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A NOTE ON THE APPROXIMATE SOLUTIONS TO STOCHASTIC DIFFERENTIAL DELAY EQUATION†

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Abstract. The main aim of this paper is to discuss the difference between the Euler-Maruyama's approximate solutions and the accurate solution to stochastic differential delay equation. To make the theory more understandable, we impose the non-uniform Lipschitz condition and weakened linear growth condition. Furthermore, we give the pth moment continuous of the approximate solution for the delay equation.

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1. Introduction

In the study of stochastic system, a more realistic model would include some of the past states of the system. Stochastic functional differential equation gives a mathematical formulation for such system. In addition, in the study of the stochastic differential delay equations, If there is not any explicit solution then how we can obtain the approximate solution is a very important matter. One of the special but important class of stochastic functional differential equations is the stochastic differential delay equations. In 2016, Kim [5] considered the following stochastic differential delay equation

$$dx(t) = F(x(t), x(t-\tau), t)dt + G(x(t), x(t-\tau), t)dB(t) \tag{1}$$

on $t \in [t_0, T]$ and defined the Euler-Maruyama approximation to the delay equation (1) as follows: For each integer $n \geq 1/\tau$, define $x_n(t)$ on $[-\tau, T]$ by

$$x_n(t_0 + \theta) = \xi(\theta) \quad \text{for} -\tau \leq \theta \leq 0$$

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and

$$\begin{aligned} x_n(t) &= x_n(t_0+k/n) \\ &\quad + \int_{t_0+k/n}^{t} F(x_n(t_0+k/n), x_n(t_0+(k-1)/n), s)ds \\ &\quad + \int_{t_0+k/n}^{t} G(x_n(t_0+k/n), x_n(t_0+(k-1)/n), s)dB(s) \end{aligned} \tag{2}$$

for $t_0+k/n < t \le [t_0+(k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$.

In [5], by employing non-uniform Lipschitz condition and weakened linear growth condition, Kim established the following results for the second moment to stochastic differential delay equation. The following theorem shows that the Euler-Maruyama sequence (2) converges to the unique solution of the equation (1) and gives an estimate for difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 1.1 ([5]). *Assume that there exists a constant K and a concave function κ such that*

(i) (non-uniform Lipschitz condition) For all $t \in [t_0, T]$, and all $x, y, \overline{x}, \overline{y} \in R^d$

$$|F(x,y,t) - F(\overline{x},\overline{y},t)|^2 \vee |G(x,y,t) - G(\overline{x},\overline{y},t)|^2 \le \kappa(|x-\overline{x}|^2 + |y-\overline{y}|^2);$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$ and $\int_{0+} du/\kappa(u) = \infty$.

(ii) (weakened linear growth condition) there is a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(0,0,t)|^2 \vee |G(0,0,t)|^2 \le K.$$

Also, assume that $\delta(\cdot)$ is Lipschitz continuous, that is there is a positive constant α such that

$$|\delta(t) - \delta(s)|^2 \le \alpha(t-s)$$

if $t_0 \le s < t \le T$. Then, for every $n > 1+\alpha$, the difference between the Euler-Maruyama approximate solution $x_n(t)$ defined by (2) and the accurate solution $x(t)$ of equation (1) can be estimate as

$$E\Big(\sup_{t_0 \le t \le T} |x(t) - x_n(t)|^2\Big) \le \Big[2\alpha_3\gamma + 4\alpha_3(T-t_0+4)(\widehat{J}_1 + \widehat{J}_3)\Big] e^{8\alpha_3\gamma}$$

where $\gamma = (T-t_0)(T-t_0+4)$,

$$\widehat{J}_1 = C_3[T-t_0]\frac{1}{n}, \quad \widehat{J}_3 = [C_3(T-t_0) + 2(\beta \vee C_3)\tau]\frac{1+\alpha}{n},$$

and C_3 is defined in [5].

For results related to the stochastic differential equation, see [1]-[12], and references therein for details. By using the non-uniform Lipschitz condition and weakened growth condition, Kim [5] studied the difference between the approximate and the accurate solution to stochastic differential delay equation

(SDDEs). Motivated by the results, we established some exponential estimate for the pth moment and estimated on difference between the approximate solutions and the unique solution to stochastic differential delay equation that can be obtained from the conditions. When we try to carry over this procedure to the this delay equation, we used the Euler-Maruyama sequence approximation procedure.

2. Preliminary

Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets), where $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$. And let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$.

Also, let $C([-\tau, 0]; R^d)$ denote the family of continuous R^d-valued functions φ defined on $[-\tau, 0]$ with norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi|$.

In the result [9], they considered the following non-Lipschitz condition and non-linear growth condition:

(iii) (Non-Lipschitz condition) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \leq \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$ and $\int_{0+} du/\kappa(u) = \infty$.

(iv) (Non-linear growth condition) $f(0, t), g(0, t) \in L^2$ and for all $t \in [t_0, T]$, it follows that

$$|f(0, t)|^2 \vee |g(0, t)|^2 \leq K,$$

where $K > 0$ is a constant. Moreover, the authors established the following results for d-dimensional stochastic functional differential equation.

Theorem 2.1 ([9]). *Assume that the non-Lipschitz condition and non-linear growth condition hold. Then, there exists a unique solution to the equation*

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \quad \text{on } t_0 \leq t \leq T, \tag{3}$$

with initial data.

For more results related to some stochastic differential delay equation, see [2], [3], [6] - [12], and references therein for details.

On the other hand, we consider a special class of stochastic functional differential delay equation

$$dx(t) = F(x(t), x(t-\tau), t)dt + G(x(t), x(t-\tau), t)dB(t) \tag{4}$$

on $t \in [t_0, T]$, where $F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$ are Borel measurable. If we define

$$f(\varphi, t) = F(\varphi(0), \varphi(-\tau), t) \quad \text{and} \quad g(\varphi, t) = G(\varphi(0), \varphi(-\tau), t)$$

for $(\varphi, t) \in C([-\tau, 0]; R^d) \times [t_0, T]$, then equation (4) can be written as the equation (3). So we can apply the existence-and-uniqueness theorem established in the previous theorem to the delay equation (4).

Let us now prepare a few lemmas in order to show the main result.

Lemma 2.2 (Moment inequality, [7]). *If* $p \geq 2, g \in \mathcal{M}^2([0,T]; R^{d\times m})$ *such that* $E\int_0^T |g(s)|^p \,\mathrm{d}s < \infty$, *then*

$$E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\, ds.$$

In particular, $E|\int_0^T g(s)\,\mathrm{d}B(s)|^2 = E\int_0^T |g(s)|^2\, ds$ *when* $p = 2$.

Lemma 2.3 (Moment inequality, [7]). *Under the same assumptions as Lemma 2.2, we have*

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\, ds.$$

3. Approximate solutions

Let us begin with the discussion of the following stochastic differential delay equation

$$dx(t) = F(x(t), x(t-\tau), t)dt + G(x(t), x(t-\tau), t)dB(t) \tag{5}$$

on $t \in [t_0, T]$, where $F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$ are Borel measurable. Moreover, the initial value is followed:

$$\begin{aligned} &x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \quad \text{is an}\, \mathcal{F}_{t_0} - \text{measurable} \\ &BC([-\tau, 0]; R^d) - \text{value random variable such that}\, \xi \in \mathcal{M}^2([-\tau, 0]; R^d). \end{aligned} \tag{6}$$

Moreover, we impose the non-uniform Lipschitz condition and weakened linear growth condition:

(v) (Non-uniform Lipschitz condition) For all $t \in [t_0, T]$, and all $x, y, \overline{x}, \overline{y} \in R^d$

$$|F(x, y, t) - F(\overline{x}, \overline{y}, t)|^2 \vee |G(x, y, t) - G(\overline{x}, \overline{y}, t)|^2 \leq \kappa(|x - \overline{x}|^2 + |y - \overline{y}|^2) \tag{7}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$ and $\int_{0+} du/\kappa(u) = \infty$.

(vi) (Weakened linear growth condition) There is a $K > 0$ such that for all $(x, y, t) \in R^d \times R^d \times [t_0, T]$,

$$|F(0, 0, t)|^2 \vee |G(0, 0, t)|^2 \leq K. \tag{8}$$

Let us now turn to the Euler-Maruyama approximation procedure. Consider the stochastic differential delay equation (5) with initial data (6). It is in this spirit we define the Euler-Maruyama approximation procedure as follows: For each integer $n \geq 1/\tau$, define $x_n(t)$ on $[t_0 - \tau, T]$ by

$$x_n(t_0 + \theta) = \xi(\theta) \quad \text{for} - \tau \leq \theta \leq 0$$

and

$$\begin{aligned} x_n(t) &= x_n(t_0+k/n) \\ &\quad + \int_{t_0+k/n}^t F(x_n(t_0+k/n), x_n(t_0+(k-1)/n), s)ds \\ &\quad + \int_{t_0+k/n}^t G(x_n(t_0+k/n), x_n(t_0+(k-1)/n), s)dB(s) \end{aligned} \tag{9}$$

for $t_0+k/n < t \le [t_0+(k+1)/n] \wedge T$, $k = 0,1,2,\cdots$. Moreover, if we define $\widehat{x}_n(t_0) = x_n(t_0)$, $\widetilde{x}_n(t_0) = x_n(t_0-1/n)$,

$$\widehat{x}_n(t) = x_n(t_0+k/n), \quad \text{and} \quad \widetilde{x}_n(t) = x_n(t_0+(k-1)/n)$$

for $t_0+k/n < t \le [t_0+(k+1)/n] \wedge T$, $k = 0,1,2,\cdots$, it then follows from (9) that

$$x_n(t) = \xi(0) + \int_{t_0}^t F(\widehat{x}_n(s), \widetilde{x}_n(s), s)ds + \int_{t_0}^t G(\widehat{x}_n(s), \widetilde{x}_n(s), s)dB(s). \tag{10}$$

From now on, $x_n(t)$ means the Euler-Maruyama approximation (9). The following lemma shows that the Euler-Maruyama approximation sequence is bounded in L^p.

Lemma 3.1. *Let (7) and (8) hold and $p \ge 2$. Then, for all $n \ge 1/\tau$, we have*

$$\begin{aligned} &E\Big(\sup_{t_0-\tau\le s\le t} |x_n(s)|^p\Big) \\ &\le C_k := \big((3^{p-1}+1)E\|\xi\|^p + C_1C_2\big)\exp(2^{2p-1}3^{p-1}\alpha^{p/2}C_2(T-t_0)^{-1}) \end{aligned} \tag{11}$$

for all $t \ge t_0$, where $C_1 = 6^{p-1}(2^{(p-2)/2}\alpha^{p/2} + K^{p/2})$ and $C_2 = (T-t_0)^p + [(p^3/2(p-1))^{p/2}](T-t_0)^{p/2}$.

Proof. Fix $n \ge 1$ arbitrarily. It is easy to see from the equation (10) that

$$\begin{aligned} |x_n(s)|^p &\le 3^{p-1}|\xi(0)|^p + 3^{p-1}\Big|\int_{t_0}^t F(\widehat{x}_n(s), \widetilde{x}_n(s), s)ds\Big|^p \\ &\quad + 3^{p-1}\Big|\int_{t_0}^t G(\widehat{x}_n(s), \widetilde{x}_n(s), s)dB(s)\Big|^p \end{aligned} \tag{12}$$

for $t_0 \le t \le T$. By Hölder's inequality and Lemma 2.3, it is easy to see from (12) that for $t_0 \le t \le T$,

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t} |x_n(s)|^p\Big) \\ &\le 3^{p-1}E|\xi(0)|^p + [3(T-t_0)]^{p-1}E\int_{t_0}^t |F(\widehat{x}_n(s), \widetilde{x}_n(s), s)|^p ds \\ &\quad + 3^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{\frac{p}{2}}(T-t_0)^{\frac{p-2}{2}}E\int_{t_0}^t |G(\widehat{x}_n(s), \widetilde{x}_n(s), s)|^p ds. \end{aligned}$$

By the condition (7) and (8), we obtain

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x_n(s)|^p\Big)\\
&\le 3^{p-1}E|\xi(0)|^p+[6(T-t_0)]^{p-1}E\int_{t_0}^{t}\{[\kappa(|\widehat{x}_n(s)|^2+|\widetilde{x}_n(s)|^2)]^{\frac{p}{2}}+K^{\frac{p}{2}}\}ds\\
&+6^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{\frac{p}{2}}(T-t_0)^{\frac{p-2}{2}}E\int_{t_0}^{t}\{[\kappa(|\widehat{x}_n(s)|^2+|\widetilde{x}_n(s)|^2)]^{\frac{p}{2}}+K^{\frac{p}{2}}\}ds.
\end{aligned}
$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a positive constant α such that $\kappa(u)\le\alpha(1+u)$ for all $u\ge 0$ and recalling the definition of $\widehat{x}_n(s)$ and $\widetilde{x}_n(s)$, we then see that

$$
\begin{aligned}
&E\Big(\sup_{t_0\le s\le t}|x_n(s)|^p\Big)\\
&\le 3^{p-1}E|\xi(0)|^p+C_1C_2\\
&\quad+2^{2p-2}3^{p-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}\int_{t_0}^{t}E\Big(\sup_{t_0-\tau\le r\le s}|x_n(r)|^p\Big)ds,
\end{aligned}
$$

where $C_1=6^{p-1}(2^{(p-2)/2}\alpha^{p/2}+K^{p/2})$ and $C_2=(T-t_0)^p+[(p^3/2(p-1))^{p/2}](T-t_0)^{p/2}$. Consequently

$$
\begin{aligned}
&E\Big(\sup_{t_0-\tau\le s\le t}|x_n(s)|^p\Big)\\
&\le E\|\xi\|^p+E\Big(\sup_{t_0\le s\le t}|x_n(s)|^p\Big)\\
&\le (1+3^{p-1})E|\xi(0)|^p+C_1C_2\\
&\quad+2^{2p-2}3^{p-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}\int_{t_0}^{t}E\Big(\sup_{t_0-\tau\le r\le s}|x_n(r)|^p\Big)ds,
\end{aligned}
$$

An application of the Gronwall inequality implies that

$$
E\Big(\sup_{t_0-\tau\le s\le t}|x_n(s)|^2\Big)\le\big((1+3^{p-1})E|\xi(0)|^p+C_1C_2\big)\,e^{2^{2p-2}3^{p-1}\alpha^{\frac{p}{2}}C_2},
$$

and the desired inequality follows immediately. Thus the proof is complete. □

As an application of Lemma 3.1 we show the continuity of the p-th moment of the Euler-Maruyama's approximate solution.

Theorem 3.2. *Let (7) and (8) hold and $p\ge 2$. Then, for any $t_0\le s<t\le T$ with $t-s<1$, we have*

$$
E\Big(|x_n(t)-x_n(s)|^p\Big)\le 4^{p-1}\Big[K^{\frac{p}{2}}+2^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}+2^{p-1}\alpha^{\frac{p}{2}}C_k\Big]C_3(t-s)^p, \qquad (13)
$$

where C_k is defined in Lemma 3.1 and $C_3=1+(p(p-1)/2)^{p/2}(t-s)^{-p/2}$.

Proof. It is easy to see from the equation (10) that

$$x_n(t) - x_n(s) = \int_s^t F(\widehat{x}_n(r), \widetilde{x}_n(r), r)dr + \int_s^t G(\widehat{x}_n(r), \widetilde{x}_n(r), r)dB(r).$$

By Hölder's inequality and Lemma 2.2, it is easy to note that for $t_0 \le t \le T$,

$$\begin{aligned} & E\Big(|x_n(t) - x_n(s)|^p\Big) \\ & \le (2(t-s))^{p-1} E\int_s^t |F(\widehat{x}_n(r), \widetilde{x}_n(r), r)|^p dr \\ & + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} (t-s)^{\frac{p-2}{2}} E\int_s^t |G(\widehat{x}_n(r), \widetilde{x}_n(r), r)|^p dr. \end{aligned}$$

By the condition (7) and (8), we obtain

$$\begin{aligned} & E\Big(|x_n(t) - x_n(s)|^p\Big) \\ & \le 4^{p-1}K^{\frac{p}{2}}C_3(t-s)^p + 4^{p-1}C_3(t-s)^{p-1}\int_s^t \left[\kappa(|\widehat{x}_n(r)|^2 + |\widetilde{x}_n(r)|^2)\right]^{\frac{p}{2}} dr, \end{aligned}$$

where $C_3 = 1 + (p(p-1)/2)^{p/2}(t-s)^{-p/2}$.

Given that $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constant α such that $\kappa(u) \le \alpha(1+u)$ for all $u \ge 0$. Therefore

$$\begin{aligned} E\Big(|x_n(t) - x_n(s)|^p\Big) & \le 4^{p-1}K^{\frac{p}{2}}C_3(t-s)^p + 4^{p-1}2^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}C_3(t-s)^p \\ & \quad + 8^{p-1}\alpha^{\frac{p}{2}}C_3(t-s)^{p-1}\int_s^t E\Big(\sup_{t_0-\tau\le r\le s}|x_n(r)|^p\Big)ds. \end{aligned}$$

Hence, by Lemma 3.1,

$$E\Big(|x_n(t) - x_n(s)|^p\Big) \le 4^{p-1}\left[K^{\frac{p}{2}} + 2^{\frac{p-2}{2}}\alpha^{\frac{p}{2}} + 2^{p-1}\alpha^{\frac{p}{2}}C_k\right]C_3(t-s)^p$$

and the desired inequality follows immediately. Thus the proof is complete. □

Moreover, under non-uniform Lipschitz condition (7) and weakened linear growth condition (8), we are still able to show that the solution of the delay equation (5) is bounded in L^p, that is, the pth moment of the solution satisfies

$$E\Big(\sup_{t_0-\tau\le s\le t}|x(s)|^p\Big) \le C_l. \tag{14}$$

In view of Theorem 3.2, we could know that the continuity of the pth moment of the solution of equation (5) satisfies

$$E\Big(|x(t) - x(s)|^p\Big) \le C_m(t-s)^p, \tag{15}$$

This means that the pth moment of the solution is continuous. But the details are left to the reader.

The following theorem shows that the Euler-Maruyama approximate solution of the equation (9) gives an estimate for the difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 3.3. *Let (7) and (8) hold and $p \geq 2$. Assume that the initial data $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ is uniformly Lipschitz L^p-continuous, that is, there is a positive constant β such that*

$$E|\xi(\theta_1) - \xi(\theta_2)|^p \leq \beta(\theta_2 - \theta_1)^p \tag{16}$$

if $-\tau \leq \theta_1 < \theta_2 \leq 0$. Then, the difference between the Euler-Maruyama approximate solution $x_n(t)$ and the accurate solution $x(t)$ of equation (5) can be estimate as

$$E\Big(\sup_{t_0 \leq t \leq T} |x(t) - x_n(t)|^p\Big)$$
$$\leq \left[1 + 2^{p-1}(C_m + (\beta \vee C_m)2^p)n^{-p}\right] C_2C_4 \exp\left(2^pC_2C_4\right),$$

where $C_2 = (T-t_0)^p + [(p^3/2(p-1))^{p/2}](T-t_0)^{p/2}, C_4 = 2^{p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}$.

Proof. By Hölder's inequality, we can derive that

$$|x(s) - x_n(s)|^p$$
$$\leq [2(t-t_0)]^{p-1}\int_{t_0}^t |F(x(s), x(s-\tau), s) - F(\widehat{x}_n(s), \widetilde{x}_n(s), s)|^p ds$$
$$+2^{p-1}\left|\int_{t_0}^t G(x(s), x(s-\tau), s) - G(\widehat{x}_n(s), \widetilde{x}_n(s), s) ds\right|^p.$$

By Lemma 2.3, the condition (7) and (8), we then see that

$$E\Big(\sup_{t_0 \leq s \leq t} |x(s) - x_n(s)|^p\Big)$$
$$\leq 2^{p-1}C_2(T-t_0)^{-1}E\int_{t_0}^t \left[\kappa(|x(s) - \widehat{x}_n(s)|^2 + |x(s-\tau) - \widetilde{x}_n(s)|^2)\right]^{\frac{p}{2}} ds.$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constant α such that $\kappa(u) \leq \alpha(1+u)$ for all $u \geq 0$. Therefore

$$E\Big(\sup_{t_0 \leq s \leq t} |x(s) - x_n(s)|^p\Big) \leq 2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2 \tag{17}$$
$$+2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}E\int_{t_0}^t [|x(s) - \widehat{x}_n(s)|^p + |x(s-\tau) - \widetilde{x}_n(s)|^p] ds.$$

Define $\widehat{x}(t_0) = x(t_0)$, $\widetilde{x}(t_0) = x(t_0 - 1/n)$, $\widehat{x}(t) = x(t_0 + k/n)$, and $\widetilde{x}(t) = x(t_0 + k/n - 1/n)$ for $t_0 + k/n < t \leq [t_0 + (k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$, it then follows from (17) that

$$E\Big(\sup_{t_0 \leq s \leq t} |x(s) - x_n(s)|^p\Big)$$
$$\leq 2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2 + 4^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}[J_1 + J_2]$$

$$+2^{2p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}\int_{t_0}^{t}E\Big(\sup_{t_0\le r\le s}|x(r)-x_n(r)|^2\Big)ds,$$

where

$$J_1=\int_{t_0}^{t}E|x(s)-\widehat{x}(s)|^p ds \quad \text{and} \quad J_2=\int_{t_0}^{t}E|x(s-\tau)-\widetilde{x}(s)|^p ds.$$

An application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^p\Big)\le 2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2$$
$$+4^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}[J_1+J_2]\exp\left(2^{2p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2\right). \quad (18)$$

We now estimate J_1 and J_2. By the condition (15), we can estimate

$$\begin{aligned} J_1 &= \int_{t_0}^{t}E|x(s)-x(t_0+k/n)|^p ds \qquad (19)\\ &= \sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}E|x(s)-x(t_0+k/n)|^p ds\\ &\le C_m\left(\frac{1}{n}\right)^p\sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}ds\\ &= C_m\left(\frac{1}{n}\right)^p[T-t_0]. \end{aligned}$$

Also, by the condition (15) and (16), we can estimate

$$\begin{aligned} J_2 &= \int_{t_0}^{t}E|x(s-\tau)-x(t_0+k/n-1/n)|^p ds\\ &\le \sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}E|x(s-\tau)-x(t_0+k/n-1/n)|^p ds\\ &\le \sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge \tau}(\beta\vee C_m)\left(\frac{2}{n}-\tau\right)^p ds. \end{aligned}$$

It is easy to show that

$$J_2\le(\beta\vee C_m)\left(\frac{2}{n}\right)^p(T-t_0) \quad (20)$$

if $-\tau\le s<t\le\tau,\quad t-s\le 1.$

Substituting (19) and (20) into (18) yields that

$$E\Big(\sup_{t_0\le s\le t}|x(s)-x_n(s)|^p\Big)$$
$$\le 2^{p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}\left[1+2^{p-1}(C_m+(\beta\vee C_m)2^p)n^{-p}\right]C_2\exp\left(2^{2p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}C_2\right).$$

Thus the proof is complete. □

In the case when both functions F and G are independent of t, the Euler-Maruyama approximate solutions can be defined by a simpler form, that is (5) can be replaced by

$$x_n(t_0+\theta) = \xi(\theta) \quad \text{for} -\tau \le \theta \le 0$$

and

$$\begin{aligned} x_n(t) &= x_n(t_0+k/n) \\ &+F(x_n(t_0+k/n), x_n(t_0+(k-1)/n))[t-t_0-k/n] \\ &+G(x_n(t_0+k/n), x_n(t_0+(k-1)/n))[B(t)-B(t_0+k/n)] \end{aligned} \tag{21}$$

for $t_0+k/n < t \le [t_0+(k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$.

Let us second discuss the Euler-Maruyama approximation procedure. Consider the following stochastic differential delay equation

$$dy(t) = F(y(t), y(t-\delta(t)), t)dt + G(y(t), y(t-\delta(t)), t)dB(t) \tag{22}$$

on $t \in [t_0, T]$ with initial data, where $\delta : [t_0, T] \to [0, \tau]$, $F : R^d \times R^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to R^{d\times m}$ are Borel measurable. In the case when the time delay function $\delta(t)$ is Lipschitz continuous, the Euler-Maruyama approximate sequence of the equation (22) can be definde as follows: For each integer $n \ge 1$, define $y_n(t)$ on $[t_0 - \tau, T]$ by

$$y_n(t_0+\theta) = \xi(\theta) \quad \text{for} -\tau \le \theta \le 0$$

and

$$\begin{aligned} y_n(t) &= y_n(t_0+k/n) \\ &+\int_{t_0+k/n}^{t} F(y_n(t_0+k/n), y_n(t_0+k/n-\delta(s)), s)ds \\ &+\int_{t_0+k/n}^{t} G(y_n(t_0+k/n), y_n(t_0+k/n-\delta(s)), s)dB(s) \end{aligned} \tag{23}$$

for $t_0+k/n < t \le [t_0+(k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$.

Moreover, under non-uniform Lipschitz condition (7) and weakened linear growth condition (8), we are still able to show that the Euler-Maruyama approximation sequence (23) is bounded in L^2.

From now on, $y_n(t)$ means the Euler-Maruyama approximation (23). The following lemma shows that the Euler-Maruyama approximation sequence is bounded in L^p.

Lemma 3.4. *Let (7) and (8) hold and $p \ge 2$. Then, for all $n \ge 1/\tau$, we have*

$$E\Big(\sup_{t_0-\tau\le s\le t} |y_n(s)|^p\Big) \le C_k$$

for all $t \ge t_0$, where C_k is defined in Lemma 3.1.

Proof. The proof is similar to the proof of Lemma 3.1, but the details are left to the reader. □

As an application of Lemma 3.4 we show the continuity of the p-th moment of the Euler-Maruyama's approximate sequence.

Theorem 3.5. *Let (7) and (8) hold and* $p \geq 2$. *Then, for any* $t_0 \leq s < t \leq T$ *with* $t - s < 1$, *we have*

$$E\Big(|y_n(t) - y_n(s)|^p\Big) \leq 4^{p-1}\left[K^{\frac{p}{2}} + 2^{\frac{p-2}{2}}\alpha^{\frac{p}{2}} + 2^{p-1}\alpha^{\frac{p}{2}}C_k\right] C_3(t-s)^p, \qquad (24)$$

where C_k *is defined in Lemma 3.1 and* $C_3 = 1 + (p(p-1)/2)^{p/2}(t-s)^{-p/2}$.

Proof. The proof is similar to the proof of Theorem 3.2, but the details are left to the reader. □

Moreover, under non-uniform Lipschitz condition (7) and weakened linear growth condition (8), we are still able to show that the solution of the delay equation (22) is bounded in L^p, that is, the pth moment of the solution satisfies

$$E\Big(\sup_{t_0-\tau\leq s\leq t}|y(s)|^p\Big) \leq C_{l_1}. \qquad (25)$$

In view of Theorem 3.5, we could know that the continuity of the pth moment of the solution of equation (22) satisfies

$$E\Big(|y(t) - y(s)|^p\Big) \leq C_{m_1}(t-s)^p. \qquad (26)$$

This means that the pth moment of the solution is continuous.

The following theorem estimates the difference between Euler-Maruyama approximate sequence and the accurate solution of equation (22).

Theorem 3.6. *In addition to the assumptions of Theorem 3.3. Then the difference between the Euler-Maruyama approximate solution* $y_n(t)$ *defined by (23) and the accurate solution* $y(t)$ *of equation (22) can be estimate as*

$$\begin{aligned}&E\Big(\sup_{t_0\leq s\leq t}|y(s) - y_n(s)|^p\Big)\\ &\leq \left[1 + 2^{p-1}(C_{m_1} + 2(\beta\vee C_{m_1}))n^{-p}\right] C_2 \exp\left(2^pC_2C_4\right),\end{aligned}$$

where $C_2 = (T-t_0)^p + [(p^3/2(p-1))^{p/2}](T-t_0)^{p/2}, C_4 = 2^{p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}$.

Proof. This theorem can be proved in the same way as in the proof of Theorem 3.3 with a little bit careful consideration on the estimation of the integral. If we define $\widehat{y}_n(t) = y_n(t_0 + k/n)$, $\widetilde{y}_n(t) = y_n(t_0 + k/n - \delta(s))$ for $t_0 + k/n < t \leq [t_0 + (k+1)/n] \wedge T$, $k = 0, 1, 2, \cdots$, by Hölder's inequality, we can derive that

$$\begin{aligned}&|y(s) - y_n(s)|^p\\ &\leq [2(t-t_0)]^{p-1}\int_{t_0}^t |F(y(s), y(s-\delta(s)), s) - F(\widehat{y}_n(s), \widetilde{y}_n(s), s)|^p ds\\ &\quad +2^{p-1}\left|\int_{t_0}^t G(y(s), y(s-\delta(s)), s) - G(\widehat{y}_n(s), \widetilde{y}_n(s), s) ds\right|^p.\end{aligned}$$

By Lemma 2.3, the condition (7) and (8), we then see that

$$E\Big(\sup_{t_0\le s\le t}|y(s)-y_n(s)|^p\Big)$$
$$\le 2^{p-1}C_2(T-t_0)^{-1}E\int_{t_0}^{t}\big[\kappa(|y(s)-\widehat{y}_n(s)|^2+|y(s-\delta(s))-\widetilde{y}_n(s)|^2)\big]^{\frac{p}{2}}\,ds.$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a positive constant α such that $\kappa(u)\le\alpha(1+u)$ for all $u\ge 0$. Therefore

$$E\Big(\sup_{t_0\le s\le t}|y(s)-y_n(s)|^p\Big)\le 2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2 \tag{27}$$
$$+2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}E\int_{t_0}^{t}[|y(s)-\widehat{y}_n(s)|^p+|x(s-\delta(s))-\widetilde{y}_n(s)|^p]ds.$$

Define $\widehat{y}(t_0)=y(t_0)$, $\widetilde{y}(t_0)=y(t_0-\delta(t_0))$, $\widehat{y}(t)=y(t_0+k/n)$, and $\widetilde{y}(t)=y(t_0+k/n-\delta(s))$ for $t_0+k/n<t\le[t_0+(k+1)/n]\wedge T$, $k=0,1,2,\cdots$. It then follows from (27) that

$$E\Big(\sup_{t_0\le s\le t}|y(s)-y_n(s)|^p\Big)$$
$$\le 2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2+4^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}[M_1+M_2]$$
$$+2^{2p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}\int_{t_0}^{t}E\Big(\sup_{t_0\le r\le s}|y(r)-y_n(r)|^2\Big)ds,$$

where

$$M_1=\int_{t_0}^{t}E|y(s)-\widehat{y}(s)|^pds\quad\text{and}\quad M_2=\int_{t_0}^{t}E|y(s-\delta(s))-\widetilde{y}(s)|^pds.$$

An application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0\le s\le t}|y(s)-y_n(s)|^p\Big)\le\{2^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2$$
$$+4^{p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2(T-t_0)^{-1}[M_1+M_2]\}\exp\Big(2^{2p-1}3^{\frac{p}{2}-1}\alpha^{\frac{p}{2}}C_2\Big). \tag{28}$$

We now estimate M_1 and M_2. By the condition (26), we can estimate

$$\begin{aligned} M_1 &= \int_{t_0}^{t}E|y(s)-y(t_0+k/n)|^pds \qquad (29)\\ &= \sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}E|y(s)-y(t_0+k/n)|^pds\\ &\le C_{m_1}n^{-p}\sum_{k\ge 0}\int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T}ds\\ &= C_{m_1}n^{-p}[T-t_0]. \end{aligned}$$

Also, by the condition (15) and (26), we can estimate

$$\begin{aligned} M_2 &= \int_{t_0}^{t} E|y(s-\delta(s)) - x(t_0+k/n-\delta(s))|^p ds \qquad (30) \\ &\le \sum_{k\ge 0} \int_{t_0+k/n}^{[t_0+(k+1)/n]\wedge T} E|y(s-\delta(s)) - y(t_0+k/n-\delta(s))|^p ds \\ &\le C_{m_1} n^{-p}(T-t_0) + 2(\beta \vee C_{m_1}) n^{-p}(T-t_0). \end{aligned}$$

Substituting (29) and (30) into (28) yields that

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t} |y(s)-y_n(s)|^p\Big) \\ &\le 2^{p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}} \left[1+2^{p-1}(C_{m_1}+2(\beta\vee C_{m_1}))n^{-p}\right] C_2 \exp\left(2^{2p-1}3^{\frac{p-2}{2}}\alpha^{\frac{p}{2}}C_2\right). \end{aligned}$$

Thus the proof is complete. □

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