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# REGULAR FUNCTIONS FOR DIFFERENT KINDS OF CONJUGATIONS IN THE BICOMPLEX NUMBER FIELD

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Abstract. In this paper, using three types of conjugations in a bicomplex number filed $\mathcal{T}$, we provide some basic definitions of bicomplex number and definitions of regular functions for each differential operators. And we investigate the corresponding Cauchy-Riemann systems and the corresponding Cauchy theorems in $\mathcal{T}$ in Clifford analysis.

## 1. Introduction

In 1971, Naser [8] researched properties of hyperholomorphic functions. Naser [8] gave some results for harmonicity of hyperholomorphic functions, integral formulas, et cetera in a quaternion field $\mathcal{A}$ as a noncommutative extension of the complex numbers.

Rochon and Shapiro [9] studied about basic definitions and algebraic properties of bicomplex and hyperbolic numbers in 2004. Rochon and Shapiro [9] specified some moduli using three types of bicomplex conjugation and algebraic structures in Clifford algebras. In 2012, Luna-Elizarrarás and Shapiro [7] described how to define functions in bicomplex number field $\mathcal{T}$. And Luna-Elizarrarás and Shapiro [7] showed properties and generalizations of the theory of bicomplex functions.

Lim and Shon [4] have researched some properties of hyperholomorphic functions on $\mathcal{A} \times \mathcal{A}$. Lim and Shon [6] represented hyperholomorphy on octonionic functions in Clifford analysis in 2013. Lim and Shon [6] researched properties of hyperholomorphic functions and hyper-conjugate harmonic functions in octonion field. Lim and Shon [5] have studied some properties on regular functions and given applications of the extension problem in dual quaternion field.

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In 2013, Jung and Shon [1] showed several properties of hyperholomorphic functions valued ternary numbers. And Jung and Shon [1] investigated some properties and theorems on dual reduced quaternion in Clifford analysis.

Kang and Shon [2] defined and provided some properties of left regular functions in the quaternion field $\mathcal{A}$ and generalized quaternion in 2015. In the same year, Kim and Shon [3] researched a corresponding Cauchy-Riemann system and a Cauchy theorem on bicomplex numbers in Clifford analysis using a bicomplex differential operator.

## 2. Preliminaries

### 2.1. Notations of bicomplex numbers

We consider the following matrices:

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i$ is an usual complex number. Then we know the matrices satisfy the followings:

$$e_1e_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = e_2e_1,$$

and $ke_2 = e_2k$ $(k \in \mathbb{R})$. Then we defined a bicomplex number $z$ as an extension of the complex number,

$$z := z_0 + e_2z_1,$$

where $z_0$ and $z_1$ are usual complex numbers. Putting $z_0 = x_0 + e_1x_1, \ z_1 = x_2 + e_1x_3$ with $x_i \in \mathbb{R}$ $(i = 0, 1, 2, 3)$, we have

$$\begin{aligned} z &= (x_0 + e_1x_1) + e_2(x_2 + e_1x_3) \\ &= x_0 + e_1x_1 + e_2x_2 + e_2e_1x_3. \end{aligned}$$

We set $e_3 := e_1e_2 = e_2e_1$. And we denote $z$ as follows:

$$z = x_0 + e_1x_1 + e_2x_2 + e_3x_3.$$

Let $\mathcal{T}$ be a bicomplex number field. Then the field $\mathcal{T}$ is a four dimensional commutative field over $\mathbb{R}$,

$$\mathcal{T} = \{z_0 + e_2z_1 \mid z_0, z_1 \in \mathbb{C}\} = \{z \mid z = \sum_0^3 e_jx_j, \ x_j \in \mathbb{R}\},$$

identified with $\mathbb{R}^4$ and $\mathbb{C}^2$. The field $\mathcal{T}$ is generated by $e_0, \ e_1, \ e_2$ and $e_3$ satisfying the followings:

$$\begin{aligned} &e_0 = id., \ e_1^2 = e_2^2 = -1, \ e_3^2 = 1, \\ &e_1e_3 = e_3e_1 = -e_2, \\ &e_2e_3 = e_3e_2 = -e_1. \end{aligned}$$

Let $z$ and $w$ be bicomplex numbers and each $z$ and $w$ are denoted by $z = z_0 + e_2 z_1 = e_0 x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3,\ w = z_2 + e_2 z_3 = e_0 y_0 + e_1 y_1 + e_2 y_2 + e_3 y_3$ for $z_t \in \mathbb{C}$ $(t = 0,1,2,3)$ and $x_t,\ y_t \in \mathbb{R}$ $(t = 0,1,2,3)$. Then the field $\mathcal{T}$ is closed by the addition and the multiplication. By direct computations,

$$z + w = e_0(x_0 + y_0) + e_1(x_1 + y_1) + e_2(x_2 + y_2) + e_3(x_3 + y_3) = w + z,$$

$$\begin{aligned} zw = e_0(x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3) + e_1(x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2) \\ +e_2(x_0y_2 - x_1y_3 + x_2y_0 - x_3y_1) + e_3(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0) = wz. \end{aligned}$$

Thus, $z + w = w + z \in \mathcal{T}$ and $zw = wz \in \mathcal{T}$.

We know that there are several conjugations in the field $\mathcal{T}$ with respect to $e_1,\ e_2$ or both of them.

## 2.2. $z^{\#}$: The 1st kind of conjugation with respect to $e_2$

The 1st kind of conjugation $z^{\#}$ is determined by the formula:

$$z^{\#} = (z_0 + e_2 z_1)^{\#} := z_0 - e_2 z_1 \text{ for all } z_0, z_1 \in \mathbb{C}.$$

*Remark* 1. (Properties of the 1st bicomplex conjugation $z^{\#}$)
For any $z,\ w \in \mathcal{T}$, the following properties are satisfied:

(a) $(z + w)^{\#} = z^{\#} + w^{\#}$.
(b) $(z - w)^{\#} = z^{\#} - w^{\#}$.
(c) $(z \cdot w)^{\#} = z^{\#} \cdot w^{\#}$.
(d) $(z^{\#})^{\#} = z$.
(e) The absolute value by using the 1st kind of conjugation $z^{\#}$ is defined by

$$\begin{aligned} |z|_{\#}^2 := z \cdot z^{\#} &= (z_0 + e_2 z_1)(z_0 - e_2 z_1) \\ &= z_0^2 + z_1^2. \end{aligned}$$

(f) The bicomplex number $z$ has the unique inverse for $z^{\#}$:

$$z^{-1} = \frac{z^{\#}}{|z|_{\#}^2} \ (|z|_{\#} \neq 0).$$

In the field $\mathcal{T}$, we consider three kinds of conjugation of bicomplex number $z$ with respect to $e_1$, $e_2$ or both of them. The 1st kind of conjugation $z^{\#}$ is a conjugation of $z$ with respect to $e_2$. Then we know other conjugations of $z$ as follows:

## 2.3. The 2nd and 3rd kinds of conjugation

The 2nd kind of conjugation $z^*$ with respect to $e_1$ is determined by the formula:

$$z^* = (z_0 + e_2 z_1)^* := \bar{z}_0 + e_2 \bar{z}_1 \text{ for all } z_0, z_1 \in \mathbb{C}.$$

As the composition for $e_1$ and $e_2$ of the above two conjugations $z^{\#}$ and $z^*$, the 3rd kind of conjugation $z^\dagger$ is determined by the formula:

$$z^\dagger = (z_0 + e_2 z_1)^\dagger := ((z_0 + e_2 z_1)^*)^{\#} = ((z_0 + e_2 z_1)^{\#})^* = \bar{z}_0 + e_2 \bar{z}_1$$

for all $z_0,\ z_1 \in \mathbb{C}$.

*Remark* 2. (Properties of the 2nd and 3rd bicomplex conjugations)
For any $z,\ w \in \mathcal{T}$, the following properties for $z^*$ ($z^\dagger$) are satisfied:

(a) $(z+w)^* = z^* + w^*$ ($(z+w)^\dagger = z^\dagger + w^\dagger$).
(b) $(z-w)^* = z^* - w^*$ ($(z-w)^\dagger = z^\dagger - w^\dagger$).
(c) $(z \cdot w)^* = z^* \cdot w^*$ ($(z \cdot w)^\dagger = z^\dagger \cdot w^\dagger$).
(d) $(z^*)^* = z$ ($(z^\dagger)^\dagger = z$).
(e) The absolute values by using the 2nd and 3rd bicomplex conjugation $z^*$ and $z^\dagger$ are defined by

$$\begin{aligned} |z|_*^2 := z \cdot z^* &= (z_0 + e_2 z_1)(\bar{z}_0 + e_2 \bar{z}_1) \\ &= z_0 \bar{z}_0 + e_2(z_0 \bar{z}_1 + z_1 \bar{z}_0) - z_1 \bar{z}_1 \\ &= |z_0|^2 - |z_1|^2 + e_2\{2\mathrm{Re}(z_0 \bar{z}_1)\} \end{aligned}$$

and

$$\begin{aligned} |z|_\dagger^2 := z \cdot z^\dagger &= (z_0 + e_2 z_1)(\bar{z}_0 + e_2 \bar{z}_1) \\ &= z_0 \bar{z}_0 + e_2(z_0 \bar{z}_1 + z_1 \bar{z}_0) - z_1 \bar{z}_1 \\ &= |z_0|^2 + |z_1|^2 - e_2\{2\mathrm{Im}(z_0 \bar{z}_1)\}. \end{aligned}$$

(f) The bicomplex number $z$ has the unique inverse for $z^\dagger$:

$$z^{-1} = \frac{z^\dagger}{|z|_\dagger^2} \ (|z|_\dagger \neq 0).$$

And the bicomplex number $z$ has the unique inverse for $z^*$:

$$z^{-1} = \frac{z^*}{|z|_*^2} \ (|z|_* \neq 0).$$

Let $\Omega$ be a bounded open set in $\mathbb{C}^2$. A bicomplex function $f : \Omega \to \mathcal{T}$ is defined by

$$\begin{aligned} f(z) &= u_0(x_0, x_1, x_2, x_3) + e_1 u_1(x_0, x_1, x_2, x_3) \\ &\quad + e_2 u_2(x_0, x_1, x_2, x_3) + e_3 u_3(x_0, x_1, x_2, x_3) \\ &= f_0(z_0, z_1) + e_2 f_1(z_0, z_1), \end{aligned}$$

where $u_t$ $(t = 0, 1, 2, 3)$ are real functions and $f_0, f_1$ are complex functions of two complex variables $z_0$ and $z_1$.

We consider bicomplex differential operators as follows:

$$D := \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial z_0} - e_2 \frac{\partial}{\partial z_1}\right),$$
$$D^{\#} = \frac{\partial}{\partial z^{\#}} = \frac{1}{2}\left(\frac{\partial}{\partial z_0} + e_2 \frac{\partial}{\partial z_1}\right),$$

where $\dfrac{\partial}{\partial z_t}$ $(t = 0, 1)$ is an usual complex differential operator. In 2015, Kim and Shon [3] have shown the corresponding Cauchy-Riemann system and the Cauchy theorem in the field $\mathcal{T}$. Now, we investigated new bicomplex differential operators and found the corresponding Cauchy-Riemann systems and the Cauchy theorems for each differential operators in next sections.

## 3. The corresponding Cauchy-Riemann systems

Let $\Omega$ be a bounded open set in $\mathbb{C}^2$. A function $f(z) = f_0(z) + e_2 f_1(z)$ is said to be a regular function in $\Omega$ if

(a) $f \in C^1(\Omega)$,
(b) $D^{\#} f = 0$ in $\Omega$.

We obtain a result by direct computation of above equation (b) as follows:

$$D^{\#} f = \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1}\right) + e_2\left(\frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1}\right)\right\} = 0.$$

Thus, the equation (b) is equivalent to the following system:

$$\frac{\partial f_0}{\partial z_0} = \frac{\partial f_1}{\partial z_1} \quad \text{and} \quad \frac{\partial f_1}{\partial z_0} = -\frac{\partial f_0}{\partial z_1}. \tag{1}$$

The system (1) is called a corresponding Cauchy-Riemmann system in $\mathcal{T}$.

We know three kinds of conjugations for bicomplex number $z$. Similarly, we obtain other differential operators respected to each conjugation. Now, we consider the following bicomplex differential operators:

$$D^* = \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial \bar{z_0}} - e_2 \frac{\partial}{\partial \bar{z_1}}\right),$$
$$D^\dagger = \frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial \bar{z_0}} + e_2 \frac{\partial}{\partial \bar{z_1}}\right).$$

Since $D^*$ is the conjugation of $D$ respected to $e_1$, we call this operator as the 2nd conjugation of $D$. Similarly, the operator $D^\dagger$ is called as the 3rd conjugation of $D$.

**Definition 1.** (The regular function by the 2nd conjugation)
Let $\Omega$ be a bounded open set in $\mathbb{C}^2$. A function $f$ is said to be the 2nd regular function in $\Omega$ if

(a) $f \in C^1(\Omega)$,
(b) $D^* f = 0$ in $\Omega$.

By simple computations, we show the result of the equation (b) of Definition 1 as follows.

$$\begin{aligned} D^* f &= \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_0} - e_2 \frac{\partial}{\partial \bar{z}_1}\right)(f_0 + e_2 f_1) \\ &= \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial \bar{z}_0} + \frac{\partial f_1}{\partial \bar{z}_1}\right) + e_2\left(\frac{\partial f_1}{\partial \bar{z}_0} - \frac{\partial f_0}{\partial \bar{z}_1}\right)\right\} = 0. \end{aligned}$$

Thus, $D^* f = 0$ is equivalent to the following system:

$$\frac{\partial f_0}{\partial \bar{z}_0} = -\frac{\partial f_1}{\partial \bar{z}_1} \quad \text{and} \quad \frac{\partial f_1}{\partial \bar{z}_0} = \frac{\partial f_0}{\partial \bar{z}_1}. \tag{2}$$

Similarly in the case of the regular function, the system (2) is called by the 2nd corresponding Cauchy-Riemann system. For convenience, we call the differential operator $D^{\#}$ as the 1st conjugation of $D$. So, a bicomplex function $f$ satisfying the corresponding Cauchy-Riemann system (1) is regarded as the 1st regular function in $\Omega$. And that system (1) is called as the 1st corresponding Cauchy-Riemann system.

**Definition 2.** (The regular function by the 3rd conjugation)
Let $\Omega$ be a bounded open set in $\mathbb{C}^2$. A function $f$ is said to be the 3rd regular function in $\Omega$ if

(a) $f \in C^1(\Omega)$,
(b) $D^\dagger f = 0$ in $\Omega$.

Similarly, in the cases of the 1st regular function and the 2nd regular function,

$$\begin{aligned} D^\dagger f &= \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}_0} + e_2 \frac{\partial}{\partial \bar{z}_1}\right)(f_0 + e_2 f_1) \\ &= \frac{1}{2}\left\{\left(\frac{\partial f_0}{\partial \bar{z}_0} - \frac{\partial f_1}{\partial \bar{z}_1}\right) + e_2\left(\frac{\partial f_1}{\partial \bar{z}_0} + \frac{\partial f_0}{\partial \bar{z}_1}\right)\right\} = 0. \end{aligned}$$

The equation $D^\dagger f = 0$ is equivalent to

$$\frac{\partial f_0}{\partial \bar{z}_0} = \frac{\partial f_1}{\partial \bar{z}_1} \quad \text{and} \quad \frac{\partial f_1}{\partial \bar{z}_0} = -\frac{\partial f_0}{\partial \bar{z}_1}. \tag{3}$$

This system (3) is called the 3rd corresponding Cauchy-Riemann system in the bicomplex number field.

In next section, we show some properties of the each regular function (the 1st, 2nd and 3rd) in bicomplex number.

## 4. The corresponding Cauchy theorems

Kim and Shon [3] have shown the following theorem.

**Theorem 4.1.** *(The 1st corresponding Cauchy theorem)*
*Let $\Omega$ be a open set in $\mathbb{C}^2$. If $f = f_0 + e_2 f_1$ is a 1st regular function in $\Omega$, then for a domain $D \subset \Omega$ with smooth boundary $\partial D$,*

$$\int_{\partial D} \omega^{\#} f = 0,$$

*where $\omega^{\#} = dz_0 \wedge d\bar{z_0} \wedge d\bar{z_1} + e_2 dz_1 \wedge d\bar{z_0} \wedge d\bar{z_1}$.*

Now, we call $\omega^{\#}$ as a kernel for the 1st corresponding Cauchy theorem. Since we know the other types of the conjugation of differential operator, we find kernels of the 2nd and 3rd corresponding Cauchy theorem. So we obtain the following theorems.

**Theorem 4.2.** *(The 2nd corresponding Cauchy theorem)*
*Let $\Omega$ be a open set in $\mathbb{C}^2$. If $f = f_0 + e_2 f_1$ is the 2nd regular function in $\Omega$, then for a domain $D \subset \Omega$ with smooth boundary $\partial D$,*

$$\int_{\partial D} \omega^{*} f = 0,$$

*where $\omega^{*} = dz_0 \wedge dz_1 \wedge d\bar{z_1} + e_2 dz_0 \wedge dz_1 \wedge d\bar{z_0}$.*

*Proof.* Similarly in a case of the 1st corresponding Cauchy theorem, we obtain the following by the simple multiplication:

$$\begin{aligned} \omega^* f &= (dz_0 \wedge dz_1 \wedge d\bar{z_1} + e_2 dz_0 \wedge dz_1 \wedge d\bar{z_0})(f_0 + e_2 f_1) \\ &= f_0 dz_0 \wedge dz_1 \wedge d\bar{z_1} - f_1 dz_0 \wedge dz_1 \wedge d\bar{z_0} \\ &\quad + e_2 (f_0 dz_0 \wedge dz_1 \wedge d\bar{z_0} + f_1 dz_0 \wedge dz_1 \wedge d\bar{z_1}). \end{aligned}$$

Then,

$$\begin{aligned} d(\omega^* f) &= \left( \frac{\partial}{\partial z_0} dz_0 + \frac{\partial}{\partial z_1} dz_1 + \frac{\partial}{\partial \bar{z_0}} d\bar{z_0} + \frac{\partial}{\partial \bar{z_1}} d\bar{z_1} \right) \cdot \\ &\quad \{ f_0 dz_0 \wedge dz_1 \wedge d\bar{z_1} - f_1 dz_0 \wedge dz_1 \wedge d\bar{z_0} \\ &\quad + e_2 \left( f_0 dz_0 \wedge dz_1 \wedge d\bar{z_0} + f_1 dz_0 \wedge dz_1 \wedge d\bar{z_1} \right) \} \\ &= \left( \frac{\partial f_0}{\partial \bar{z_0}} + \frac{\partial f_1}{\partial \bar{z_1}} \right) dz_0 \wedge dz_1 \wedge d\bar{z_0} \wedge d\bar{z_1} \\ &\quad + e_2 \left( \frac{\partial f_1}{\partial \bar{z_0}} - \frac{\partial f_0}{\partial \bar{z_1}} \right) dz_0 \wedge dz_1 \wedge d\bar{z_0} \wedge d\bar{z_1} \\ &= \left\{ - \left( \frac{\partial f_1}{\partial z_0} + \frac{\partial f_0}{\partial z_1} \right) + e_2 \left( \frac{\partial f_0}{\partial z_0} - \frac{\partial f_1}{\partial z_1} \right) \right\} dz_0 \wedge dz_1 \wedge d\bar{z_0} \wedge d\bar{z_1}. \end{aligned}$$

Since the function $f$ is the 2nd regular function in $\Omega$, $f$ satisfies (2). Thus,

$$d(\omega^* f) = 0.$$

By Stokes theorem,

$$\int_D d(\omega^* f) = \int_{\partial D} \omega^* f = 0.$$

□

**Theorem 4.3.** *(The 3rd corresponding Cauchy theorem)*
*Let $\Omega$ be a open set in $\mathbb{C}^2$. If a function $f = f_0 + e_2 f_1$ is the 3rd regular in $\Omega$, then for a domain $D \subset \Omega$ with smooth boundary $\partial D$,*

$$\int_{\partial D} \omega^\dagger f = 0,$$

*where $\omega^\dagger = dz_0 \wedge dz_1 \wedge d\bar{z_0} + e_2 dz_0 \wedge dz_1 \wedge d\bar{z_1}$.*

*Proof.* By the direct computation,

$$\begin{aligned} \omega^\dagger f &= (dz_0 \wedge dz_1 \wedge d\bar{z_0} + e_2 dz_0 \wedge dz_1 \wedge d\bar{z_1})(f_0 + e_2 f_1) \\ &= f_0 dz_0 \wedge dz_1 \wedge d\bar{z_0} - f_1 dz_0 \wedge dz_1 \wedge d\bar{z_1} \\ &\quad + e_2(f_0 dz_0 \wedge dz_1 \wedge d\bar{z_1} + f_1 dz_0 \wedge dz_1 \wedge d\bar{z_0}). \end{aligned}$$

And we have

$$\begin{aligned} d(\omega^\dagger f) &= \left( \frac{\partial}{\partial z_0} dz_0 + \frac{\partial}{\partial z_1} dz_1 + \frac{\partial}{\partial \bar{z_0}} d\bar{z_0} + \frac{\partial}{\partial \bar{z_1}} d\bar{z_1} \right) \cdot \\ &\quad \{ f_0 dz_0 \wedge dz_1 \wedge d\bar{z_0} - f_1 dz_0 \wedge dz_1 \wedge d\bar{z_1} \\ &\quad + e_2(f_0 dz_0 \wedge dz_1 \wedge d\bar{z_1} + f_1 dz_0 \wedge dz_1 \wedge d\bar{z_0}) \} \\ &= \left\{ -\left( \frac{\partial f_1}{\partial \bar{z_0}} + \frac{\partial f_0}{\partial \bar{z_1}} \right) + e_2 \left( \frac{\partial f_0}{\partial \bar{z_0}} - \frac{\partial f_1}{\partial \bar{z_1}} \right) \right\} dz_0 \wedge dz_1 \wedge d\bar{z_0} \wedge d\bar{z_1}. \end{aligned}$$

Since the function $f$ is the 3rd regular function in $\Omega$, $f$ satisfies (3). Then,

$$d(\omega^\dagger f) = 0.$$

By Stokes theorem,

$$\int_D d(\omega^\dagger f) = \int_{\partial D} \omega^\dagger f = 0.$$

The proof is done. □

We call $\omega^*$ (and $\omega^\dagger$) the kernel for the 1st (and 3rd) corresponding Cauchy theorem likewise the 2nd corresponding Cauchy theorem.

## References

[1] H. S. Jung and K. H. Shon, *Functions and Differential Operators in The Dual Reduced Quaternion Field*, East Asian Math. J. **29** (2013), 293-302.

[2] H. U. Kang and K. H. Shon, *Relations of L-regular Functions on Quaternions in Clifford Analysis*, East Asian Math. J. **20** (2013), 667-675.

[3] J. E. Kim and K. H. Shon, *Properties of Regular Functions with Values in Bicomplex Numbers*, Bull. Korean Math. Soc. **Preprint**.

[4] S. J. Lim and K. H. Shon, *Dual Quaternion Functions and Its Applications*, East Asian Math. J. **28** (2012), 553-559.

[5] ______, *Dual Quaternion Functions and Its Applications*, J. of Applied Mathematics **Art. ID 583813** (2013), 6 pages.

[6] ______, *Hyperholomorphic Functions and Hyper-conjugate Harmonic Functions of Octonion Variables*, J. of Inequalities and Applications **77** (2013), 8 pages.

[7] M. E. Luna-Elizarrarás and M. Shapiro, *Bicomplex Numbers and their Elementary Functions*, CUBO A Mathematical J. **14** (2012), 61-80.

[8] M. Naser, *Hyperholomorphic functions*, Siberian Math. J. **12** (1971), 959-968.

[9] D. Rochon and M. Shapiro, *On Algebraic Properties of Bicomplex and Hyperbolic Numbers*, Anal. Univ. Oradea (2004), 28 pages.

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