 2_{[i_2]} \cdots l_{[i_l]}\}$ and $T = \{1_{[j'_1]}, 2_{[j'_2]} \cdots l_{[j'_l]}\}$. Hence, $(x, S), (y, T) \in s_l$. Define $(g_1, g_2 \cdots, g_l; \pi) \in G\wr S_l$ corresponds to d by $\pi(t) = s$ if $t_{[i]}$ is connected to $s_{[j']}$ by an edge with colour g_t in d. Since the G-diagram d is in the standard form, x, y and $(g_1, g_2 \cdots, g_l; \pi)$ are unique. Thus, every G-diagram d can be uniquely represented as $(x, S)\times(y, T)\times(g_1, g_2 \cdots, g_l; \pi)$ in $s_l \times s_l \times (G\wr S_l)$. Conversely, for every element $(x, S) \times (y, T) \times (g_1, g_2 \cdots, g_l; \pi) \in s_l \times s_l \times (G \wr S_l)$ we can associate unique partition G-diagram $d \in P_{\mathbf{k}\cup\mathbf{k}'}(G)$. □

For every $l \in \{0, 1, \cdots, k\}$, V_l and $F[G \wr S_l]$ are vector space with basis set s_l and $G \wr S_l$ respectively. So, $\bigoplus_{l=0}^{k} V_l \otimes_F V_l \otimes_F F[G \wr S_l]$ is a vector space with basis set $\amalg_{l=0}^{k} s_l \times s_l \times G \wr S_l$.

Remark 4.3. As vector space, $P_{\mathbf{k}\cup\mathbf{k}'}(q, G)$ is isomorphic to $\bigoplus_{l=0}^{k} V_l \otimes_F V_l \otimes_F F[G \wr S_l]$ (by above Lemma 4.2).

For $l \in \{0, 1, \cdots, k\}$, define $\phi_l : V_l \otimes_k V_l \to K[G \wr S_l]$ as follows: Let (x, S) and (y, T) be two elements in s_l. Define

$$\phi_l((x, S), (y, T)) = \begin{cases} q^{|H|}(e; \pi) & \text{if there exist a } \pi \in S_l \text{ such that the block of} \\ & x \cdot y \text{ (if } \neq 0 \text{ and) containing the } i\text{th block of } S \\ & \text{contains the unique } \pi(i)\text{th block of } T,\ (i = 1, 2, \cdots, l) \\ 0 & \textit{otherwise} \end{cases}$$

where H be the set of all blocks on $\mathbf{k} \setminus S \cup T$ which are obtained from the blocks of $x \cdot y$ by deleting the elements of $S \cup T$. By Lemma 4.3 in [14], ϕ_l is a bilinear map.

Lemma 4.4. *Let d, d' be two G-diagrams. If $d = (u, R) \otimes (x, S) \otimes (g^1; \pi_1)$, $d' = (y, T)\otimes(v, Q)\otimes(g^2; \pi_2) \in V_l \otimes_F V_l \otimes_F F[G\wr S_l]$, where $g^i = (g^i_1, g^i_2, \cdots, g^i_l)$, $(i = 1, 2)$ then $dd' = (u, R) \otimes (v, Q) \otimes (g^1; \pi_1)\phi_l((x, S), (y, T))(g^2; \pi_2)$ modulo $J_{l-1} = \bigoplus_{j=0}^{l-1} V_j \otimes_F V_j \otimes_F F[G \wr S_j]$.*

Proof. Let $dd' = \delta q^r d''$. We claim that $(u, R)\otimes(v, Q)\otimes(g^1; \pi_1)\phi_l((x, S), (y, T))(g^2; \pi_2)$ is exactly equal to $\delta q^r d''$, in $P_{\mathbf{k}\cup\mathbf{k}'}(q, G)$ modulo J_{l-1}.

Case(1): Suppose $\phi_l((x, S), (y, T)) = 0$. Then by definition of ϕ_l, $x \cdot y$ is zero or any one of the following is true:

1. there exits a block of $x \cdot y$ which contains either more than one element of S(or T),
2. there exits a block of $x \cdot y$ which contains a single element of S (res. T) but no element of T (res. S),

which implies that $dd' = 0$ or $pn(dd') < l$. Therefore, $dd' \in J_{l-1}$.

Case(2): Suppose $\phi((x,S)(y,T)) = q^{|H|}(e;\pi)$ where π is defined as in the definition of ϕ_l. Since $d_{\mathbf{k}'} = x$ and $d'_{\mathbf{k}} = y$, we have $|H|$ is equal to the number of middle components. So, it is sufficient to prove that $(u,R)\otimes(v,Q)\otimes(g^1;\pi_1)(e;\pi)(g^2;\pi_2) = d''$. That is,

$$(u,R)\otimes(v,Q)\otimes(g_1^1 g^2_{(\pi_1\pi)(1)},\cdots,g_l^1 g^2_{(\pi_1\pi)(l)};\pi_1\pi\pi_2) = d''.$$

Clearly, $d''_{\mathbf{k}} = u, d''_{\mathbf{k}'} = v$. By the definition of ϕ_l, there are exactly l blocks $C_1, C_2, \cdots, C_l$ of $x\cdot y$ in which each block contains exactly one block in S and one block in T. Now consider a block C_i in $x\cdot y$, then there is a block $i_{[s]} \in S$ and $\pi(i)_{[t]} \in T$ which is contained in C_i. Moreover, the block $i_{[s]}$ is connected to $\pi(i)_{[t]}$ by an edge which is colored by e. Then, there is a block in d which contains $\pi_1^{-1}(i)_{[r]} \in R$ and $i_{[s]} \in S$ and that edge is colored by $g_j^1 = g^1_{\pi_1^{-1}(i)}$ and there is a block in d' which contains $\pi(i)_{[t]} \in T$ and $\pi_2(\pi(i))_{[p]} \in Q$ and that edge is colored by $g^2_{\pi(i)}$). Hence, there is a block in d'' which contains both $\pi_1^{-1}(i)_{[r]} \in R$ and $\pi_2(\pi(i))_{[p]} \in Q$ and the edge is colored by $g^1_{\pi_1^{-1}(i)} g^2_{\pi(i)}$. That is, there is a block in d'' which contains both $j_{[r]} \in R$ and $\pi_2(\pi(\pi_1(j)))_{[p]} \in Q$ and the edge is colored by $g_j^1 g^2_{\pi(\pi_1(j))}$. Therefore, $(u,R)\otimes(v,Q)\otimes(g_1^1 g^2_{(\pi_1\pi)(1)},\cdots,g_l^1 g^2_{(\pi_1\pi)(l)};\pi_1\pi\pi_2) = d''$. □

Lemma 4.5. *Let l and m be two non-negative integers such that $l < m$. Suppose $d = (u,R)\otimes(x,S)\otimes(g^1;\pi_1) \in V_m\otimes_F V_m\otimes_F F[G\wr S_m]$, and $d' = (y,T)\otimes(v,Q)\otimes(g^2;\pi_2) \in V_l\otimes_F V_l\otimes_F F[G\wr S_l]$. Then $dd' = q^{|H|}(w,E)\otimes(z,G)\otimes(g;\tau)$ in $V_l\otimes_F V_l\otimes_F F[G\wr S_l]$ modulo J_{l-1}, where $(g;\tau) = (g^3;\pi_1')(g^2;\pi_2)$ for some $(g^3;\pi_1') \in G\wr S_l$.*

Proof. By lemma 4.2, if we consider d and d' as a diagrams, then $pn(dd') \le l$. Suppose $pn(dd') = l$ that is, $|E| = l$. Then $|G| = l$. Since $|Q| = l$ and G is obtained from Q, which implies that $(z,G) = (v,Q)$. Hence, by Lemma 4.2 and Lemma 4.4 we have $(g;\tau) = (g^3;\pi_1')(g^2;\pi_2)$ for some $(g^3;\pi_1') \in G\wr S_l$. Therefore, $dd' \in V_l\otimes_F V_l\otimes_F F[G\wr S_l]$ Suppose $pn(dd') < l$ that is, $|E| < l$, then obviously $dd' \in J_{l-1}$. □

Lemma 4.6. *If $d = (x,S)\otimes(y,T)\otimes(g_1,g_2\cdots g_l;\pi) \in V_l\otimes_F V_l\otimes_F F[G\wr S_l]$, then $\eta(d) = (y,T)\otimes(x,S)\otimes((g^{-1}_{\pi^{-1}(1)},\cdots,g^{-1}_{\pi^{-1}(l)});\pi^{-1})$.*

Proof. For every $i \in \{1,2,\cdots,l\}$, there is a block $i_{[s]} \in S$ which is connected to $\pi(i)_{[t]} = j_{[t]} \in T$ by an edge colored by g_i in d. Which imply that the block $j_{[t]} \in T$ which is connected to $\pi^{-1}(j)_{[s]} \in S$ by an edge colored by $g^{-1}_{\pi^{-1}(j)}$ in $\eta(d)$ (since the orientation of edge is changed). Therefore, by definition of η, $\eta(d) = (y,T)\otimes(x,S)\otimes((g^{-1}_{\pi^{-1}(1)},\cdots,g^{-1}_{\pi^{-1}(l)});\pi^{-1})$. □

Lemma 4.7. *Let $* : F[G\wr S_l] \to F[G\wr S_l]$ be the involution on $F[G\wr S_l]$ which is defined by $(g_1,g_2,\cdots,g_l;\pi) \mapsto ((g^{-1}_{\pi^{-1}(1)},\cdots,g^{-1}_{\pi^{-1}(l)});\pi^{-1})$. for all $(g_1,g_2,\cdots,g_l;\pi) \in G\wr S_l$. Then $(\phi_l(v_1,v_2))^* = \phi_l(v_2,v_1)$ for all $v_1,v_2 \in V_l$.*

Proof. Let $v_1 = (x,S)$ and $v_2 = (y,T)$. Suppose $\phi_l(v_1,v_2) = 0$. Since $x\cdot y = y\cdot x$, then by definition of ϕ_l, $\phi_l(v_2, v_1) = 0$. If $\phi_l(v_1, v_2) \neq 0$, then $\phi_l(v_1, v_2) = q^{|H|}(e;\pi)$. So, there is a block C_i of $x \cdot y$ which contains both $i_{[s]} \in S$ and $\pi(i)_{[t]} \in T$ with edge colored by e. Since C_i is block of $y \cdot x$, then C_i contains both $\pi^{-1}(i)_{[s]} \in S$ and $i_{[t]} \in T$ with edge labeled by e. Therefore, $\phi_l(v_2, v_1) = q^{|H|}(e;\pi^{-1})$. By definition of involution $*$, the result follows. □

Theorem 4.8. *The G-Edge Colored Partition algebras $P_{\mathbf{k}\cup\mathbf{k}'}(q,G))$ are cellular with involution η if $F[G \wr S_l]$ is cellular with involution $*$ for all $l \in \{0, 1, \cdots, k\}$.*

Proof. Put $j_{-1} = 0$ and $G \wr S_0 = \{1\}$. By Remark 4.3, the edge colored partition algebra $P_{\mathbf{k}\cup\mathbf{k}'}(q,G)$ has decomposition as direct sum of vector space

$$P_{\mathbf{k}\cup\mathbf{k}'}(q,G) = \bigoplus_{l=0}^{k} V_l \otimes_F V_l \otimes_F F[G \wr S_l].$$

Since $F[G \wr S_l]$ is cellular with involution $(g_1, g_2, \cdots, g_l; \pi) \mapsto ((g^{-1}_{\pi^{-1}(1)}, \cdots, g^{-1}_{\pi^{-1}(l)}); \pi^{-1})$, there is a cell chain $J_1^{(l)} \subset \cdots \subset J_{s_l}^{(l)} = F[G \wr S_l]$ for all l. By Lemma 4.2, Lemma 4.4 and Lemma 4.5, $V_l \otimes V_l \otimes J_j^l + J_{l-1}$ is an ideal of $P_{\mathbf{k}\cup\mathbf{k}'}(q,G)$, for every l. Moreover,

$$\begin{aligned} V_1 \otimes V_1 \otimes J_1^{(1)} &\subset \cdots \subset V_1 \otimes V_1 \otimes J_{s_1}^{(1)} \subset V_1 \otimes V_1 \otimes F[G \wr S_1] \oplus V_2 \otimes V_2 \otimes J_1^{(2)} \\ &\subset \cdots \subset V_1 \otimes V_1 \otimes F[G \wr S_1] \oplus V_2 \otimes V_2 \otimes F[G \wr S_2] \\ &\subset \cdots \subset \bigoplus_{l=1}^{k-1} V_l \otimes_F V_l \otimes_F F[G \wr S_l] \oplus V_k \otimes V_k \otimes J_{S_k}^k = P_{\mathbf{k}\cup\mathbf{k}'}(q,G). \end{aligned}$$

By Lemma 4.6 and Lemma 4.7, it satisfied all the condition of Lemma 2.3. Hence $P_{\mathbf{k}\cup\mathbf{k}'}(q,G)$ is cellular. □

Cellular algebras are cyclic cellular if all the cell modules are cyclic. In [6], T.Geetha and F. M. Goodman have proved that if A is cyclic cellular then $A \wr S_n$ is cyclic cellular.

Corollary 4.9([6]). *If $F[G]$ is cyclic cellular then G-Edge colored partition algebras are cellular.*

Corollary 4.10. *The partition algebra is cellular.*

Proof. Take G is trivial group. □

In general, $F[G \wr S_n]$ is not cellular for any arbitrary group G. And even the group algebra $F[G]$ is not a cellular, since cellular algebra is always split but general field are not splitting field for arbitrary group. Moreover $F[G \wr S_n] = (F[G]) \wr S_n$ and if $F[G]$ is quasi hereditary then $F[G \wr S_n]$ is also quasi hereditary whenever $n! \in F$. Since cellular algebras are more close to quasi-hereditary, so in a similar way we can ask that if $F[G]$ is cellular, whether $F[G \wr S_n]$ is cellular ?. Suppose if G is cyclic group of order r and F is a field which contains primitive r^{th} roots of unity, then by Theorem 4.15, $F[(Z/rZ) \wr S_n]$ have a cellular structure.

Cellular basis for $F[(\mathbb{Z}/r\mathbb{Z})\wr S_n]$

The Ariki-Koike Hecke algrbras $\mathcal{H}$ were introduced by Ariki and Koike in [1], as deformation of $\mathbb{Z}/r\mathbb{Z}\wr S_n$. Moreover, these algebras are a generalization of Iwahori-Hecke algebras of type A and B. For Hecke algebra of Symmetric group $\mathcal{H}(S_n)$ (deformation of S_n), the Kazhdan-Lusztig basis became a cellular basis. Graham and Lehrer in [5] constructed a cellular basis for $\mathcal{H}$ through the Kazhdan-Lusztig basis of $\mathcal{H}(S_n)$. Dipper, James and Mathas in [4], have described a different cellular basis for the Ariki-Koike Hecke algrbras $\mathcal{H}$. We prefer this basis because it has many combinatorial and representation theoretic properties and it is more natural generalization from the cellular basis of group algebra of symmetric group. Let ζ be an invertible element of the field F, and $Q_1, Q_2, \cdots, Q_r$ arbitrary elements of F.

Definition 4.11([1]). The **Ariki-Koike** algebra $\mathscr{H} = \mathcal{H}_{\zeta,\mathcal{F}}$ is the unital associative F-algebra with generator $T_0, T_1, \cdots, T_{n-1}$ and relations

$$\begin{aligned}
(T_0 - Q_1)\cdots(T_0 - Q_r) &= 0 \\
(T_i - \zeta)(T_i + 1) &= 0 && \text{for } 1 \le i < n, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_iT_j &= T_jT_i && \text{for } 0 \le i < j-1 < n-1, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && \text{for } 1 \le i < n-1.
\end{aligned}$$

Remark 4.12([1]). Suppose a field F contains a primitive r^{th} root of unity ω and if $\zeta = 1, Q_s = \omega^s$ for $1 \le s \le r$, then $\mathscr{H} \cong F[(\mathbb{Z}/r\mathbb{Z})\wr S_n]$

Definition 4.13.

(i) A partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of non-negative integers such that $\lambda_1 \ge \lambda_2 \ge \cdots$ and $|\lambda| = \sum_{i\le 1} \lambda_i = n$.

(ii) A multi-partition of n is an ordered r-tuple of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)})$ with $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. We denote $\lambda \vdash n$ if λ is a multi-partition of n. Denote $I(n)$ be the set of all multi-partitions of n. and $M(\lambda)$ be the set of all standard tableau of shape λ.

Define e be the smallest positive integer such that $1 + \zeta + \zeta^2 + \cdots + \zeta^{(e-1)} = 0$ if no such positive integer exists we set $e = 0$.

Definition 4.14. A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is *e-restricted* if $\lambda_i - \lambda_{(i+1)} < e$ for $i \ge 1$, unless $e = 0$ in which case we stipulate that all partition are 0-restricted. A multipartiton $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(r)}) \vdash n$ is *e-restricted* if each partition $\lambda^{(s)}$ is e-restricted for $1 \le s \le r$.

Note that, if $\zeta = 1$, then e must be characteristic of underlying field F. Otherwise q is a primitive e^{th} root of unity.

Theorem 4.15([4]). *Let F is a any field which contains r^{th} root of unity ω and $*$ be the involution on $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$ which is defined by $(g_1, g_2, \cdots, g_l; \pi) \mapsto ((g^{-1}_{\pi^{-1}(1)}, \cdots, g^{-1}_{\pi^{-1}(l)}); \pi^{-1})$. for all $(g_1, g_2, \cdots, g_l; \pi) \in G \wr S_l$. If $\zeta = 1$ and $Q_k = \omega^k$ for $k = 1, 2, \cdots, r$. Then*

i) $\{C^{\lambda}_{\mathfrak{s},\mathfrak{t}} | \mathfrak{s}, \mathfrak{t} \in M(\lambda), \lambda \in I(n)\}$ *is a cellular basis for $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$.*

ii) *Suppose for each $\lambda \vdash n$, $\Delta(\lambda)$ is the cell module of $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$, then $\{\Delta(\lambda) | \lambda \in I(n)$ and λ is e-restricted $\}$ is a complete set of pairwise non-isomorphic irreducible $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$-modules.*

Next we are going to classify the representation of $P_{\mathbf{k}\cup\mathbf{k}'}(q, (\mathbb{Z}/r\mathbb{Z}))$ by using cellularity of $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$.

Theorem 4.16. *Let F be field of characteristic p (or 0) which contains a primitive r^{th} roots of unity. Then the standard modules of $P_{\mathbf{k}\cup\mathbf{k}'}(q, (\mathbb{Z}/r\mathbb{Z}))$ are $W(l, \lambda) = V_l \otimes v_l \otimes \Delta(\lambda)$ where $l \in \mathbf{k} \cup \{0\}$, $\lambda \in I(l)$, v_l is fixed non zero vector of V_l and $\Delta(\lambda)$ is standard modules of $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_l]$.*

Theorem 4.17. *Let F be field of characteristic p (or 0) which contains a primitive r^{th} roots of unity. If $q \neq 0$, then the non isomorphic simple $P_{\mathbf{k}\cup\mathbf{k}'}(q, (\mathbb{Z}/r\mathbb{Z}))$-modules are parameterized by $\{(m, \lambda) \mid 0 \leq m \leq k, \lambda \in I(m)$ and λ is p-restricted $\}$.*

Proof. From the above corollary and general theory of cellularity, the irreducible $P_{\mathbf{k}\cup\mathbf{k}'}(q, (\mathbb{Z}/r\mathbb{Z}))$-module are parameterized by $\{(l, \lambda) | \Phi_{(l,\lambda)} \neq 0\}$, where $\Phi_{(l,\lambda)}$is a bilinear form on $W(l, \lambda) \times W(l, \lambda)$ to $F[\mathbb{Z}/r\mathbb{Z} \wr S_l]$. Suppose $l \neq 0$. Then the bilinear form $\Phi_{(l,\lambda)} \neq 0$ if and only if the corresponding linear form Φ_λ for the cellular algebra $F[(\mathbb{Z}/r\mathbb{Z}) \wr S_n]$ is not zero. By the corollary, $\Phi_\lambda \neq 0$ if and only if λ is p-restricted. If $l = 0$, then $\Phi_{(l,\lambda)} \neq 0$ if and only if $q \neq 0$. Hence proved the corollary. □

The quasi-hereditary algebras are typically cellular algebras. This algebra were introduced by Cline, Parshall and Scott in [3] to study the highest-weight categories in the representation theory of Lie algebra.

Definition 4.18. Let A be an F-algebra. An ideal J in A is called a *hereditary ideal* if J is idempotents, $J(rad(A))J = 0$ and J is a projective left(or right) A-module. The algebra A is called quasi-hereditary provided there is a finite chain $0 = J_0 \subset \cdots \subset J_t \subset \cdots \subset J_m = A$ of ideal in A such that J_i/J_{j-1} is a hereditary ideal in A/J_{j-1} for all j.

Theorem 4.19. *Suppose F is field of characteristic zero which contains primitive r^{th} roots of unity. If $q \neq 0$, then $P_{\mathbf{k}\cup\mathbf{k}'}(q, (Z/rZ))$ is quasi-hereditary.*

Proof. Since F is field of characteristic zero which contains primitive r^{th} roots of unity. And by Theorem 4.17, for $0 \leq m \leq k, \lambda \in I(m)$ if and only if $\Phi_{(m,\lambda)} \neq 0$. The result follows from the Remark 3.10 of [5]. □

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