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# SOME PROPERTIES OF LATTICE-BASED $K$- AND $M$-MAPS

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**Abstract.** The recent papers [7, 12] developed two maps named by an $LA$- [7] and an $LMA$-map [12]. The present paper studies their properties and further, develops generalized versions of an $LA$- and an $LMA$-map, which makes the $LA$- [7] and the $LMA$-map [12] improved.

## 1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}^n$, and $\mathbf{R}^n$ represent the set of natural numbers, points in the Euclidean $n$D space with integer coordinates, $n$D real numbers, respectively.

To digitize subspaces of $\mathbf{R}^n$ into some digital spaces in $\mathbf{Z}^n$, we have often used graph theory and locally finite topological structures such as Marucs-Wyse ($M$-, for short) topology, Khalimsky ($K$-, for brevity) topology, and so forth [1, 6, 18, 21]. Motivated by the Alexandroff topology [1], $M$-topology, denoted by $(\mathbf{Z}^2, \gamma)$, was established in [21] to study topological properties of the subspace $(X, \gamma_X)$ induced by $(\mathbf{Z}^2, \gamma)$. Using an $M$-continuous map, we used to study $M$-topological spaces. Based on an $M$-topological graph derived from $M$-topology, the recent paper [8] developed an $MA$-map which is broader than an $M$-continuous map.

To study the digitization of subspaces $(X, E^2_X)$ of $(\mathbf{R}^2, E^2)$ in the $M$-topological approach, the papers [10, 12, 14] proposed an $MA$-digitization and studies it properties, which can be used to study some properties

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of a *lattice-based M-adjacency* ($LMA$-, for short) map. This approach plays an important role in $MA$-digitizing subspaces of $\mathbf{R}^2$.

Meanwhile, to study the digitization of subspaces $(X, E^n_X)$ of $(\mathbf{R}^n, E^n)$ in a $K$-topological approach, the paper [7, 14] proposed a $KA$-digitization and further, the paper [7] proposed the notion of *lattice-based K-adjacency* ($LA$-, for short) map, which also contributed to a $KA$-digitization of subspaces of $\mathbf{R}^n$.

The rest of the paper proceeds as follows: Section 2 provides some basic notions from digital topology. Section 3 investigates some properties of an $MA$-digitization of subspaces of $\mathbf{R}^2$ and further, generalizes the notion of an $LMA$-map, which improves the original version of an $LMA$-map in [12]. Section 4 studies some properties of a $KA$-digitization of subspaces of $\mathbf{R}^n$ and further, generalizes the $LA$-map in [7], which improves the $LA$-map. Section 5 concludes the paper with concluding remarks and a further work.

## 2. Preliminaries

Since the notion of digital connectivity plays an important role in both digital topology and digital geometry, this section starts with basic notions of the graph theoretical approach, *i.e. Rosenfeld model* [19, 20]. In relation to the study of digital spaces in $\mathbf{Z}^n$, if we follow the graph theoretical approach, then a digital picture is usually represented as a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$ [17], a black points set $X \subset \mathbf{Z}^n$ is the set of points we regard as belonging to the image depicted, $k$ represents as an adjacency relation for $X$ and $\bar{k}$ represents an adjacency relation for white points set $\mathbf{Z}^n \setminus X$ and $k \neq \bar{k}$ except $(\mathbf{Z}, 2, 2, X)$ [17]. We say that the pair $(X, k)$ is a digital image in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$. To study multi-dimensional digital spaces in $\mathbf{Z}^n$, we need to use the $k$-adjacency relations of $\mathbf{Z}^n$ [3, 5, 6] as a generalization of $k$-adjacency relations of $\mathbf{Z}^n$ [19] as follows:
for a natural number $m$ with $1 \leq m \leq n$, two distinct points

$$p = (p_i)_{i \in [1,n]_{\mathbf{Z}}} \text{ and } q = (q_i)_{i \in [1,n]_{\mathbf{Z}}} \text{ in } \mathbf{Z}^n$$

are called $k(m, n)$- ($k$-, for short)adjacent if at most $m$ of their coordinates differ by $\pm 1$ and all others coincide.

Namely, these $k(m, n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the two numbers $m, n \in \mathbf{N}$ [4] (see also [3, 5, 6]).

In terms of the above operator, the $k$-adjacency relations of $\mathbf{Z}^n$ [8] are obtained, as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.1}$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$.

For $\{a,b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a,b]_{\mathbf{Z}} = \{a \le n \le b \,|\, n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z}, 2, 2, [a,b]_{\mathbf{Z}})$ [17]. However, the present paper is not concerned with the $\bar{k}$-adjacency of $\mathbf{Z}^n \setminus X$.

For a digital image $(X,k)$ in $\mathbf{Z}^n$, two points $x, y \in X$ are $k$-connected if there is a $k$-path from $x$ to $y$ in $X \subset \mathbf{Z}^n$. We say that a digital image $(X,k)$ is $k$-connected with $|X| \ge 1$, if for two distinct points $x, y \in X$ there is a $k$-path in $X$ connecting these points. Besides, a singleton digital image is assumed to be $k$-connected. For a $k$-adjacency relation of (2.1), a simple $k$-path from $x$ to $y$ in $\mathbf{Z}^n$ is assumed to be the injective sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that $x_i$ and $x_j$ are $k$-adjacent if and only if $|i-j| = 1$ [17] and further, $x_0 = x$ and $x_l = y$. The length of this simple $k$-path, denoted by $l_k(x,y)$, is the number $l$.

To study both digital continuity and various properties of a digital space, we have often used the following digital $k$-neighborhood. Using the above adjacency relations of (2.1), we say that a digital $k$-neighborhood of $p \in \mathbf{Z}^n$ is the set [19] $N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\} \subset \mathbf{Z}^n$. Furthermore, we often use the notation [17] $N_k^*(p) := N_k(p) \cup \{p\}$.

For a digital image $(X,k)$ a digital $k$-neighborhood of $x_0 \in X$ with radius $\varepsilon$ is defined in $X$ to be the following subset of $X$ [4]

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \le \varepsilon\} \cup \{x_0\}, \tag{2.2}$$

where $l_k(x_0, x)$ is the length of a shortest simple $k$-path from $x_0$ to $x$ and $\varepsilon \in \mathbf{N}$. Indeed, for $X \subset \mathbf{Z}^n$ we obtain [3]

$$N_k(x, 1) = N_k^*(x) \cap X. \tag{2.3}$$

For a function $f : (X, k_0) \to (Y, k_1)$, to map every $k_0$-connected subset of $(X, k_0)$ into a $k_1$-connected subset of $(Y, k_1)$, the paper [20] established the notion of digital continuity.

## 3. Generalization of an $LMA$-map

The recent paper [12] established the notion of an $LMA$-map associated with both $M$-topology and an $M$-localized neighborhood in [10].

Let us now review basic concepts from $M$-topology. The $M$-topology on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced by the set $\{U\}$ in (3.1) as a base [21], where for each point $p = (x, y) \in \mathbf{Z}^2$

$$U := \left\{ \begin{array}{l} N_4^*(p) \text{ if } x+y \text{ is even, and} \\ \{p\} : \text{ else.} \end{array} \right\} \tag{3.1}$$

In view of (3.1), $(\mathbf{Z}^2, \gamma)$ is a $T_{\frac{1}{2}}$ space and further, a semi-$T_{\frac{1}{2}}$ space [2]. Hence each singleton from $(\mathbf{Z}^2, \gamma)$ either a closed set (denoted by $\Diamond$ in the Fig. 1 and 2) or an open set(denoted by an ordinary dot in the Fig. 1 and 2). Since $(\mathbf{Z}^2, \gamma)$ is an Alexandroff space, for a point $p \in X$ the paper will denote by $SN_M(p)$ the *smallest open set* containing the point $p \in \mathbf{Z}^2$.

For two $M$-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be $M$-continuous at a point $x \in X$ [8] if

$$f(SN_M(x)) \subset SN_M(f(x)). \tag{3.2}$$

Furthermore, a map $f : X \to Y$ is $M$-continuous if it is $M$-continuous at every point $x \in X$.

Using $M$-continuous maps, we obtain the $M$-topological category, denoted by $MTC$ [10, 8], consisting of the following two sets:

- the set of $M$-topological spaces $(X, \gamma_X)$ as objects, denoted by $Ob(MTC)$,
- for every ordered pair of objects $(X, \gamma_X)$ and $(Y, \gamma_Y)$, the set of all $M$-continuous maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms.

Meanwhile, by using an $M$-topological graph derived from $M$-topology, the recent paper [8] developed the so-called $MA$-map which is broader than an $M$-continuous map. Indeed, an $M$-topological graph is not a topological space but a just digital graph on $\mathbf{Z}^2$. This approach can be substantially helpful to study geometric transformations of $M$-topological spaces $(X, \gamma_X)$. For a space $(X, \gamma_X) := X$ we consider an *$MA$-relation* of a point $p \in X$ as follows:

**Definition 1.** [8] *For $(X, \gamma_X)$ we say that two distinct points $x$ and $y$ in $X$ are $M$-adjacent in $(X, \gamma_X)$ if $y \in SN_M(x)$ or $x \in SN_M(y)$.*

In Definition 1 we say that the two points $p, q$ have an $MA$-relation or $p$ is $MA$-related to $q$. Hereafter, a set $X$ with $M$-adjacency as a forgetful $M$-topological space $(X, \gamma_X) := X$ is called an $MA$-space (or a space if there is no danger of ambiguity) in this paper.

For a point $p \in \mathbf{Z}^2$ we obtain the following $MA$-neighborhood of a point $p \in \mathbf{Z}^2$ [8].

$$MA(p) = N_4(p). \tag{3.3}$$

By using the property of (3.3), we can represent an *$MA$-relation* of two points $p, q$ of $X$ as follows. For an $MA$-space $(X, \gamma_X) := X$ put $MA_X(p) := MA(p) \cap X$ [8]. We say that for two distinct points $p, q \in X$ they are $M$-adjacent to each other if $q \in MA_X(p)$ (or $p \in MA_X(q)$).

**Definition 2.** [8] *For an $MA$-space $(X, \gamma_X) := X$ and a point $p \in X$ we define an $MA$-neighborhood of $p$ to be the set $MA_X(p) \cup \{p\} := MN_X(p) \subset X$.*

Hereafter, in an $MA$-space $(X, \gamma_X)$ we use the notation $MN(p)$ instead of $MN_X(p)$ if there is no danger of ambiguity. Indeed, we obtain the following [8]:

$$MN(p) = N_4(p, 1). \tag{3.4}$$

For a space $(X, \gamma_X) := X$ and each point $x \in X$, owing to the Alexandroff topological structure of $(X, \gamma_X)$, it is clear that each point $x \in X$ always has $MN(x) \subset X$ so that we have the following version.

**Definition 3.** [8] *For two $MA$-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, we say that a function $f : X \to Y$ is an $MA$-map at a point $x \in X$ if*

$$f(MN(x)) \subset MN(f(x)).$$

*Furthermore, we say that a map $f : X \to Y$ is an $MA$-map if the map $f$ is an $MA$-map at every point $x \in X$.*

Using $MA$-maps, we obtain an *$MA$-category* [8], denoted by $MAC$, consisting of the following two sets.

(1) The set of $MA$-spaces $(X, \gamma_X)$ as objects, denoted by $Ob(MAC)$,

(2) For every ordered pair of objects $(X, \gamma_X)$ and $(Y, \gamma_Y)$, the set of all $MA$-maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms.

To study an $MA$-digitization of $\mathbf{R}^2$, for a point $p \in \mathbf{Z}^2$, we have used an $M$-localized neighborhood of the given point $p$, denoted by $N_M(p)$ (see Definition 4). And the structure of this neighborhood is substantially related to the $M$-topological structure (see Fig. 1).

**Definition 4.** [10] *In* $\mathbf{R}^2$, *for a point* $p := (p_1, p_2) \in \mathbf{Z}^2$, *we define the following neighborhood of* $p$*: for* $i \in \{1, 2\}$

$$N_M(p) := \begin{cases} \{(t_1, t_2) \,|\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \\ \quad \text{if } p = (p_1, p_2) \text{ is a double even point, and} \\ \{(t_1, t_2) \,|\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \setminus \{(p_1 \pm \frac{1}{2}, p_2 \pm \frac{1}{2})\} \\ \quad \text{if } p = (p_1, p_2) \text{ is an even point, and} \\ \{(t_1, t_2) \,|\, t_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2})\} \text{ if } p = (p_1, p_2) \text{ is an odd point} \end{cases}$$

*which is called an* $M$*-localized neighborhood of* $p$ *associated with* $(\mathbf{Z}^2, \gamma)$.

In Fig. 1 (a)-(c), depending on the given point $p \in \mathbf{Z}^2$, we obtain their corresponding $M$-localized neighborhoods $N_M(p) \subset \mathbf{R}^2$ [10].

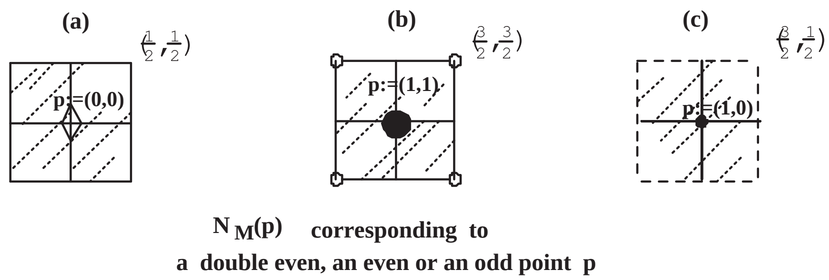


FIGURE 1. Configuration of an $M$-localized neighborhood of a point $p \in \mathbf{Z}^2$ [10]

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**Definition 5.** [10] *For two points* $x$, $y \in (\mathbf{R}^2, E^2)$, *we say that* $x$ *is related to* $y$ *if* $x$, $y \in N_M(p)$ *for some point* $p \in \mathbf{Z}^2$, *denoted by* $x \sim_M y$.

The relation " $\sim_M$" of Definition 5 is an equivalence relation [10]. Besides, the set $\{N_M(p) \,|\, p \in \mathbf{Z}^2\}$ is a partition of $\mathbf{R}^2$ associated with $M$-topology[10].

After defining an $M$-digitization of $(X, E^2_X) \subset (\mathbf{R}^2, E^2)$ (see Definition 6), we shall propose an $MA$-digitization of $X$ (see Definition 8).

**Definition 6.** *For a non-empty space* $(X, E^2_X) \subset (\mathbf{R}^2, E^2)$, *we define an* $M$*-digitization of* $X$, *denoted by* $D_M(X)$, *to be the space under* $M$*-topology*

$$D_M(X) := \{p \in \mathbf{Z}^2 \,|\, N_M(p) \cap X \neq \emptyset\}.$$

**Definition 7.** [14] *For a space $(X, E_X^2)$ and two points $p, q \in X$, we say that the point $p$ is related to $q$ if there is a point $x \in D_M(X)$ such that $p, q \in N_M(x)$.*

After digitizing $X$ in the $M$-topological approach, we define the following:

**Definition 8.** [12] *We say that $D_{MA}(X)$ is the set $D_M(X)$ with $M$-adjacency.*

Indeed, it is clear that $D_{MA}(X)$ is an $MA$-space with the set $D_M(X)$ and $M$-adjacency on $D_M(X)$.

Combining an $M$-localized neighborhood of Definition 4 with an $MA$-map, we define the following map which is used to classify subspaces of $\mathbf{R}^2$ associated with $M$-topology.

**Definition 9.** [12] *Consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ such that*

$$\begin{cases} (1)\, D_{MA}(F) : D_{MA}(X) \to D_{MA}(Y) \text{ is an } MA\text{-map,} \\ \quad \text{where } D_{MA}(F) := f \text{ is the restriction of } F \text{ to } D_{MA}(X) \\ \quad \text{with the codomain } D_{MA}(Y); \\ (2)\, \text{For any point } p \in D_M(X), F(N_M(p) \cap X) \subset N_M(f(p)) \cap Y. \end{cases}$$

*Then we say that the map $F$ is a lattice-based $MA$- ($LMA$-, for short) map.*

Motivated by the $L(K)$-property of [9], let us now develop the notion of a generalized $LMA$-map as follows:

**Definition 10.** *Consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ such that $D_{MA}(F) := f : D_{MA}(X) \to D_{MA}(Y)$ is an $MA$-map, where $D_{MA}(F) := f$ is induced by $F$ satisfying that for any point $p \in D_{MA}(X)$,*

$$\left\{ \begin{array}{l} F(N_M(p) \cap X) \subset N_M(f(p)) \cap Y \text{ and } f \text{ maps } p \text{ to } q_i, \\ \text{where } \{q_i \in \mathbf{Z}^2 \mid N_M(q_i) \cap F(N_M(p) \cap X) \neq \emptyset\} \subset D_{MA}(Y). \end{array} \right\}$$

*Then we say that the map $F$ is a generalized $LMA$-map.*

**Proposition 3.1.** *The map of Definition 10 is a generalization(an improvement) of an $LMA$-map.*

*Proof:* To prove the assertion, we need to recall the process of an $MA$-digitization of $(X, E_X^2)$. According to an algorithm [14, 10] for digitizing a space $X \in Ob(LMAC)$ into an $MA$-space $D_{MA}(X) \in Ob(MAC)$ (see

Fig. 2), we can proceed with an $MA$-digitization of $(X, E_X^2)$ in such a way [12, 14].

(Step 1) Take the points $p \in \mathbf{Z}^2$ such that $N_M(p) \cap X \neq \emptyset$ and put $X' := \{p \in \mathbf{Z}^2 \mid N_M(p) \cap X \neq \emptyset\}$.

(Step 2) For each point $p \in X'$ take $N_M(p) \subset \mathbf{R}^2$ and further, consider $N_M(p) \cap X$.

(Step 3) Delete the set $\mathbf{R}^2 \setminus \cup_{p \in X'} N_M(p)$ from $\mathbf{R}^2$.

(Step 4) Recognize the set $N_M(p) \cap X$ to be the singleton set $\{p\} \subset \mathbf{Z}^2$ with $N_M(p) \cap X := p \in D_M(X)$.

(Step 5) Adopting $M$-adjacency into the space $(D_M(X), \gamma_{D_M(X)}) \in Ob(MTC)$, we finally obtain $D_{MA}(X) \in Ob(MAC)$(see Definition 8).

Then the digitized space $D_{MA}(X)$ is called an $MA$-space $(X', \gamma_X)$ (see Fig. 2).

To prove the main assertion, we need to consider the following two cases:

(Case 1): Consider the case $D_{MA}(X)$ is a subset of $(X, E_X^2)$ (see Fig.2(1)). It is clear that an $LMA$-map is equivalently considered to be the map of Definition 10.

For instance, in $(\mathbf{R}^2, E^2)$, assume that $X = \cup_{i \in \{1,2,3,4\}} N_M(x_i)$, $Y = \cup_{i \in \{1,2\}} N_M(y_i)$ (see Fig. 2(1)). Let us consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ in Fig. 2(1) given by $F(\{x_1, x_2\}) = \{y_1\}, F(\{x_3, x_4\}) = \{y_2\}$ and further, for each point $x_i \in D_{MA}(X) := X' = \{x_1, x_2, x_3, x_4\}$ we further assume that $F(N_M(x_i) \cap X) \subset N_M(f(x_i)) \cap Y$ (see Fig. 2(1)), where the map $f$ is the restriction of $F$ to $X'$ with the codomain $D_{MA}(Y) := Y'$. Namely, we obtain $f : D_{MA}(X) \to D_{MA}(Y)$ given by $f(\{x_1, x_2\}) = \{y_1\}, f(\{x_3, x_4\}) = \{y_2\}$. Then both the maps $F$ and $f$ satisfy the property of Definition 9. Furthermore these maps also satisfy the property of Definition 10 as follows; $D_{MA}(F) : D_{MA}(X) \to D_{MA}(Y)$ is an $MA$-map, where $D_{MA}(F) := f$ is induced by $F$ satisfying that for any point $p \in D_{MA}(X), F(N_M(p) \cap X) \subset N_M(f(p)) \cap Y$.

Conversely, the property of Definition 10 can support that of Definition 9.

(Case 2): Consider the case $D_{MA}(X)$ is not a subset of $(X, E_X^2)$ (see Fig.2(2)). At this moment we can see that $D_{MA}(X)$(resp. $D_{MA}(Y)$) is not a subset of $(X, E_X^2)$ (resp. $(Y, E_Y^2)$). Owing to this situation, we cannot use the property of Definition 9 into this case. However, according to the property of Definition 10, we can consider the map $F : (X, E_X^2) \to (Y, E_Y^2)$ such that $D_{MA}(F) : D_{MA}(X) \to D_{MA}(Y)$ is an $MA$-map, where, $D_{MA}(F) := f$ is induced by $F$ satisfying that for any point $p \in D_{MA}(X), F(N_M(p) \cap X) \subset N_M(f(p)) \cap Y$.

For instance, consider the map $G : (X, E_X^2) \to (Y, E_Y^2)$ in Fig.2(2) with the following assumption. First of all, we need to remind that $D_{MA}(X)$ is not a subset of $(X, E_X^2)$. Next, consider the map $G$ (see Fig.2(2)) given by $G(N_M(x_i) \cap X) \subset N_M(g(x_i)) \cap Y$, where the map $g$ is the $MA$-map from $D_{MA}(X)$ to $D_{MA}(Y)$ induced by the given map $G$ in such a way: $D_{MA}(G) : D_{MA}(X) \to D_{MA}(Y)$ is an $MA$-map, where $D_{MA}(G) := g$ is induced by $G$ satisfying that for any point $p \in D_{MA}(X), G(N_M(p) \cap X) \subset N_M(g(p)) \cap Y$. This implies that the map $G$ is a generalized $LMA$-map. But it is clear that the property of Definition 10 cannot support that of Definition 9. $\square$

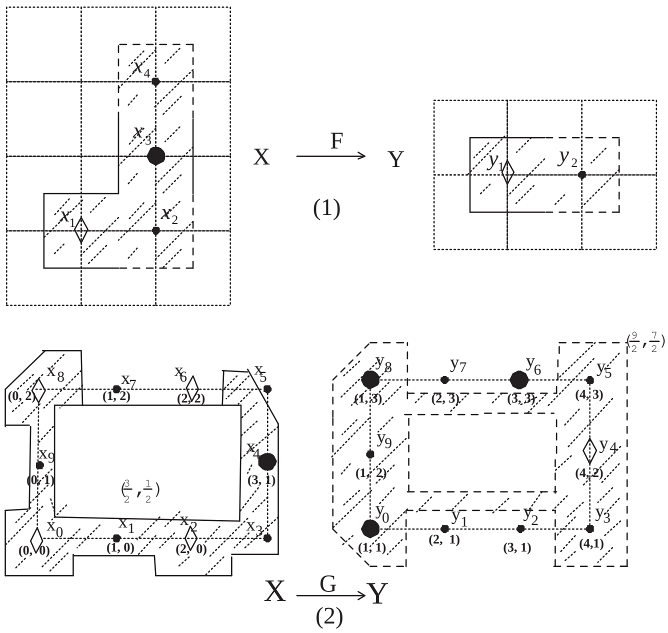


FIGURE 2. Configuration of an $LMA$-map

In view of Definitions 9 and 10, we can see some merits of a generalized $LMA$-map as follows:

**Remark 3.2.** *Comparing an $LMA$-map with a generalized $LMA$-map, we see that the latter can be considered to an improved version of an $LMA$-map. Hence, hereafter we will use the generalized $LMA$-map of Definition 10 instead of the $LMA$-map of Definition 9(see below). Furthermore, for convenience, we will call it just an $LMA$-map instead of*

*a generalized $LMA$-map. Thus we have the following: when considering spaces $(X, E_X^2)$ as objects and $LMA$-maps as morphisms, we denote by $LMAC$ the $LMA$-category consisting of the following two sets:*
*(∗ 1) the set of spaces $(X, E_X^2) := X$ as objects of $LMAC$ denoted by $Ob(LMAC)$;*
*(∗ 2) the set of $LMA$-maps between all pairs of elements in $Ob(LMAC)$ as morphisms of $LMAC$ denoted by $Mor(LMAC)$.*

**Remark 3.3.** *The $LMA$-map of the paper* [12] *can be replaced by the current generalized $LMA$-map.*

## 4. Generalization of an $LA$-map

The recent paper [7] established the notion of an $LA$-map associated with $KA$-digitization [14]. Indeed, the notion is associated with a $K$-topological graph derived from $K$-topology. Let us now recall the $n$D $K$-topological space, denoted by $(\mathbf{Z}^n, \kappa^n)$. It turns out that this topological space is an Alexandroff space [1] and a semi-$T_{\frac{1}{2}}$ space [2]. For a set $X \subset \mathbf{Z}^n$ we consider the subspace $(X, \kappa_X^n)$ induced by $(\mathbf{Z}^n, \kappa^n)$. Under $(X, \kappa_X^n)$, for a point $p \in X$ the paper will denote by $SN_K(p)$ the smallest open neighborhood of the given point $p$.

For two $K$-topological spaces $(X, \kappa_X^n) := X$ and $(Y, \kappa_Y^n) := Y$, if a function $f : X \to Y$ satisfies the following property

$$f(SN_K(x)) \subset SN_K(f(x)), \tag{4.1}$$

then we say that the map $f$ is $K$-continuous at a point $x \in X$ [16, 11] if $f$ is continuous at the point $x$ from the viewpoint of $K$-topology. Furthermore, a map $f : X \to Y$ is Khalimsky ($K$-, for short) continuous if it is $K$-continuous at every point $x \in X$.

By using $K$-continuous maps, we obtain a category of $K$-topological spaces, denoted by $KTC$ [11], consisting of the following two sets:
• the set of $(X, \kappa_X^n)$ as objects, denoted by $Ob(KTC)$,
• for every ordered pair of objects $(X, \kappa_X^n)$ and $(Y, \kappa_Y^n)$, the set of all $K$-continuous maps $f : (X, \kappa_X^n) \to (Y, \kappa_Y^n)$ as morphisms.

By using the $K$-topological graph derived from $K$-topology, the recent paper [11] developed the so-called $A$-map (see Definition 13 in the present paper) which is broader than a $K$-continuous map. This approach can be substantially helpful to study geometric transformations of $K$-topological spaces $(X, \kappa_X^n)$.

**Definition 11.** [16] *For $(X, \kappa_X^n)$ we say that two distinct points $x$ and $y$ in $X$ are $K$-adjacent if $y \in SN_K(x)$ or $x \in SN_K(y)$.*

Considering $(X, \kappa_X^n)$ with $K$-adjacency, we call it a *topological graph derived from $K$-topology* (for short *$K$-topological graph or $KA$-space*). Namely, a *$K$-topological graph* is not a topological space but a just digital graph on $\mathbf{Z}^n$.
In relation to the establishment of an $A$-map, we will use the following $K$-adjacency neighborhood of a point $p \in X$.

**Definition 12.** [11] *For a $KA$-space $(X, \kappa_X^n) := X$ and a point $p \in X$ we define a $K$-adjacency neighborhood of $p$ to be the set $A_X(p) \cup \{p\} := AN_X(p)$ which is called an $A$-neighborhood of $p$.*

Hereafter, we will briefly use $AN(p)$ instead of $AN_X(p)$ if there is no danger of ambiguity. For a $KA$-space $(X, \kappa_X^n) := X$ and each point $x \in X$, since for every $x \in X$ there are always $AN(x) \subset X$ and $AN(f(x)) \subset Y$, we can develop an $A$-map (see Definition 12).

**Definition 13.** [11] *For two $KA$-spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$, we say that a function $f : X \to Y$ is an $A$-map at a point $x \in X$ if*

$$f(AN(x)) \subset AN(f(x)).$$

*Furthermore, we say that a map $f : X \to Y$ is an $A$-map if the map $f$ is an $A$-map at every point $x \in X$.*

Using $A$-maps, we establish the *$KA$-category* [11], denoted by $KAC$, consisting of the following two sets.

(1) The set of $KA$-spaces as objects, denoted by $Ob(KAC)$,

(2) For every ordered pair of objects $(X, \kappa_X^n)$ and $(Y, \kappa_Y^n)$, the set of all $A$-maps $f : (X, \kappa_X^n) \to (Y, \kappa_Y^n)$ as morphisms.

The paper [7] investigates some properties of a $KA$-digitization of $(X, E_X^n)$. To do this work, for a point $p \in \mathbf{Z}^n$, the papers [7, 14] uses a $K$-localized neighborhood of the given point $p$, denoted by $N_K(p)$, which is substantially related to the $K$-topological structure (see Definition 14).

**Definition 14.** [7] *In $\mathbf{R}^n$, for each point $p := (p_i)_{i \in [1,n]_\mathbf{Z}} \in \mathbf{Z}^n$, we define the set $N_K(p) := \{(x_i)_{i \in [1,n]_\mathbf{Z}}\}$ which is called the local $K$-neighborhood of $p$ associated with $(\mathbf{Z}^n, T^n)$, where*

$$\left\{ \begin{array}{l} \textit{if } p_i = 2m, \ \ \textit{then } x_i \in [2m - \frac{1}{2}, 2m + \frac{1}{2}] \\ \textit{if } p_i = 2m+1, \ \ \textit{then } x_i \in (2m + \frac{1}{2}, 2m + \frac{3}{2}). \end{array} \right\}$$

In Fig.3, for a pure closed point, a mixed point and a pure open point $p$, we obtain their corresponding $K$-localized neighborhoods $N_K(p) \subset \mathbf{R}^n$ [7].

**Definition 15.** [7] *For two points $x, y \in (\mathbf{R}^n, E^n)$, we say that $x$ is related to $y$ if $x, y \in N_K(p)$ for some point $p \in \mathbf{Z}^n$, denoted by $x \sim_K y$.*

The relation " $\sim_K$" of Definition 15 is an equivalence relation[7]. Besides, the set $\{N_K(p) \,|\, p \in \mathbf{Z}^n\}$ is a partition of $\mathbf{R}^n$ associated with $K$-topology [7].

Let us recall the $K$-digitization of a non-empty space $(X, E^n_X)$.

**Definition 16.** [7] *For a non-empty space $(X, E^n_X)$ we define a $K$-digitization of $X$, denoted by $D_K(X)$, to be the space with $K$-topology*

$$D_K(X) := \{p \in \mathbf{Z}^n \,|\, N_K(p) \cap X \neq \emptyset\}.$$

**Definition 17.** [7] *For a space $(X, E^n_X)$ and two points $p, q \in X$, we say that the point $p$ is related to $q$ if there is a point $x \in D_K(X)$ such that $p, q \in N_K(x)$.*

It is clear that the relation of Definition 17 is an equivalence relation [7].

After digitizing $X$ in the $K$-topological approach, we define the following:

**Definition 18.** [7] *We say that $D_{KA}(X)$ is the set $D_K(X)$ with $K$-adjacency.*

In Definition 17, $D_{KA}(X)$ is called a $K$-topological graph derived from $K$-topology ( a $KA$-space for short). More precisely, we can consider $D_{KA}(X)$ as a $K$-topological graph of which a vertex set is $D_K(X)$ and an edge of between two points in $D_K(X)$ is defined in terms of $K$-adjacency of Definition 11. Thus we obtain $D_{KA}(X) \in Ob(KAC)$ as a topological graph derived from $K$-topology.

Combining a $K$-localized neighborhood of Definition 14 with an $A$-map, the paper [7] established the following:

**Definition 19.** [7]

$$\begin{cases} (1)\, D_{KA}(F) : D_{KA}(X) \to D_{KA}(Y) \text{ is an } A\text{-map,} \\ \quad \text{where } D_{KA}(F) := f \text{ is the restriction of } F \text{ to } D_{KA}(X) \\ \quad \text{with the codomain } D_{KA}(Y); \\ (2)\, \text{For any point } p \in D_K(X), F(N_K(p) \cap X) \subset N_K(f(p)) \cap Y. \end{cases}$$

*Then we say that the map $F$ is a lattice-based $K$-adjacency map (an $LA$-map, for short).*

The paper [11] denotes by $LAC$ the category consisting of the following two sets:
($*$ 1) the set of spaces $(X, E^n_X) := X$ as objects of $LAC$ denoted by $Ob(LAC)$;
($*$ 2) the set of $LA$-maps of every ordered pair of elements in $Ob(LAC)$ as morphisms of $LAC$ denoted by $Mor(LAC)$.

Combining a $K$-localized neighborhood of Definition 14 with an $A$-map, motivated by the $L(K)$-property of [9], let us establish the notion of a generalized $LA$-map in such a way.

**Definition 20.** *Consider the map $F : (X, E^n_X) \to (Y, E^n_Y)$ such that $D_{KA}(F) := f : D_{KA}(X) \to D_{KA}(Y)$ is an $A$-map, where $D_{KA}(F) := f$ is induced by $F$ satisfying that for any point $p \in D_{KA}(X)$,*

$$\left\{\begin{array}{l} F(N_K(p) \cap X) \subset N_K(f(p)) \cap Y, \text{ and } f \text{ maps } p \text{ to } q_i, \\ \text{where } \{q_i \in \mathbf{Z}^n \mid N_K(q_i) \cap F(N_K(p) \cap X) \neq \emptyset\} \subset D_{KA}(Y). \end{array}\right\}$$

*Then we say that the map $F$ is a generalized $LA$-map.*

Let us now compare between the maps of Definitions 19 and 20 as follows:

**Proposition 4.1.** *The map of Definition 20 is a generalization(an improvement) of an $LA$-map.*

*Proof:* To prove the assertion, we need to recall the process of $KA$-digitization of $(X, E^n_X)$. According to an algorithm for $KA$-digitization of $(X, E^n_X)$ [7], we obtain $D_K(X) \in Ob(KTC)$ and $D_{KA}(X) \in Ob(KAC)$ (see Definition 18). To be specific, proceed with the following algorithm for establishing $D_{KA}(X) \in Ob(KAC)$ from $(X, E^n_X) \in Ob(LAC)$ [7] as follows;

(Step 1) Take all points $p \in \mathbf{Z}^n$ such that $N_K(p) \cap X \neq \emptyset$ and put $X' := \{p \in \mathbf{Z}^n \mid N_K(p) \cap X \neq \emptyset\}$.

(Step 2) For each point $p \in X'$ take $N_K(p) \subset \mathbf{R}^n$ and further, consider $N_K(p) \cap X$.

(Step 3) Delete the set $\mathbf{R}^n \setminus \cup_{p \in X'} N_K(p)$ from $\mathbf{R}^n$.

(Step 4) Recognize the set $N_K(p) \cap X$ to be a singleton $\{p\} \subset \mathbf{Z}^n$ with $N_K(p) \cap X := p \in D_K(X)$.

(Step 5) After adopting $K$-adjacency into the space $(D_K(X), \kappa^n_{D_K(X)})$, we finally obtain $D_{KA}(X) \in Ob(KAC)$.

To prove that the map of Definition 20 is a generalization of an $LA$-map, we need to consider the following two cases:

(Case 1): Consider the case $D_{KA}(X)$ is a subset of $(X, E^n_X)$ (see Fig.3(1)). It is clear that an $LA$-map is equivalently considered to be the map of Definition 20.
For instance, In Fig.3(1), consider the map $F : (X, E^2_X) \to (Y, E^2_Y)$ given by $F(\{x_1, x_2\}) = \{y_1\}, F(\{x_3, x_4\}) = \{y_2\}$ and further, take the map $D_{KA}(F) := f : D_{KA}(X) \to D_{KA}(Y)$ induced by $F$ in terms of $f(\{x_1, x_2\}) = \{y_1\}, f(\{x_3, x_4\}) = \{y_2\}$. Then the maps $F$ and $f$ satisfy the property of Definition 19. Furthermore, these two maps also satisfy the following:

$F((N_K(x_1) \cup N_K(x_2)) \cap X) \subset N_K(y_1) \cap Y$ and $F((N_K(x_3) \cup N_K(x_4)) \cap X) \subset N_K(y_2) \cap Y$. Since $D_{KA}(X)$ is a subset of $(X, E^n_X)$, it is clear that the given map $F$ is an $LA$-map. Furthermore, we obtain $D_{MA}(F) := f : D_{KA}(X) \to D_{KA}(Y)$ given by $f(\{x_1, x_2\}) = y_1, f(\{x_3, x_4\}) = \{y_2\}$ so that both $F$ and $f$ satisfy the following:
$D_{KA}(F) : D_{KA}(X) \to D_{KA}(Y)$ is an $A$-map, where $D_{KA}(F) := f$ induced by $F$ satisfying that for any point $p \in D_{KA}(X), F(N_K(p) \cap X) \subset N_K(f(p)) \cap Y$.
Conversely, the property of Definition 20 can also support the property of Definition 19.

(Case 2): Consider the case $D_{KA}(X)$ is not a subset of $(X, E^n_X)$ (see Fig.3(2)). Owing to this assumption, we cannot use the property of Definition 19 into this case. However, by using the property of Definition 20, we can consider the map $F : (X, E^n_X) \to (Y, E^n_Y)$ such that $D_{KA}(F) : D_{KA}(X) \to D_{KA}(Y)$ is an $A$-map, where $D_{KA}(F) := f$ induced by $F$ satisfying that for any point $p \in D_{KA}(X), F(N_K(p) \cap X) \subset N_K(f(p)) \cap Y$.

For instance, consider the map $G : (Z, E^2_Z) \to (W, E^2_W)$ in Fig.3(2) with the following assumption. First of all, we need to remind that $D_{KA}(Z)$(resp. $D_{KA}(W)$) is not a subset of $(Z, E^2_Z)$(resp. $(W, E^2_W)$). Next, consider the map $G$ (see Fig.3(2)) given by $G(N_K(z_i) \cap Z) \subset N_K(g(z_i)) \cap Y$, where the map $g$ is an $A$-map from $D_{KA}(Z)$ to $D_{KA}(W)$ induced by the given map $G$ in such a way:
$D_{KA}(G) : D_{KA}(Z) \to D_{KA}(W)$ is an $A$-map, where $D_{KA}(G) := g$ induced by $G$ satisfying that for any point $p \in D_{KA}(Z), G(N_K(p) \cap Z) \subset N_K(g(p)) \cap W$. This implies that the map $G$ is a generalized version of an $LA$-map. But it is clear that the property of Definition 20 cannot support the property of Definition 19 because $D_{KA}(Z)$ is not a subset of $(Z, E^2_Z)$. $\square$

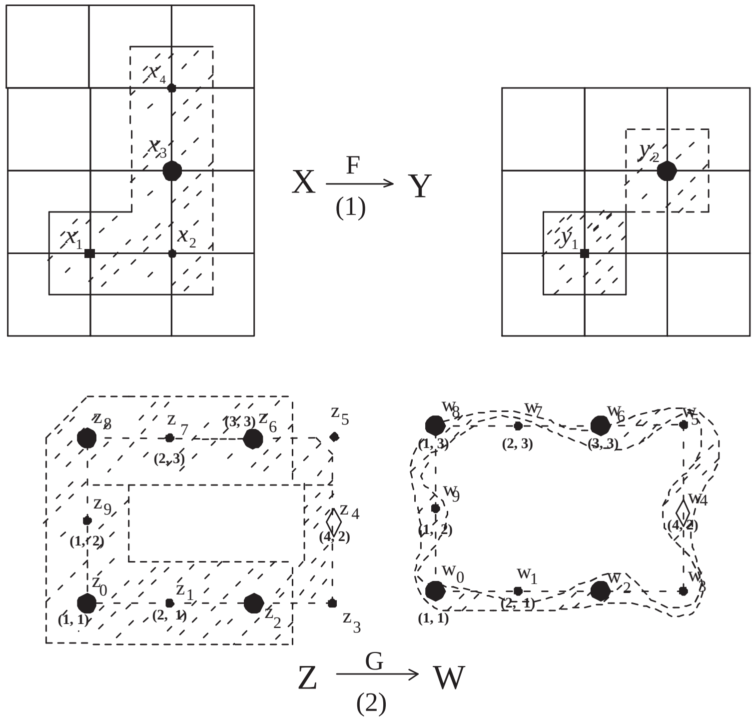


Figure 3. Comparison of an $LA$-map and a generalized $LA$-map (see (1)-(2)).

In view of Definitions 19 and 20, we can see some benefits of a generalized $LA$-map, as follows:

**Remark 4.2.** *The generalized $LA$-map of Definition 20 can be considered to an improved version of an $LA$-map. Hence, hereafter we will use the generalized $LA$-map of Definition 20 instead of the $LA$-map of Definition 19. Furthermore, for convenience, we will call it just an $LA$-map instead of a generalized $LA$-map. Thus we have the following:*
*when considering spaces $(X, E_X^n)$ as objects and $LA$-maps as morphisms, we denote by $LAC$ the $LA$-category consisting of the following two sets:*
*(∗ 1) the set of spaces $(X, E_X^n) := X$ as objects of $LAC$ denoted by $Ob(LAC)$;*
*(∗ 2) the set of $LA$-maps between all pairs of elements in $Ob(LAC)$ as morphisms of $LAC$ denoted by $Mor(LAC)$.*

**Remark 4.3.** *The $LA$-map of the paper* [7] *can be replaced by the current generalized $LA$-map.*

## 5. Summary and further works

We have studied the notions of an $LMA$-map in $LMAC$ and an $LA$-map in $LKAC$. Furthermore, we proposed generalizations of both an $LMA$- and an $LA$-map, which makes both an $LMA$- and an $LA$-map improved. As a further work, by using these improved notions, we can establish the notions of new homotopies in $LMAC$ and $LKAC$.

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