East Asian Math. J.
Vol. 32 (2016), No. 5, pp. 651–658
http://dx.doi.org/10.7858/eamj.2016.045

# GENERAL DECAY OF SOLUTIONS OF NONLINEAR VISCOELASTIC WAVE EQUATION

Kiyeon Shin and Sujin Kang

Abstract. In a bounded domain, we consider

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u d\tau + u_t = |u|^p u,$$

where $p > 0$ and $g$ is a nonnegative and decaying function. We establish a general decay result which is not necessarily of exponential or polynomial type.

## 1. Introduction

In this paper, we consider the following problem ;

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u\, d\tau + a(x)u_t = |u|^p u, & x \in \Omega,\ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $p > 0$ is a constant, $g$ is positive function satisfying some conditions to be satisfied later, $a(x) = 1$ and $\Omega$ is bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$.

The viscoelastic wave equation has been consider by many authors during the past decades. Cavalcanti et al. [3] studied for a function $a : \Omega \to \mathbb{R}^+$ which may be a null in a part of the domain $\Omega$. Under the conditions $a(x) \geq a_0 > 0$ on $\omega(\subset \Omega)$ which satisfies some geometry restrictions, the authors established results on exponential rate of decay with conditions $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, $t \geq 0$. Berrimi and Messaoudi [1] introduced a different functional which allows a weak condition than that of Cavalcanti et al. [3]. For other related works, we refer the readers [5], [6] and [8].

In the case of $a(x) = 0$, Berrimi and Messaoudi [2] showed, under the condition of $g'(t) \leq -\xi g(t)$ $(\xi > 0)$, that the solution is global and decays in a polynomial or an exponential function when the initial data is small enough.

Received May 27, 2016; Accepted August 27, 2016.
2010 *Mathematics Subject Classification.* 35L05, 35L15, 35L70, 35B25.
*Key words and phrases.* General decay, Relaxation, Viscoelastic.
This work was supported by a 2-Year Research Grant of Pusan National University.


(pISSN 1226-6973, eISSN 2287-2833)

Then Messaoudi [7] improved these results by establishing a general decay of energy which is similar to the relaxation function.

We show, in case of $a(x) = 1$, that the solution energy decays at a similar rate of decay of relaxation function, which is not necessarily decaying in a polynomial or exponential fashion.

This paper is organized as follows ; In Section 2, we present some notations and materials to be needed for our works. And, Section 3 contains the statements and proofs of our main results.

## 2. Preliminaries

In this section, we present some necessary materials in the proof of our main results. Also, for the sake of completeness we state, without a proof, the global existence result of Cavalcanti and Oquendo [4]. For the relaxation function $g$, we assume the followings ;

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing $C^1$–function satisfying
$g(0) > 0$, and $1 - \int_0^\infty g(s)\,ds = l > 0$.

(H2) There exists a positive differentiable function $\xi(t)$ satisfying
i) $g'(t) \le -\xi(t)g(t)$ for $t \ge 0$,
ii) $|\xi'(t)/\xi(t)| \le k$, $\xi(t) > 0$, and $\xi'(t) \le 0$ for $t > 0$.

(H3) For the nonlinear term, we assume
$p > 0$, for $n = 1, 2$ and $0 < p \le \frac{2}{n-2}$, for $n \ge 3$.

*Remark* 1. Since $\xi$ is nonincreasing, $\xi(t) \le \xi(0) := M$

We will use the embeddings $H_0^1 \hookrightarrow L^p$ for $p \le \frac{2n}{n-2}$ $(n \ge 3)$, $p \ge 2$ $(n = 1, 2)$ and $L^q \hookrightarrow L^p$ $(p < q)$ with the same embedding constant $C$.

We introduce the modified energy functional

$$E(t) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g * \nabla u)(t) - \frac{1}{p+2}\|u(t)\|_{p+2}^{p+2} + \frac{1}{2}\|u_t\|_2^2$$

where $(g * u)(t) = \int_0^t g(t-\tau)||u(t) - u(\tau)||_2^2 d\tau$.

**Lemma 2.1.** *We suppose that (H1) and (H2) hold and that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. If $u$ is the solution of (1), then the energy functional $E$ satisfies*

$$E'(t) = \frac{1}{2}(g' * \nabla u)(t) - \frac{1}{2}g(t)||\nabla u(t)||_2^2 - ||u_t||_2^2 \le \frac{1}{2}(g' * \nabla u)(t) \le 0,$$

*for almost all $t \in [0, T]$.*

*Proof.* Multiplying (1) by $u_t$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+2}\int_\Omega |u(t)|^{p+2} dx\right\}$$
$$- \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u_t(t) dx d\tau = -\int_\Omega |u_t|^2 dx \qquad (2)$$

For the last term on the left side of (2), we obtain

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\nabla u_t(t)dxd\tau \\
&= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2 dxd\tau\right] \\
&\quad +\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(\tau)\int_\Omega |\nabla u(t)|^2 dxd\tau\right] \\
&\quad +\frac{1}{2}\int_0^t g'(t-\tau)\int_\Omega |\nabla u(\tau)-\nabla u(t)|^2 dxd\tau \\
&\quad -\frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2 dxd\tau
\end{aligned} \tag{3}$$

Inserting (3) into (2), we obtain

$$E'(t) = \frac{1}{2}(g'*\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \|u_t\|_2^2 \le \frac{1}{2}(g'*\nabla u)(t) \le 0.$$

□

We set $J(t) = (1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + (g*\nabla u)(t) - \|u(t)\|_{p+2}^{p+2}$.

**Lemma 2.2.** *Suppose (H1), (H3), $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ hold such that $\beta = \frac{C^{p+2}}{l}(\frac{2(p+2)}{pl}E(0))^{\frac{p}{2}} < 1$ and $J(0) > 0$. Then $J(t) > 0$, $\forall t > 0$.*

*Proof.* See [7]. □

**Proposition 2.3.** *Suppose that the conditions of Lemma 2.2 are satisfied. Then the solution of (1) is global and bounded.*

*Proof.* See [2]. □

## 3. Decay of solution

In this section, we state and prove main result. For this purpose, we set $L(t) = E(t) + \varepsilon_1 I(t) + \varepsilon_2 K(t)$, where $\varepsilon_1$ and $\varepsilon_2$ are positive constants and $I(t) = \xi(t)\int_\Omega uu_t dx$, $K(t) = -\xi(t)\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx$.

**Lemma 3.1.** *For $u \in H_0^1(\Omega)$,*

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 dx \le (1-l)C_p^2(g*\nabla u)(t).$$

*Proof.* By applying the Cauchy-Schwartz inequality and Poincaré's constant $C_p$, we obtain Lemma 3.1. □

**Lemma 3.2.** *Suppose $u$ is the solution of (1). Then we have $\frac{1}{2}E(t) \le L(t) \le 2E(t)$.*

*Proof.* By Lemma 3.1, (H1), we obtain Lemma 3.2. by using $\xi(t) \le M$. ([7]) □

**Lemma 3.3.** *Suppose that (H1)–(H3) hold and that* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$*. If* $u$ *is the solution of* (1)*, then* $I(t)$ *satisfies*

$$\begin{aligned} I'(t) \quad \leq \quad & \xi(t)\left[1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right]\int_\Omega (u_t)^2dx + \frac{(3-l)(1-l)}{2l}\xi(t)(g*\nabla u)(t) \\ & -\frac{(3-l)l}{16}\xi(t)\int_\Omega |\nabla u|^2dx + \xi(t)\int_\Omega |u|^{p+2}dx. \end{aligned} \tag{4}$$

*Proof.* By using (1), (H1), (H2) and Young's inequality,

$$\begin{aligned} I'(t) \quad = \quad & \xi(t)\int_\Omega u_t^2dx + \xi'(t)\int_\Omega uu_tdx - \xi(t)\int_\Omega |\nabla u|^2dx + \xi(t)\int_\Omega |u|^{p+2}dx \\ & -\xi(t)\int_\Omega u\int_0^t g(t-\tau)\Delta u(x,t)d\tau dx - \xi\int_\Omega uu_tdx \\ \leq \quad & \xi(t)\left[1+\frac{1}{4\alpha_1}\left|\frac{\xi'(t)}{\xi(t)}\right| + \frac{1}{4\alpha_2}\right]\int_\Omega |u_t|^2dx + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g*\nabla u)(t) \\ & -\frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2\left|\frac{\xi'(t)}{\xi(t)}\right|\alpha_1C_p^2 - \alpha_2C_p^2\right]\xi\int_\Omega |\nabla u|^2dx \\ & +\xi(t)\int_\Omega |u|^{p+2}dx \\ \leq \quad & \xi(t)\left[1+\frac{k}{4\alpha_1}+\frac{1}{4\alpha_2}\right]\int_\Omega |u_t|^2dx + \frac{1}{2}(1+\frac{1}{\eta})(1-l)\xi(t)(g*\nabla u)(t) \\ & -\frac{1}{2}\left[1-(1+\eta)(1-l)^2 - (2k\alpha_1+\alpha_2)C_p^2\right]\xi\int_\Omega |\nabla u|^2dx \\ & +\xi(t)\int_\Omega |u|^{p+2}dx. \end{aligned} \tag{5}$$

By choosing $\eta = \dfrac{l}{2(1-l)}$, $\alpha_1 = \dfrac{l(3-l)}{16C_p^2k}$, $\alpha_2 = \dfrac{l(3-l)}{4C_p^2}$ and (5),
(since $0 < l < 1$, $\eta$, $\alpha_1$, $\alpha_2 > 0$)

$$\begin{aligned} I'(t) \quad \leq \quad & \xi(t)\left[1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right]\int_\Omega (u_t)^2dx + \frac{(3-l)(1-l)}{2l}\xi(t)(g*\nabla u)(t) \\ & -\frac{(3-l)l}{16}\xi(t)\int_\Omega |\nabla u|^2dx + \xi(t)\int_\Omega |u|^{p+2}dx. \end{aligned}$$

□

**Lemma 3.4.** *Suppose that (H1)–(H3) hold and that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If $u$ is the solution of (1), then $K(t)$ satisfies*

$$\begin{aligned} K'(t) \quad \leq \quad & \delta\xi(t)\left[1-2(1-l)^2+C^{2p+2}\left(\frac{2(p+2)E(0)}{pl}\right)^p\right]\int_\Omega |\nabla u_t|^2dx \\ & +C_\delta\xi(t)(g*\nabla u)(t)-\frac{g(0)}{4\delta}C_p^2\xi(t)(g'*\nabla u)(t) \\ & +\left[\delta(k+2)-\int_0^t g(s)ds\right]\xi(t)\int_\Omega |u_t|^2dx \end{aligned}$$

*for all $\delta > 0$, where $C_\delta$ is a constant depending on $\delta$.*

*Proof.* By (1),

$$\begin{aligned} K'(t) \quad = \quad & \xi(t)\left[K_1(t)-K_2(t)+K_3(t)-K_4(t)+K_5(t)-\int_0^t g(s)ds\int_\Omega u_t^2dx\right] \\ & +\xi'(t)K_6(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} K_1(t) &= \int_\Omega \nabla u(t)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx, \\ K_2(t) &= \int_\Omega \left(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)dx, \\ K_3(t) &= \int_\Omega u_t\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)dx, \\ K_4(t) &= \int_\Omega |u|^p u\left(\int_0^t g(t-\tau)(u(t)-u(\tau)d\tau)\right)dx, \\ K_5(t) &= \int_\Omega -u_t\left(\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau\right)dx, \\ K_6(t) &= \int_\Omega u_t\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)dx. \end{aligned}$$

For $K_1(t)$ – $K_3(t)$,

$$K_1(t) \leq \delta\int_\Omega |\nabla u(t)|^2dx+\frac{1-l}{4\delta}(g*\nabla u)(t), \tag{7}$$

$$\begin{aligned}
K_2(t) &\leq \delta \int_\Omega \left| \int_0^t g(t-\tau)\nabla u(\tau)\, d\tau \right|^2 dx \\
&\quad + \frac{1}{4\delta} \int_\Omega \left| \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau \right|^2 dx \\
&\leq \delta \int_\Omega \left( \int_0^t [g(t-\tau)(|\nabla u(t) - \nabla u(\tau)| + |\nabla u(t)|]\, d\tau \right)^2 dx \\
&\quad + \frac{1}{4\delta} \int_\Omega \left| \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau \right|^2 dx \\
&\leq (2\delta + \frac{1}{4\delta})(1-l)(g * \nabla u)(t) + 2\delta(1-l)^2 \int_\Omega |\nabla u|^2 dx, \qquad (8) \\
K_3(t) &\leq \delta \int_\Omega |u_t|^2 dx + \frac{1-l}{4\delta}(g * \nabla u)(t). \qquad (9)
\end{aligned}$$

And, by Lemma 2.2,

$$\begin{aligned}
K_4(t) &\leq \int_\Omega \left[ \delta |u|^{2p+2} + \frac{1}{4\delta} \left( \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau \right)^2 \right] dx \qquad (10) \\
&\leq \delta C^{2p+2} \left( \frac{2(p+2)}{pl} E(0) \right)^p \int_\Omega |\nabla u(t)|^2 dx + \frac{(1-l)C_p^2}{4\delta}(g * \nabla u)(t).
\end{aligned}$$

Also, for $K_5(t) - K_6(t)$, by using Lemma 3.1 and the Young's inequality,

$$\begin{aligned}
K_5(t) &\leq \delta \int_\Omega |u_t|^2 dx - \frac{g(0)C_p^2}{4\delta}(g' * \nabla u)(t), \qquad (11) \\
K_6(t) &\leq \delta \int_\Omega |u_t|^2 dx + \frac{C_p^2}{4\delta}(g * \nabla u)(t). \qquad (12)
\end{aligned}$$

When we put equations (7)–(12) into equation (6), we have the desired result. □

**Theorem 3.5.** *Suppose that (H1)–(H3) hold and that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If $u$ is the solution of (1), then for each $t_0 > 0$ there exist positive constants $K$ and $\lambda$ such that the solution of (1) satisfies $E(t) \leq K e^{-\lambda \int_{t_0}^t \xi(s) ds}$, for $t \geq t_0$.*

*Proof.* Since $g$ is positive, continuous and $g(0) > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds =: g_0 > 0, \qquad t \geq t_0. \qquad (13)$$

By the definition of $L(t)$, Lemma 2.2, Lemma 3.3–3.4 and (10),

$$\begin{aligned}
L'(t) \leq & -\left[\varepsilon_2\{g_0-\delta(k+2)\}-\varepsilon_1\{1+\frac{4C_p^2(k^2+1)}{l(3-l)}\}\right]\xi(t)\int_\Omega |u_t|^2dx \\
& -\left[\frac{\varepsilon_1 l(3-l)}{16}-\varepsilon_2\delta\{1+2(1-l)^2+C^{2p+2}(\frac{2(p+2)E(0)}{pl})^p\}\right]\xi(t)\|\nabla u\|_2^2 \\
& -\{\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_p^2M-\frac{(3-l)(1-l)}{2l}\varepsilon_1-\varepsilon_2K_\delta\}\xi(t)(g*\nabla u)(t) \\
& +\varepsilon_1\xi(t)\int_\Omega |u|^{p+2}dx. \qquad (14)
\end{aligned}$$

We choose $\delta$ sufficiently small so that $g_0-\delta(k+2)>\frac{1}{2}g_0$ and

$$\frac{16\delta}{l(3-l)}\left[1+2(1-l)^2+C^{2p+2}\left(\frac{2(p+2)E(0)}{pl}\right)^p\right]<\frac{g_0}{4\left(1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right)}.$$

Hence, for a fixed $\delta$, we may choose two positive constants $\varepsilon_1$ and $\varepsilon_2$ satisfying

$$\frac{g_0}{4\left(1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right)}\varepsilon_2<\varepsilon_1<\frac{g_0}{2\left(1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right)}\varepsilon_2,$$

$$\frac{1}{2}-\frac{\varepsilon_2 g(0)C_p^2M}{4\delta}>\frac{(3-l)(1-l)}{2l}\varepsilon_1+\varepsilon_2K_\delta.$$

Then we will make

$$\begin{aligned}
k_1 &= \varepsilon_2\{g_0-\delta(k+2)\}-\varepsilon_1\left[1+\frac{4C_p^2(k^2+1)}{l(3-l)}\right]>0, \\
k_2 &= \frac{\varepsilon_1 l(3-l)}{16}-\varepsilon_2\delta\left[1+2(1-l)^2+C^{2p+2}\left(\frac{2(p+2)E(0)}{pl}\right)^p\right]>0, \\
k_3 &= \frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_p^2M-\frac{(3-l)(1-l)}{2l}\varepsilon_1-\varepsilon_2K_\delta>0.
\end{aligned}$$

Thus, by (14) and Lemma 3.2,

$$\begin{aligned}
L'(t) &\leq -k_1\xi(t)\int_\Omega u_t^2\,dx-k_2\xi(t)\|\nabla u(t)\|_2^2-k_3\xi(t)(g*\nabla u)(t)+\varepsilon_1\xi(t)\int_\Omega |u|^{p+2}\,dx \\
&\leq -\beta_1\xi(t)\left[\frac{1}{2}(1-\int_0^t g(s)\,ds)\|\nabla u(t)\|_2^2+\frac{1}{2}(g*\nabla u)(t)\right. \\
&\qquad \left.-\frac{1}{p+2}\int_\Omega |u|^{p+2}\,dx+\frac{1}{2}\|u_t\|_2^2\right] \\
&\leq -\beta_1\xi(t)E(t)\leq -\frac{\beta_1}{2}\xi(t)L(t), \qquad (15)
\end{aligned}$$

for $\forall\, t\geq t_0$ and $\beta_1>0$. Integrating (15), we have

$$L(t)\leq L(t_0)e^{-\frac{\beta_1}{2}\int_{t_0}^t \xi(s)\,ds}.$$

Then, by Lemma 3.2,

$$E(t) \leq 2L(t_0)e^{-\frac{\beta_1}{2}\int_{t_0}^t \xi(s)\,ds} =: Ke^{-\lambda\int_{t_0}^t \xi(s)ds},$$

for $t \geq t_0$. This completes the proof. $\square$

## References

[1] S. Berrimi and S.A. Messaoudi, *Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping*, Electron. J. Differential Equations, **88** (2004), 1–10.

[2] S. Berrimi and S.A. Messaoudi, *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonl. Anal., **64** (2006), 2314–2331.

[3] M.M. Cavalcanti, V.M. Domingos Cavalcanti and J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Differential Equations, **44** (2002), 1–14.

[4] M.M. Cavalcanti and H.P. Oquendo, *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim., **42** no.4 (2003), 1310–1324.

[5] W.J. Liu, *General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms*, J. Math. Phys., **50** no.11 (2009), 113506.

[6] W.J. Liu and J. Yu, it On decay and blow-up of the solution for a viscoelastic wave equation with boundary damping and source terms, Nonl. Anal., **74** no.6 (2011), 2175–2190.

[7] S.A. Messaoudi, *General decay of the solution energy in a viscoelastic equation with a nonlinear source*, Nonl. Anal., **69** (2008), 2589–2598.

[8] S.T. Wu, *General decay and blow-up of solutions for a viscoelastic equation with a nonlinear boundary damping-source interactions*, Z. Angew. Math. Phys., **63** no.1 (2012), 65–106.

Kiyeon Shin
Department of Mathematics, Pusan National University, Pusan, 609-735, Korea
*E-mail address*: kyshin@pusan.ac.kr

Sujin Kang
Department of Nanomaterials Engineering, Pusan National University,
Pusan, 609-735, Korea
*E-mail address*: sjnisj@pusan.ac.kr