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DERIVATIONS OF MV-ALGEBRAS FROM HYPER MV-ALGEBRAS

M. Hamidi* and R. A. Borzooei

Abstract. In this paper, we investigate some new results in MV-algebras and (strong) hyper MV-algebras. We show that for any infinite countable set M, we can construct an MV-algebra and a strong hyper MV-algebra on M. Specially, for any infinite totally bounded set, we can construct a strong hyper MV-algebra on it. Then by considering the concept of fundamental relation on hyper MV-algebras, we define the notion of fundamental MV-algebra and prove that any MV-algebra is a fundamental MV-algebra. In practical, we show that any infinite countable MV-algebra is a fundamental MV-algebra of itself, but it is not correct for finite MV-algebras.

1. Introduction

MV-algebras introduced by C. C. Chang [2] in 1958 provide an algebraic proof of completeness theorem of infinite valued Lukasewicz propositional calculus. The hyper structure theory was introduced by F. Marty [12] at the 8th congress of Scandinavian Mathematicians in 1934. Since then many researches have worked in this areas. Recently in [5], Sh. Ghorbani, et al. applied the hyperstructure to MV-algebras and introduced the concept of a hyper MV-algebra which is a generalization of an MV-algebra and investigated some related results. Based on [6, 7], L. Torkzadeh , et al. [15], discussed hyper MV-ideals in hyper MV-algebras. In [13, 14], Davvaz et al. are defined the concept of fundamental relation on hyper MV-algebras. Now, in this paper, we prove that any MV-algebra is a fundamental MV-algebra. But, we show that any finite MV-algebra is not a fundamental MV-algebra of itself.

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*Corresponding author

2. Preliminaries

Definition 2.1. [3, 13] *Let M be a set with a binary operation "$\oplus$", a unary operation "$*$" and a constant "0". Then, $(M,\oplus,*,0)$ is called an MV-algebra if it satisfies the conditions (MV1): $x\oplus(y\oplus z)=(x\oplus y)\oplus z$, (MV2): $x\oplus y=y\oplus x$, (MV3): $x\oplus 0=x$, (MV4): $(x^*)^*=x$, (MV5): $x\oplus 0^*=0^*$, (MV6): $(x^*\oplus y)^*\oplus y=(y^*\oplus x)^*\oplus x$. Let $(M,\oplus,*,0)$ be an MV-algebra. For any $x,y\in M$, a relation "$\lesssim$" which is defined by $x\lesssim y \Longleftrightarrow x^*\oplus y=0^*$ is a partial order and is called the natural order (See* [3]*). We call $(M,\oplus,*,0,\lesssim)$ is an MV-natural partial ordered and an MV-natural total ordered is an MV-chain. Let $(M,\oplus,*,0)$ and $(M',\oplus',*',0')$ be two MV-algebras. A mapping $f:M\to M'$ is called a homomorphism from M into M', if for any $x,y\in X$, $f(x\oplus y)=f(x)\oplus' f(y), f(0)=0'$ and $f(x^*)=(f(x))^{*'}$. The homomorphism f, is called an isomorphism, if it is onto and one to one.*

Definition 2.2. [4] *Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H. Functions $*_{i_H}: H\times H\longrightarrow P^*(H)$, where $i\in\{1,2,\ldots,n\}$, are called binary hyperoperations. For all $x,y\in H$, $*_{i_H}(x,y)$ is called the hyperproduct of x and y and structure $(H,*_H)$ is called a hypergroupoid. For any two nonempty subsets A and B of hypergropoid H and $x\in H$, we define $A*_H B=\bigcup_{a\in A,b\in B} a*_H b$, $A*_H x=\bigcup_{a\in A} a*_H x$ and $x*_H B=\bigcup_{b\in B} x*_H b$.*

Definition 2.3. [13, 14] *Let M be a non-empty set, endowed with a binary hyperoperation "$\oplus$", a unary operation "$*$" and a constant "0". Then, $(M,\oplus,*,0)$ is called a* hyper MV-algebra *if satisfies the following axioms, (HMV1): $x\oplus(y\oplus z)=(x\oplus y)\oplus z$, (HMV2): $x\oplus y=y\oplus x$, (HMV3): $(x^*)^*=x$, (HMV4): $(x^*\oplus y)^*\oplus y=(y^*\oplus x)^*\oplus x$, (HMV5): $0^*\in x\oplus 0^*$, (HMV6): $x\in x\oplus 0$, and we say that hyper MV-algebra M, is a* strong hyper MV-algebra*, if it satisfies the axiom (HMV7): if $x\ll y$ and $y\ll x$, then $x=y$, for all $x,y,z\in M$, where $x\ll y$ is defined by $0^*\in x^*\oplus y$. For every subsets A and B of M, we define $A\ll B\Longleftrightarrow \exists a\in A$ and $\exists b\in B$ such that $a\ll b$ and $A^*=\{a^*\mid a\in A\}$. Let $(M,\oplus,*,0)$ be a hyper MV-algebra and R be an equivalence relation on M. If A and B are nonempty subsets of M, then $A\overline{R}B$ means that for all $a\in A$, there exists $b\in B$ such that aRb and for all $b'\in B$, there exists $a'\in A$ such that $b'Ra'$, $A\overline{\overline{R}}B$ means that for all $a\in A$, and $b\in B$, we have aRb, R is called* regular on the right (on the left) *if for* *all $x \in M$, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b))$, R is called strongly regular on the right (on the left) if for all $x \in M$, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b))$, R is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left, R is called good if $(a \circ b)R0$ and $(b \circ a)R0$ imply aRb, for all $a, b \in M$.*

A totally ordered set $(X, 0)$ is said to be well ordered (or have a well-founded order) if every nonempty subset of X, has a least element. Every finite totally ordered set is well ordered.

Theorem 2.4. [10] (*Zermelo's Well-Ordering Theorem*) *Every set can be well-ordered.*

Lemma 2.5. [8] Let X be an infinite set. Then for any set $\{a, b\}$, we have $|X \times \{a, b\}| = |X|$.

Theorem 2.6. [1] *Let X and Y be two sets such that $|X| = |Y|$. If $(Y, \leq, 0)$ is a well-ordered set, then there exists a binary order relation "$\leq$" on X and $x_0 \in X$, such that $(X, \leq, x_0)$ is a well-ordered set.*

3. Constructing of Some MV-algebras

In this section, we get some results that we need in the next sections. Specially, we construct an MV-algebra and a strong hyper MV-algebra from a nonempty countable set and any totally ordered set with maximum element. We show that the MV-algebras and the hyper MV-algebras with the same cardinal are isomorphism.

Lemma 3.1. *Let X and Y be two sets such that $|X| = |Y|$. If $(X, \oplus_X, *_X, 0_X)$ is an MV-algebra, then there exist a binary operation "$\oplus_Y$", a unary operation "$*_Y$" and constant "0_Y" on Y, such that $(Y, \oplus_Y, *_Y, 0_Y)$ is an MV-algebra and $(X, \oplus_X, *_X, 0_X) \cong (Y, \oplus_Y, *_Y, 0_Y)$.*

Proof. Since $|X| = |Y|$, then there exists a bijection $\varphi : X \longrightarrow Y$. For any $y_1, y_2 \in Y$, we define the binary operation "$\oplus_Y$" on Y by, $y_1 \oplus_Y y_2 = \varphi(x_1 \oplus_X x_2)$, where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $\oplus_Y$ is well-defined. Moreover, for any $y \in Y$ we define the unary operation "$*$" as $y^* = \varphi(x^*)$, where $x \in X, y = \varphi(x)$ and $0_Y = \varphi(0_X)$. Since φ is a bijection, then the unary operation $*$ is well-defined. Now, by some modification we can show that $(Y, \oplus_Y, *_Y, 0_Y)$ is an MV-algebra. In the follow, we define the map $\theta : (X, \oplus_X, *_X, 0_X) \longrightarrow (Y, \oplus_Y, *_Y, \varphi(0_X^*))$ by $\theta(x) = \varphi(x)$. Since φ is a bijection then θ is a bijection. Now, it is easy to see that θ is a homomorphism and so it is an isomorphism. □

Lemma 3.2. *For any $k \in \mathbb{N}$, we can construct an MV-algebra on $\mathbb{W}_k = \{0, 1, 2, 3, \ldots, k-1\}$, which is a chain.*

Proof. Let $k \in \mathbb{N}$. We define the binary operation "$\odot$" and the unary operation "$*$", on $\mathbb{W}_k$ as follows:

$$x \odot y = \begin{cases} k-1 & ,\text{if } x+y \geq k-1 \\ x+y & ,\text{otherwise} \end{cases} \qquad \text{and} \qquad x^* = k-(x+1)$$

Clearly, 0 is the smallest element in $\mathbb{W}_k, k-1 = max(\mathbb{W}_k)$ and for any $x \in \mathbb{W}_k$, $(x^*)^* = x$. First, we show that "$\odot$" is well-defined on $\mathbb{W}_k$. Let $x = x'$ and $y = y'$. If $x + y \geq k-1$ then $x' + y' \geq k-1$ and so $x \odot y = k-1 = x' \odot y'$. Moreover, if $x + y < k-1$ then $x' + y' < k-1$ and so $x \odot y = x + y = x' + y' = x' \odot y'$. Now, we show that $(\mathbb{W}_k, \odot, *, 0)$ is an MV-algebra. Let $x, y, z \in \mathbb{W}_k$. Then,

(MV1): Case 1: $x+y \geq k-1$. Then $x+(y+z) = (x+y)+z \geq k-1$. Case 2: $x+y < k-1$. If $(x+y)+z < k-1$, then $x+(y+z) = (x+y)+z < k-1$ and if $(x+y)+z \geq k-1$, then $x+(y+z) = (x+y)+z \geq k-1$. Since in any cases, $(x+y)+z = x+(y+z)$, then $(x\odot y)\odot z = x\odot(y\odot z)$.

(MV2): Since $x + y = y + x$, then $x \odot y = y \odot x$.

(MV3): By hypothesis, $x \odot 0 = x$.

(MV4): By hypothesis, $0^* = k-1, (k-1)^* = 0$ and $(x^*)^* = x$.

(MV5): By hypothesis, $x \odot 0^* = x \odot (k-1) = k-1 = 0^*$.

(MV6): Case 1: $y < x$. Then, clearly $k-(x+1)+y < k-1$ and

$$(x^*\odot y)^*\odot y = ((k-(x+1))\odot y)^*\odot y = (k-(x+1)+y)^*\odot y = (x-y)\odot y = x$$

Moreover, in this case we have $k-(1+y)+x \geq (k-1)$ and so

$$(y^* \odot x)^* \odot x = ((k-(1+y)) \odot x)^* \odot x = (k-1)^* \odot x = 0 \odot x = x$$

Case 2: $y > x$. Then, clearly $k-(x+1)+y \geq k-1$ and

$$(x^* \odot y)^* \odot y = ((k-(x+1)) \odot y)^* \odot y = (k-1)^* \odot y = 0 \odot y = y$$

Moreover, in this case we have $k-(y+1)+x < k-1$ and so

$$(y^*\odot x)^*\odot x = ((k-(y+1))\odot x)^*\odot x = (k-(y+1)+x)^*\odot x = (y-x)\odot x = y$$

Case 3: $y = x$. Then, clearly $(x^* \odot y)^* \odot y = (y^* \odot x)^* \odot x$. Therefore, $(\mathbb{W}_k, \odot, *, 0)$ is an MV-algebra.

Now, for any $x, y \in \mathbb{W}_k, x \lesssim y$ if and only if $x^* \odot y = k-1$ if and only if $(k-(x+1)) \odot y = k-1$ if and only if $(k-(x+1)) + y \geq k-1$ if and only if $x \leq y$. Therefore, $(\mathbb{W}_k, \odot, *, 0, \leq)$ is an MV-chain.

□

Theorem 3.3. *Let X be a finite set. Then there exist a binary operation "$\oplus_X$" and unary operation "$*_X$" and constant "0_X" on X, such that $(X, \oplus_X, *_X, 0_X)$, is an MV-algebra.*

Proof. Let X be a finite set. Then, there exists $k \in \mathbb{W}$ such that $|X| = |\mathbb{W}_k|$. Now, since by Lemma 3.2, $(\mathbb{W}_k, \odot, *, 0)$ is an MV-algebra, then by Lemma 3.1, there exist a binary operation "$\oplus_X$", a unary operation "$*_X$" and constant "0_X" on X, such that $(X, \oplus_X, *_X, 0_X)$, is an MV-algebra. □

Lemma 3.4. *Let $1 < n \in \mathbb{Q}$. Then there exist a binary operation "$\odot$" and a unary operation "$*$" on $E = \mathbb{Q} \cap [1\ n]$, such that $(E, \odot, *, 1)$ is an MV-algebra.*

Proof. For any $1 < n \in E$, we define the binary operation "$\odot$" and the unary operation "$*$" on E as follows:

$$x \odot y = \begin{cases} n & , \text{if } xy \geq n \\ xy & , \text{otherwise} \end{cases} \qquad \text{and} \qquad x^* = \frac{n}{x}$$

Then 1 is the smallest element in $E, n = max(E)$ and for any $x \in E$, $(x^*)^* = x$. First, we show that "$\odot$" is well-defined on E. Let $x = x_1$ and $y = y_1$. If $xy \geq n$ then $x_1 y_1 \geq n$ and so $x \odot y = n = x_1 \odot y_1$. Moreover, if $xy < n$ then $x_1 y_1 < n$ and so $x \odot y = xy = x_1 y_1 = x_1 \odot y_1$. Clearly "$*$" is well-defined. Now, we show that $(E, \odot, *, 1)$ is an MV-algebra. Let $x, y, z \in E$. Then,

<u>(MV1)</u>: If $xy \geq n$, since $z \geq 1$, then $x(yz) = (xy)z \geq n$. Now, let $xy < n$. If $(xy)z < n$, then $x(yz) = (xy)z < n$ and if $(xy)z \geq n$, then $x(yz) = (xy)z \geq n$. Since in any cases, $(xy)z = x(yz)$, then $(x \odot y) \odot z = x \odot (y \odot z)$.

<u>(MV2)</u>: Since $xy = yx$, then $x \odot y = y \odot x$.

<u>(MV3)</u>: By hypothesis, $x \odot 1 = x$.

<u>(MV4)</u>: By hypothesis, $1^* = \frac{n}{1} = n, n^* = \frac{n}{n} = 1$ and $(x^*)^* = x$.

<u>(MV5)</u>: By hypothesis, $x \odot 1^* = x \odot n = n = 1^*$.

<u>(MV6)</u>: If $y < x$, then $\frac{ny}{x} < n$ and $(x^* \odot y)^* \odot y = (\frac{n}{x} \odot y)^* \odot y = (\frac{ny}{x})^* \odot y = \frac{n}{\frac{ny}{x}} \odot y = \frac{x}{y} \odot y = x$. Moreover, in this case we have $\frac{nx}{y} > n$ and so $(y^* \odot x)^* \odot x = (\frac{n}{y} \odot x)^* \odot x = n^* \odot x = 1 \odot x = x$. If $y > x$ then, $\frac{ny}{x} > n$ and $(x^* \odot y)^* \odot y = (\frac{n}{x} \odot y)^* \odot y = n^* \odot y = 1 \odot y = y$.

Moreover, in this case we have $\frac{nx}{y} < n$ and so

$$(y^* \odot x)^* \odot x = (\frac{n}{y} \odot x)^* \odot x = (\frac{nx}{y})^* \odot x = (\frac{n}{\frac{nx}{y}})^* \odot x = \frac{y}{x} \odot x = y$$

If $y = x$, then clearly $(x^* \odot y)^* \odot y = (y^* \odot x)^* \odot x$. Therefore, $(E, \odot, *, 1)$ is an MV-algebra. □

Theorem 3.5. *Let X be an infinite countable set. Then there exists a binary operation " $\oplus$ ", a unary operation " $*$ " and constant "0" on X, such that $(X, \oplus, *, 0)$ is an MV-algebra.*

Proof. Let X be an infinite countable set. Since $E = \mathbb{Q} \cap [1\ n]$ in Lemma 3.4, is an infinite countable MV-algebra, so $|X| = |E|$. Now, by Theorem 2.6, there exist a bijection $\psi : E \longrightarrow X$, a binary relation " $\leq$ " and the smallest element $0 = \psi(1)$ on X such that $(X, \leq, 0)$ is a totally ordered set and for any $t, s \in E$ we have

$$\psi(t) \leq \psi(s) \text{ if and only if } t \leq s. \tag{1}$$

Hence, for the largest element $n \in E$ and for any $x \in X$, we have, $0 = \psi(1) \leq x \leq \psi(n)$. For any $x, y \in X$, since ψ is onto, there exist $i, j \in E$ such that $x = \psi(i)$ and $y = \psi(j)$. Now, we define a binary operation " $\oplus$ " and a unary operation " $*$ " on X as follows:

$$x \oplus y = \begin{cases} \psi(n) & ,\text{if } n \leq i \odot j \\ \psi(i \odot j) & ,\text{otherwise} \end{cases} \text{ and } x^* = \psi(i^*) = \psi(\frac{n}{i})$$

that the operation "$\odot$" is defined in Lemma 3.4. First, we show that "$\oplus$" is well-defined. Let $x = x_1$ and $y = y_1$. Then there exist $i, i_1, j, j_1 \in E$ such that $x = \psi(i), x_1 = \psi(i_1), y = \psi(j), y_1 = \psi(j_1)$. Since, ψ is a bijection, then $i = i_1$ and $j = j_1$. Now, if $i \odot j \geq n$ then $i_1 \odot j_1 \geq n$ and so $x \oplus y = \psi(n) = \psi(i_1 \odot j_1) = x_1 \oplus y_1$. Moreover, if $i \odot j < n$ then $i_1 \odot j_1 < n$ and so $x \oplus y = \psi(i \odot j) = \psi(i_1 \odot j_1) = x_1 \oplus y_1$. Since, ψ is a bijection, then clearly the operation " $*$ " is well-defined. Now, since $(E, \odot, ^*, 1)$ is an MV-algebra, then we show that $(X, \oplus, *, 0)$ is an MV-algebra. For this, let $x = \psi(i), y = \psi(j), z = \psi(k) \in X$ where $i, j, k \in E$.

(MV1): If $i \odot j \geq n$, then by Lemma 3.4, for any $k \in E$ we have, $i \odot (j \odot k) = (i \odot j) \odot k \geq n$.

Now, let $i \odot j < n$. If $(i \odot j) \odot k < n$, then $i \odot (j \odot k) = (i \odot j) \odot k < n$ and if $(i \odot j) \odot k = n$, then $i \odot (j \odot k) = (i \odot j) \odot k = n$. Since in any cases, $(i \odot j) \odot k = i \odot (j \odot k)$, and ψ is a bijection, then $\psi((i \odot j) \odot k) = \psi(i \odot (j \odot k))$ and so

$$\begin{aligned}(x \oplus y) \oplus z &= \psi(i \odot j) \oplus z = \psi((i \odot j) \odot k) = \psi(i \odot (j \odot k)) \\ &= x \oplus \psi(j \odot k) = x \oplus (y \oplus z).\end{aligned}$$

(MV2): Since $i \odot j = j \odot i$, then $x \oplus y = \psi(i \odot j) = \psi(j \odot i) = y \oplus x$.
(MV3): Since $i \odot 1 \leq n$ then, by hypothesis, $x \oplus \psi(1) = \psi(i) \oplus \psi(1) = \psi(i \odot 1) = \psi(i) = x$.
(MV4): By hypothesis, $(x^*)^* = (\psi(i^*))^* = (\psi(\frac{n}{i}))^* = \psi(\frac{n}{\frac{n}{i}}) = \psi(i) = x$.
(MV5): Since $i \odot n \geq n$ then, by hypothesis $x \oplus \psi(n) = \psi(i \odot n) = \psi(n)$.
(MV6): Since $(i^* \odot j)^* \odot j = (j^* \odot i)^* \odot i$. We consider the following cases:
Case 1: $y = \psi(j) < \psi(i) = x$. Then by (1), $j < i$ and so $\frac{n \odot j}{i} < n$. In this case

$$\begin{aligned}(x^* \oplus y)^* \oplus y &= (\psi(i^* \odot j))^* \oplus y = \psi((i^* \odot j)^* \odot j) = \psi((\frac{n}{i} \odot j)^* \odot j) \\ &= \psi(\frac{i}{j} \odot j) = \psi(i).\end{aligned}$$

Moreover, in this case we have $\frac{n \odot i}{j} > n$ and so

$$\begin{aligned}(y^* \oplus x)^* \oplus x &= \psi((j^* \odot i)^*) \oplus x = \psi((j^* \odot i)^* \odot i) \\ &= \psi((\frac{n}{j} \odot i)^* \odot i) = \psi((\frac{n \odot j}{i})^* \odot i) = \psi((n))^* \odot i) \\ &= \psi(1 \odot i) = \psi(i).\end{aligned}$$

Case 2: $y = \psi(j) > \psi(i) = x$. Then by (1), $j > i$ and so, clearly $\frac{n \odot j}{i} > n$ and

$$\begin{aligned}(x^* \oplus y)^* \oplus y &= (\psi(i^* \odot j)^*) \oplus y = \psi((i^* \odot j)^* \odot j) = \psi((\frac{n}{i} \odot j)^* \odot j) \\ &= \psi((\frac{n \odot j}{i})^* \odot j) = \psi((n^*) \odot j) = \psi(1 \odot j) = \psi(j).\end{aligned}$$

Moreover, in this case we have $\frac{ni}{j} < n$ and so

$$\begin{aligned}\psi((j^* \odot i)^* \odot i) &= \psi((\frac{n}{j} \odot i)^* \odot i) = \psi((\frac{n \odot i}{j})^* \odot i) = \psi((\frac{n}{\frac{n \odot i}{j}}) * \odot i \\ &= \psi(\frac{j}{i} \odot i) = \psi(j).\end{aligned}$$

Therefore, $(X, \oplus, *, 0)$ is an MV-algebra.

□

Corollary 3.6. *For any nonempty countable set X, we can construct an MV-algebra on X.*

Proof. Let X be a nonempty countable set. Then, $|X| = |E|$, where $E = \mathbb{Q} \cap [1\ n]$ is infinite countable set in Lemma 3.4, or there exists $k \in \mathbb{N}$ such that $|X| = |\mathbb{W}_k|$. Now, by the Theorems 3.3 and 3.5, the proof is straightforward. □

Theorem 3.7. *Let X be an infinite set. If $(X, \oplus_X, 0_X, *_X)$ is an MV-algebra, then for any set $\{a, b\}$, there exist a binary operation "$\oplus$", a unary operation "$*$" and constant "0" on X such that $(X \times \{a, b\}, \oplus, *, 0)$ is an MV-algebra and $(X, \oplus_X, *_X, 0_X) \cong (X \times \{a, b\}, \oplus, *, 0)$*

Proof. Since X is an infinite set, then by Lemma 2.5, $|X \times \{a, b\}| = |X|$. Now, by Lemma 3.1, the proof is straightforward. □

4. Constructing of Some (Strong) Hyper MV-algebras

Theorem 4.1. *Let $(M, \oplus_M, *_M, 0_M)$ and $(N, \oplus_N, *_N, 0_N)$ be two MV-algebras. Then there exist a binary hyperoperation "$\oplus$", a unary operation "$*$" and constant "0" on $M \times N$, such that $(M \times N, \oplus, *, 0)$ is a hyper MV-algebra.*

Proof. Let $(M, \oplus_M, *_M, 0_M)$ and $(N, \oplus_N, *_N, 0_N)$ be two MV-algebras. For any $(m_1, n_1), (m_2, n_2) \in M \times N$, we define the binary hyperoperation "$\oplus$" on $M \times N$ by, $(m_1, n_1) \oplus (m_2, n_2) = \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\}$ and for any $(m, n) \in M \times N$, the unary operation "$*$" by, $(m, n)^* = *(m, n) = (*_M(m), *_N(n)) = (m^{*_M}, n^{*_N})$ and constant $0 = (0_M, 0_N)$. First, we show that the hyperoperation "$\oplus$" is well defined. Let $(m_1, n_1) = (m'_1, n'_1)$ and $(m_2, n_2) = (m'_2, n'_2)$. Then,

$$(2) \qquad \begin{aligned} (m_1, n_1) \oplus (m_2, n_2) &= \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\} \\ &= \{(m'_1 \oplus_M m'_2, n'_1), (m'_1 \oplus_M m'_2, n'_2)\} \\ &= (m'_1, n'_1) \oplus (m'_2, n'_2) \end{aligned}$$

Moreover, since $(m, n) = (m', n')$ implies that $*(m, n) = *(m', n')$ then "$*$" is well-defined. Now, by some modifications we can show that $(M \times N, \oplus, *, 0)$ is a hyper MV-algebra. □

Theorem 4.2. *Let $(M, \oplus_M, *_M, 0_M, \lesssim)$ and $(N, \oplus_N, *_N, 0_N, \lesssim)$ be two MV-chains. Then there exist a binary hyperoperation "$\oplus$", a unary operation "$*$" and constant "0" on $M \times N$, such that $(M \times N, \oplus, *, 0)$ is a strong hyper MV-algebra.*

Proof. Let $(M, \oplus_M, *_M, 0_M)$ be an MV-algebra and $(N, \oplus_N, *_N, 0_N)$ be an MV-chain. Now, for any $(m_1, n_1), (m_2, n_2) \in M \times N$, we define the binary hyperoperation " $\oplus$ " on $M \times N$ by, $(m_1, n_1) \oplus (m_2, n_2) = \{(m_1 \oplus_M m_2, n_1), (m_1 \oplus_M m_2, n_2)\}$ and for any $(m, n) \in M \times N$, the unary operation " $*$ " by, $(m, n)^* = *(m, n) = (*_M(m), *_N(n)) = (m^{*_M}, n^{*_N})$ and we let constant $0 = (0_M, 0_N)$. By Theorem 4.1, $(M \times N, \oplus, *, 0)$ is a hyper MV-algebra. Now, we define a binary relation " $\ll$ " on $M \times N$ by, $(x, y) \ll (z, w)$ if and only if $(0_M, 0_N)^* \in (x, y)^* \oplus (z, w)$. We show that for any $(x, y), (z, w) \in M \times N$, if $(x, y) \ll (z, w)$ then $x \lesssim z$ and $y \lesssim w$. For this, let $(x, y) \ll (z, w)$. Then by the hypothesis,

$$\begin{aligned}(0_M, 0_N)^* &= (0_M^*, 0_N^*) \in (x, y)^* \oplus (z, w) = (x^{*_M}, y^{*_N}) \oplus (z, w) \\ &= \{(x^{*_M} \oplus_M z, y^{*_N}), (x^{*_M} \oplus_M z, w)\}\end{aligned}$$

and so $(0_M^*, 0_N^*) = (x^{*_M} \oplus_M z, y^{*_N})$ or $(0_M^*, 0_N^*) = (x^{*_M} \oplus_M z, w)$. If $(0_M^*, 0_N^*) = (x^{*_M} \oplus_M z, y^{*_N})$, then $y = 0_N, x^{*_M} \oplus_M z = 0_M^*$. Now since $(M, \oplus_M, *_M, 0_M)$ is an MV-chain, then $x \lesssim z$ and $y = 0_N \lesssim w$. If $(0_M^*, 0_N^*) = (x^{*_M} \oplus_M z, w)$, then $w = 0_N^*, x^{*_M} \oplus_M z = 0_M^*$. Now, since $(N, \oplus_N, *_N, 0_N)$ is an MV-chain, $x \lesssim z$ and $y \lesssim 0_N^* = w$. Hence, in any cases, we have, $x \lesssim z$ and $y \lesssim w$. Therefore, $(M \times N, \oplus, *, 0)$ is a strong hyper MV-algebra. □

Lemma 4.3. *Let X and Y be two sets such that $|X| = |Y|$. If $(X, \oplus_X, *_X, 0_X)$ is a (strong) hyper MV-algebra, then there exist a binary hyperoperation " $\oplus_Y$ ", a unary operation " $*_Y$ " and constant "0_Y" on Y , such that $(Y, \oplus_Y, *_Y, 0_Y)$ is a strong hyper MV-algebra and $(X, \oplus_X, *_X, 0_X) \simeq (Y, \oplus_Y, *_Y, 0_Y)$.*

Proof. The proof is similar to the proof of Lemma 3.1, by some modifications. □

Corollary 4.4. *Let $(M, \oplus_M, *_M, 0_M, \lesssim)$ be an MV-chain. Then for any set $\{a, b\}$:*
(i) there exist a binary hyperoperation "$\oplus$", a unary operation "$*$" and constant "0" on $M \times \{a, b\}$, such that $(M \times \{a, b\}, \oplus, *, 0)$ is a strong hyper MV-algebra.
(ii) If M is infinite, then there exist a binary hyperoperation " $\odot$ ", a unary operation " $*$ " and constant "0" on M, such that $(M, \odot, *, 0)$ is a strong hyper MV-algebra. and $(M \times \{a, b\}, \oplus, *, 0) \cong (M, \odot, *, 0)$.

Proof. (i) First, we define the partial relation " $\leq$ " on set $\{a, b\}$ by $\leq := \{(a, a), (b, b), (a, b)\}$. Hence $(\{a, b\}, \leq)$ is a totally ordered set. Now we consider the following binary and unary operations :

$\oplus$	a	b
a	a	b
b	b	b

and

$*$	a	b
	b	a

Then clearly $(\{a, b\}, a, \oplus, *)$ is a the smallest nontrivial MV-chain. Now, we define the binary hyperoperation " $\oplus$ " on $M \times \{a, b\}$ as follows:

$$(m_1, t) \oplus (m_2, s) = \{(m_1 \oplus_M m_2, t), (m_1 \oplus_M m_2, s)\}$$

Similar to proof of Theorem 4.2, $(M \times \{a, b\}, \oplus, *, 0)$ is a strong hyper MV-algebra.
(ii) Since M is infinite set, then by Lemmas 2.5 and 4.3, there exist a binary hyperoperation " $\odot$ ", a unary operation " $*$ " and constant "0" on M, such that $(M, \odot, *, 0)$ is a strong hyper MV-algebra. and $(M \times \{a, b\}, \oplus, *, 0) \cong (M, \odot, *, 0)$. □

Theorem 4.5. *Let $(X, \leq, x_0, y_0)$ be a totally ordered set with smallest element x_0 and greatest element y_0. Then, there exist a binary hyperoperation " $\odot$ " and a unary operation " $*$ " on X, such that $(X, \odot, *, x_0)$ is a (strong) hyper MV-algebra.*

Proof. Firstly, if $X = \{x_0, y_0\}$, then by the following tables:

$\odot$	x_0	y_0
x_0	$\{x_0, y_0\}$	$\{x_0, y_0\}$
y_0	$\{x_0, y_0\}$	$\{x_0, y_0\}$

and

$*$	x_0	y_0
	y_0	x_0

$(X, \odot, *, x_0)$ is a (strong) hyper MV-algebra. Now, let $|X| \geq 3$. For any $x, y \in X$, we define a binary hyperoperation " $\odot$ " and unary operation " $*$ " as follows:

$$x \odot y = \begin{cases} \{x_0, x, y\} & ,\text{if } x \neq y \\ \{x_0, y_0, x\} & ,\text{if } x = y \end{cases} \quad \text{and } x^* = \begin{cases} y_0 & ,\text{if } x = x_0 \\ x_0 & ,\text{if } x = y_0 \\ x & ,\text{otherwise} \end{cases}$$

First, we show " $\odot$ " is well-defined. Let $x = x'$ and $y = y'$. If $x \neq y$, then, $x \odot y = \{x_0, x, y\} = \{x_0, x', y'\} = x' \odot y'$. Now, let $x = y$. Then, $x \odot y = \{x_0, x, y_0\} = \{x_0, x', y_0\} = x' \odot y'$. Hence " $\odot$ " is well-defined. Clearly the unary operation " $*$ " is well-defined, too. Now we show that $(X, \odot, *, x_0)$ is a hyper MV-algebra. Let $x, y, z \in X$. Then,
(HMV1): Case 1: If $x = y = z$, then, $(x \odot y) \odot z = x \odot (y \odot z)$.
Case 2: If $x = y \neq z$, then, $(x \odot y) \odot z = \{x_0, x, z, y_0\} = x \odot (y \odot z)$.
Case 3: If $x \neq y = z$, then, $(x \odot y) \odot z = \{x_0, x, y, y_0\} = x \odot (y \odot z)$.

Case 4: If $x = z \neq y$, then, $(x \odot y) \odot z = \{x_0, x, y, y_0\} = x \odot (y \odot z)$.
Case 5: If $x \neq z \neq y$, then, $(x \odot y) \odot z = \{x_0, x, z, y\} = x \odot (y \odot z)$.
(HMV2): If $x \neq y$, then, $(x \odot y) = \{x_0, x, y\} = \{x_0, y, x\} = (y \odot x)$. Now let $x = y$. Then, $(x \odot y) = \{x_0, x, y_0\} = (y \odot x)$.
(HMV3): By hypothesis $(x^*)^* = (x^*) = x$.
(HMV4): Case 1: If $x = x_0$ and $y = y_0$, then,

$$(x^* \odot y)^* \odot y = \{y_0, x_0\} = (y \odot x) \odot x = (y^* \odot x)^* \odot x$$

Case 2: If $x = x_0$ and $y \neq y_0$, then, $(x^* \odot y)^* \odot y = \{y_0, x_0, y\} = (y \odot x) \odot x = (y^* \odot x)^* \odot x$.
Case 3: If $x \neq x_0$ and $y = y_0$, then, $(x^* \odot y)^* \odot y = \{y_0, x_0, x\} = (y \odot x) \odot x = (y^* \odot x)^* \odot x$.
Case 4: If $x \neq x_0$, $y \neq y_0$ and $x \neq y$, then, $(x^* \odot y)^* \odot y = \{y_0, x_0, x, y\} = (y \odot x) \odot x = (y^* \odot x)^* \odot x$.
(HMV5): By hypothesis $x \odot x_0 = \{x, x_0\}$, then $x \in x \odot x_0$.
(HMV6): By hypothesis $x \odot x_0^* = \{x, x_0^*, x_0\}$ then $x_0^* \in x \odot x_0^*$.
Therefore, $(X, \odot, *, x_0)$ is a hyper MV-algebra.
(HMV7): If $x \ll y$ and $y \ll x$, then $y_0 \in x^* \odot y$ and $y_0 \in y^* \odot x$. Since $\{x, y\} \nsubseteq \{x_0, y_0\}$, then $x^* = x$ and $y^* = y$. This implies that $y_0 \in x \odot y = y \odot x$ and by hypothesis $x = y$.
Therefore, $(X, \odot, *, x_0)$ is a strong hyper MV-algebra. □

Open Problem 4.6. *We proved that any bonded totally ordered set can be a strong hyper MV-algebra. Let X be an infinite non bounded totally ordered set. Is there a binary hyperoperation " $\oplus$ ", a unary operation " $*$ " and constant "0", such that $(X, \oplus, *, 0)$ is a (strong) hyper MV-algebra?*

5. Fundamental MV-algebras

In this section, by using the notion of fundamental relation, we define the concept of fundamental MV-algebra and we prove that any MV-algebra is a fundamental MV-algebra. Let $(M, \oplus, *, 0)$ be a hyper MV-algebra and A be a subset of M. Then with Now, in the following, the well-known idea of β^* relation on hyperstructure [4, 16, 13] is transferred and applied to hyper MV-algebras.

Let $(M, \oplus, *, 0)$ be a hyper MV-algebra and $\mathcal{L}(A)$ denote the set of all finite combinations of elements A with $\oplus$ and $*$. For example, $\mathcal{L}(\{x_1, x_2\}) = \{x_1 \oplus x_2, x_1^* \oplus x_2, (x_1 \oplus x_2,)^*, (x_1 \oplus x_2,)^* \oplus x_1, \ldots\}$. .

Then we set $\beta_1 = \{(x,x) \mid x \in M\}$ and for every integer $n \geq 1$, β_n is the relation defined as follows:

$$x\beta_n y \Longleftrightarrow \exists(a_1,a_2,\ldots,a_n) \in X^n,\ \exists u \in \mathcal{L}(a_1,a_2,\ldots,a_n) \text{ s.t } \{x,y\} \subseteq u$$

Obviously, for every $n \geq 1$, the relations β_n are symmetric, and the relation $\beta = \bigcup_{n\geq 1} \beta_n$ is reflexive and symmetric. Now, let β^* be the *transitive closure* of β. Then β^* is the smallest strongly regular equivalence relation on M, such that $\frac{M}{\beta^*}$ is an MV-algebra. (See [13]).

Theorem 5.1. [14] Let $(M_i, \oplus_i, *_i, 0_i)$ be a hyper MV-algebra and β_i^* be a fundamental relation on M_i, for any $i = 1,2,\ldots,n$. Then,

$$\frac{M_1 \times M_2 \times \ldots \times M_n}{\beta^*_{M_1\times M_2\times\ldots\times M_n}} \cong \frac{M_1}{\beta^*_1} \times \frac{M_2}{\beta^*_2} \times \ldots \times \frac{M_n}{\beta^*_n}.$$

Lemma 5.2. *Let $(M, \oplus, {}^*, 0)$ be a hyper MV-algebra. Then for the fundamental relation β^* and for any $m \in M$, we have $\beta^*(m^*) = (\beta^*(m))^*$.*

Proof. Let $m \in M$. For any $t \in M$, if $t \in \beta^*(m^*)$, then there exist $n \geq 1$, $(a_1,a_2,\ldots,a_n) \in M^n$ and $u \in \mathcal{L}(a_1,a_2,\ldots,a_n)$ such that $\{m^*,t\} \subseteq u$. Now, since $\{m,t^*\} = \{(m^*)^*,t^*\} = \{m^*,t\}^* \subseteq u^*$, then $t^* \in \beta^*(m)$ and so $\beta^*(m^*) \subseteq (\beta^*(m))^*$. Let $t \in (\beta^*(m))^*$. Then $t^* \in \beta^*(m)$ and there exist $n \geq 1$, $(a_1,a_2,\ldots,a_n) \in M^n$ and $u \in \mathcal{L}(a_1,a_2,\ldots,a_n)$ that $\{m,t^*\} \subseteq u$. Now, since $\{m^*,t\} = \{m^*,(t^*)^*\} = \{m,t^*\}^* \subseteq u^*$, then $t \in \beta^*(m^*)$ and so $(\beta^*(m))^* \subseteq \beta^*(m^*)$. □

Lemma 5.3. *Let $(X, \oplus_X, *_X, 0_X)$ and $(Y, \oplus_Y, *_Y, 0_Y)$ be two hyper MV-algebras and $f : (X, \oplus_X, *_X, 0_X) \to (Y, \oplus_Y, *_Y, 0_Y)$ be a homomorphism. Then for any $x,y \in X$, $x\beta^*_X y$ implies that $f(x)\beta^*_Y f(y)$.*

Proof. Let $(X, \oplus_X, *_X, 0_X)$ and $(Y, \oplus_Y, *_Y, 0_Y)$ be two hyper MV-algebras and $x,y \in X$. Since $x\beta^*_X y$, then there exists $u \in \mathcal{L}(X)$, such that $\{x,y\} \subseteq u$. Now, for homomorphism $f : (X, \oplus_X, *_X, 0_X) \to (Y, \oplus_Y, *_Y, 0_Y)$ we have $\{f(x),f(y)\} = f\{x,y\} \subseteq f(u) \in \mathcal{L}(Y)$. Therefore, $f(x)\beta^*_Y f(y)$. □

Example 5.4. *Let $(M_1, \oplus_1, *_1, 0)$ and $(M_2, \oplus_2, *_2, 0)$ be two hyper MV-algebras by the following tables:*

$\oplus_1$	0	1
0	$\{0,1\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

,

$*_1$	0	1
	1	0

,

$\oplus_2$	0	b	1
0	$\{0\}$	$\{b\}$	$\{b,1\}$
b	$\{b\}$	$\{b,1\}$	$\{b,1\}$
1	$\{b,1\}$	$\{b,1\}$	$\{b,1\}$

and

$*_2$	0	b	1
	1	b	0

.

Now, we define the map $f : (M_2, \oplus_2, *_2, 0) \longrightarrow (M_1, \oplus_1, *_1, 0)$ *by* $f(0) = 0$ *and* $f(1) = f(b) = 1$. *Moreover,* $\frac{(M_1,\oplus_1,*_1,0)}{\beta^*} = \{\beta^*(0) = \{0,1\}, \beta^*(0) = \{0,1\}\}$ *and* $\frac{(M_2,\oplus_2,*_2,0)}{\beta^*} = \{\beta^*(0) = \{0\}, \beta^*(b) = \{b,1\}, \beta^*(b) = \{b,1\}\}$. *Clearly* f *is a homomorphism which is not injective and* $f(b) \in \beta^*(f(1))$, *but* $b \notin \beta^*(1)$.

Lemma 5.5. *Let* $(X, \oplus_X, *_X, 0_X)$ *and* $(Y, \oplus_Y, *_Y, 0_Y)$ *be hyper* MV-*algebras and* $f : (X, \oplus_X, *_X, 0_X) \to (Y, \oplus_Y, *_Y, 0_Y)$ *be a monomorphism. Then for any* $x, y \in X$, $f(x)\beta^*_Y f(y)$ *implies that* $x\beta^*_X y$.

Proof. For any $x, y \in X$, since $f(x)\beta^*_Y f(y)$, there exists $v \in \mathcal{L}(Y)$, such that $\{f(x), f(y)\} \subseteq v$. Now, for a monomorphism $f : X \to Y$ we have $\{x, y\} = \{f^{-1}(f(x)), f^{-1}(f(y))\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(v) \in U$. Therefore, $xv^*_X y$. □

Lemma 5.6. *Let* $(X, \oplus_X, *_X, 0_X)$ *and* $(Y, \oplus_Y, *_Y, 0_Y)$ *be two hyper* MV-*algebras and* $f : (X, \oplus_X, *_X, 0_X) \to (Y, \oplus_Y, *_Y, 0_Y)$ *be an isomorphism. Then for any* $x, y \in X$, $x\beta^*_X y$ *if and only if* $f(x)\beta^*_Y f(y)$.

Proof. By Lemmas 5.3 and 5.5, the proof is straightforward. □

Theorem 5.7. *Let* X *and* Y *be two nonempty sets and* $|X| = |Y|$. *If* $(X, \oplus_X, *_X, 0_X)$ *is a (strong) hyper* MV-*algebra, then there exist a binary hyperoperation "* $\oplus_Y$ *", a unary operation "* $*_Y$ *" and constant "*0_Y*" on* Y, *such that* $(\frac{(X,\oplus_X,*_X,0_X)}{\beta^*}, \overline{\circ}) \cong (\frac{(Y,\oplus_Y,*_Y,0_Y)}{\beta^*}, \overline{\circ})$.

Proof. Since $|X| = |Y|$, then by Lemma 4.3, there exist a binary hyperoperation " $\oplus_Y$ ", a unary operation " $*_Y$ " and constant 0_Y on Y such that $(Y, \oplus_Y, *_Y, 0_Y)$ is a (strong) hyper MV-algebra. Moreover, there exists an isomorphism $f : (X, \oplus_X, *_X, 0_X) \longrightarrow (Y, \oplus_Y, *_Y, 0_Y)$, such that $f(0_X) = 0_Y$. Now, we define the map $\varphi : (\frac{(X,\oplus_X,*_X,0_X)}{\beta^*}, \overline{\oplus}) \to (\frac{(Y,\oplus_Y,*_Y,0_Y)}{\beta^*}, \overline{\oplus})$ by $\varphi(\beta^*(x)) = \beta^*(f(x))$. First, we show that for any $x_1, x_2 \in X$, $\varphi(\beta^*(x_1)\overline{\oplus}\beta^*(x_2)) = \varphi(\beta^*(x_1))\overline{\oplus}\varphi(\beta^*(x_2))$. By Lemma 5.2, for any $x \in X$,

$$\begin{aligned} \varphi(\beta^*(x_1)\overline{\oplus}\beta^*(x_2)) &= \varphi(\beta^*(x_1 \oplus_X x_2)) = \beta^*(f(x_1 \oplus_X x_2)) \\ &= \beta^*(f(x_1) \oplus_Y f(x_2)) = \beta^*(f(x_1))\overline{\oplus}\beta^*(f(x_2)) \\ &= \varphi(\beta^*(x_1))\overline{\oplus}\varphi(\beta^*(x_2)) \qquad (3) \end{aligned}$$

Since f is bijection, then φ is a bijection. Now, we show that φ is well-defined. Let $y_1, y_2 \in Y$. Then there exist the unique elements $x_1, x_2 \in X$

such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now, by Equation (3) and Lemma 5.6, $\varphi(\beta^*(x_1)) = \varphi(\beta^*(x_2))$ if and only if $\beta^*(f(x_1)) = \beta^*(f(x_2))$ if and only if $\beta^*(x_1) = \beta^*(x_2)$. Therefore, φ is well-defined and one to one and by Equation (3), is a homomorphism. Hence φ is an isomorphism. Therefore, $(\frac{(X,\oplus_X,*_X,0_X)}{\beta^*},\overline{\oplus}) \cong (\frac{(Y,\oplus_Y,*_Y,0_Y)}{\beta^*},\overline{\oplus})$. □

Definition 5.8. *An MV-algebra $(M,\oplus_M,*_M,0_M)$, is called a fundamental MV-algebra, if there exists a nontrivial hyper MV-algebra $(N,\oplus_N,*_N,0_N)$, such that* $(\frac{(N,\oplus_N,*_N,0_N)}{\beta^*},\overline{\oplus}) \cong (M,\oplus_M,*_M,0_M)$.

Theorem 5.9. *Every MV-algebra can be a fundamental MV-algebra.*

Proof. Let $(M,\oplus_M,*_M,0_M)$ be an MV-algebra. Then by Theorem 4.1, for any MV-algebra $(N,\oplus_N,*_N,0_N)$, $(M\times N,\oplus,*,(0_M,0_N))$ is a hyper MV-algebra. First, we show that for any $(a,b)\in M\times N$, $\beta^*(a,b) = \{(a,x) \mid x\in N\}$. For this let, $u = \bigoplus_{i=1}^{n}(m_i,n_i)\in \ell(M\times N)$, where $(m_i,n_i)\in M\times N$. We have

$$u = \bigoplus_{i=1}^{n}(m_i,n_i) = \{(\bigoplus_{i=1}^{n} m_i, x) \mid m_i\in M, x\in N\}$$

Now, if affect the unary operation $*$ on element u, then we obtain the type $u = \{(a,x_i) \mid a\in M \textit{ is fixed and } x_i\in N\}$. Hence, for any $(a,b),(c,d)\in M\times N$, $(a,b)\beta^*(c,d)$ if and only if $a=c$. Now, we define the map $\varphi : (\frac{(M\times N,\oplus,*,(0_M,0_N))}{\beta^*},\overline{\oplus}) \longrightarrow (M,\oplus_M,*_M,0_M)$ by $\varphi(\beta^*(m,n)) = m$. It is clear that $\beta^*(m,n) = \beta^*(m',n')$ if and only if $m = m'$ if and only if $\varphi(\beta^*(m,n)) = \varphi(\beta^*(m',n'))$. Then, φ is well defined and one to one. In follow, we show that φ is a homomorphism. For this we have,

$$\begin{aligned}\varphi(\beta^*(m,n)\overline{\oplus}\beta^*(m',n')) &= \varphi(\beta^*(m\oplus_M m',n)) = m\oplus_M m' \\ &= \varphi(\beta^*(m,n))\oplus_M \varphi(\beta^*(m',n')).\end{aligned}$$

Moreover, by Lemma 5.2, for any $m\in M$, $\varphi((\beta^*(m,n))^*) = \varphi(\beta^*(m^*,n^*))$ $= m^* = (\varphi(\beta^*(m,n))^*$ and $\varphi(\beta^*(0_M,0_N)) = 0_M$ Clearly, φ is onto. Therefore, φ is an isomorphism. □

Corollary 5.10. *From every infinite countable set we can construct a fundamental MV-algebra.*

Proof. By Corollary 3.6, there exists a binary operation "$\oplus$", a unary operation "$*$" and constant "0" such that $(M,\oplus,*,0)$ is an MV-algebra. Now by Theorem 5.9, $(M,\oplus,*,0)$ is a fundamental MV-algebra. □

Theorem 5.11. *Let* $(M, \oplus, *, 0)$ *be any finite* MV*-algebra. Then for any binary hyperoperation* "$\oplus$", *unary operation* "$*$" *and constant* "0" *on* M, *such that* $(M, \oplus, *, 0)$ *is a hyper* MV*-algebra, there is not any isomorphic between* $(M, \oplus, *, 0)$ *and* $(\frac{(M,\oplus,*,0)}{\beta^*}, \overline{\oplus})$, *that is* $(M, \oplus, *, 0) \ncong (\frac{(M,\oplus,*,0)}{\beta^*}, \overline{\oplus})$.

Proof. Let $(M, \oplus, *, 0)$ be a finite MV-algebra, $|M| = n$ and "$\oplus$" be a hyperoperation, "$*$" unary operation and "0" constant on M, such that $(M, \oplus, *, 0)$ be a hyper MV-algebra. Then there exist $x, y \in M$ such that $|x \oplus y| \geq 2$. Hence, there are $m, n \in x \oplus y$ such that $\beta^*(m) = \beta^*(n)$. Since $\frac{M}{\beta^*} = \{\beta^*(x) \mid x \in M\}$, then, $|\frac{M}{\beta^*}| < n = |M|$. Therefore, $(\frac{(M,\oplus,*,0)}{\beta^*}, \overline{\oplus}) \ncong (M, \oplus, *, 0)$. □

Now, in the follow we try to show that for any infinite countable set M, there exist an operation "$\oplus$", an unary operation $*$ and constant 0 and a hyperoperation "$\oplus$" on M, such that $(M, \oplus, *, 0)$ is an MV-algebra and $(M, \oplus, *, 0)$ is a hyper MV-algebra. Moreover, $\frac{(M,\oplus,*,0)}{\beta^*} \cong (M, \oplus, *, 0)$.

Theorem 5.12. *Let* M *be an infinite countable set. Then there exist an operation* "$\oplus$", *a unary operation* "$*$" *and constant* "0" *and a binary hyperoperation* "$\odot$" *on* M *such that* $(\frac{(M,\odot,*,0)}{\beta^*}, \overline{\oplus}) \cong (M, \oplus, *, 0)$. *That is,* M *is a fundamental* MV*-algebra of itself.*

Proof. Let M be an infinite countable set. Then by Corollary 5.10, there exist a binary operation "$\oplus_M$", a unary operation "$*$" and constant "0_M" such that $(M, \oplus_M, *, 0_M)$ is an MV-algebra. Moreover, by Corollary 4.4, there exist a binary hyperoperation "$\oplus$", a unary operation "$*$" and constant "$(0_M, a)$" such that $(M \times \{a, b\}, \oplus, *, (0_M, a))$ is a strong hyper MV-algebra and by Theorem 5.7, there exist a binary hyperoperation "$\odot$", a unary operation "$*$" and constant "0" such that $(M, \odot, *, 0)$ is a strong hyper MV-algebra and

$$\frac{(M \times \{a, b\}, \oplus, *, (0_M, a))}{\beta^*} \cong \frac{(M, \odot, *, 0)}{\beta^*} \tag{4}$$

First, we show that for any $(m, t) \in M \times \{a, b\}$, $\beta^*(m, t) = \{(m, a), (m, b)\}$. For this let $u = \bigoplus_{i=1}^{n} (m_i, n_i) \in \ell(M \times \{a, b\})$, where $(m_i, n_i) \in M \times \{a, b\}$. We have

$$u = \bigoplus_{i=1}^{n} (m_i, n_i) = \{(\bigoplus_{i=1}^{n} m_i, a), (\bigoplus_{i=1}^{n} m_i, b)\}$$

Now, if affect the unary operation $*$ on element u. Then we obtain the type $u = \{(m,a),(m,b) \mid m \in M \ is \ fixed\}$ too. Hence, for any $(m,t),(n,s) \in M \times \{a,b\}$. $(m,t)\beta^*(n,s)$ if and only if $m = n$.
Now, we define the map $\varphi : (\frac{(M\times\{a,b\},\oplus,*,(0_M,a))}{\beta^*}, \overline{\oplus}) \longrightarrow (M, \oplus_M, *_M, 0_M)$ by $\varphi(\beta^*(m,t)) = m$. It is clear that $\beta^*(m,t) = \beta^*(m',s)$ if and only if $m = m'$ if and only if $\varphi(\beta^*(m,t)) = \varphi(\beta^*(m',s))$. Then, φ is well defined and one to one. Now, we show that φ is a homomorphism.
For this we have,

$$\begin{aligned}\varphi(\beta^*(m,t)\overline{\oplus}\beta^*(m',s)) &= \varphi(\beta^*(m \oplus_M m', t)) = m \oplus_M m' \\ &= \varphi(\beta^*(m,t)) \oplus_M \varphi(\beta^*(m',s)).\end{aligned}$$

Moreover, by Lemma 5.2, for any $m \in M$, $\varphi((\beta^*(m,t))^*) = \varphi(\beta^*(m^*,t^*))$ $= m^* = (\varphi(\beta^*(m,t))^*$ and $\varphi(\beta^*(0_M,a)) = 0_M$ Clearly, φ is onto. Hence, φ is an isomorphism and so

$$(5) \qquad (\frac{(M\times\{a,b\},\oplus,*,(0_M,a))}{\beta^*}, \overline{\oplus}) \cong (M, \oplus_M, *_M, 0_M)$$

Therefore, by (4) and (5), we have

$$(M, \oplus_M, *_M, 0_M) \cong \frac{(M\times\{a,b\},\oplus,*,(0_M,a))}{\beta^*} \cong \frac{(M,\odot,*,0)}{\beta^*}$$

□

Open Problem 5.13. *If $(M,\oplus,*,0)$ is an infinite non-countable MV-algebra, then is it $(M,\oplus,*,0)$ as a fundamental MV-algebra of itself?*

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Mohammad Hamidi
Department of Mathematics, Payame Noor University, Tehran, Iran
E-mail: m.hamidi@pnu.ac.ir

Rajab Ali Borzooei
Department of Mathematics, Shahid Beheshti University, Tehran, Iran.
E-mail: borzooei@sbu.ac.ir