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ON UNIFORM SAMPLING IN SHIFT-INVARIANT SPACES ASSOCIATED WITH THE FRACTIONAL FOURIER TRANSFORM DOMAIN

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Abstract. As a generalization of the Fourier transform, the fractional Fourier transform plays an important role both in theory and in applications of signal processing. We present a new approach to reach a uniform sampling theorem in the shift-invariant spaces associated with the fractional Fourier transform domain.

1. Introduction

A fundamental problem of signal processing is to determine a signal from its partial information, samples, and to reconstruct the signal by the samples. A simple version of the classical Whittaker-Shannon-Kotelnikov (WSK) sampling theorem [8, 12, 17] states that any signal $f(t)$ band-limited to $[-\pi,\pi]$ is determined by its samples $\{f(n) : n \in \mathbb{Z}\}$, and can be reconstructed via

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}(t-n) \tag{1}$$

where $\mathrm{sinc}(t) := \frac{\sin \pi t}{\pi t}$. WSK sampling expansion (1) converges not only in $L^2(\mathbb{R})$, the space of signals with finite energy, but also absolutely and uniformly on $\mathbb{R}$. Let $PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \mathrm{supp}\, \hat{f} \subset [-\pi,\pi]\}$ be the Paley-Wiener space of signals band-limited to $[-\pi,\pi]$ where we take the Fourier transfom (FT) as $\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-it\xi}dt$ for $f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Classical sampling theory explores a fundamental question on sampling and reconstruction of signals in PW_π. See survey article [6] for details.

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Chirp-modulated signals occur often in applications such as sonar and radar. We denote by

$$\vec{f}(t) := e^{iat^2} f(t)$$

the chirp-modulated version of $f(t)$. To investigate sampling theorem of the chirp-modulated signals, the fractional Fourier transform (FrFT) is suitable more than FT, which has been proved useful for dealing with in various applications in signal processing as well as optics [11]. It has also been used to investigate sampling theorem of chirp-modulated band-limited signals, in other words, band-limited signals in FrFT domain [2, 14, 15, 18]. For $f \in L^1(\mathbb{R})$ and $\theta \in \mathbb{R}$ we let

$$\mathcal{F}_\theta[f](\xi) = \hat{f}_\theta(\xi) := \int_{\mathbb{R}} f(t) K_\theta(t,\xi) dt \tag{2}$$

be FrFT of $f(t)$ with respect to θ where

$$K_\theta(t,\xi) = \begin{cases} \delta(t-\xi) & \text{if } \theta = 2\pi n,\ n \in \mathbb{Z} \\ \delta(t+\xi) & \text{if } \theta + \pi = 2\pi n,\ n \in \mathbb{Z} \\ c(\theta) e^{ia(\theta)(t^2+\xi^2) - ib(\theta)t\xi} & \text{otherwise} \end{cases}$$

is the transformation kernel with $c(\theta) = \sqrt{\frac{1-i\cot\theta}{2\pi}}$, $a(\theta) = \frac{\cot\theta}{2}$, and $b(\theta) = \csc\theta$ [9, 10]. The FrFT can be extended in $L^2(\mathbb{R})$ by a similar density argument as in FT, and the inverse FrFT w.r.t. θ is defined by the FrFT w.r.t. $-\theta$, that is,

$$f(t) = \int_{\mathbb{R}} \hat{f}_\theta(\xi) K_{-\theta}(t,\xi) d\xi.$$

In the sequel, for the sake of simplicity we write $a(\theta)$, $b(\theta)$ and $c(\theta)$ as a, b and c, respectively. Note that the FrFT is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ and corresponds to the FT when $\theta = \frac{\pi}{2}$.

By definition, $\vec{f}(t)$ is band-limited to $[-\omega,\omega]$ in FT domain if and only if $f(t)$ is band-limited to $[-\omega|b|, \omega|b|]$ in the FrFT domain. Based on this observation, one can easily derive a sampling expansion of band-limited signals in the FrFT domain by using that of corresponding band-limited signals in the FT domain [2, 14, 15, 18]. For example, if $f(t)$ is such that $\hat{f}_\theta(\xi) \subseteq [-\pi,\pi]$ then we have

$$f(t) = e^{-iat^2} \sum_{n\in\mathbb{Z}} f(bn) e^{-ia(bn)^2} \operatorname{sinc}\frac{1}{b}(t-n).$$

On the other hand, it is well known that PW_π is a special case of shift-invariant spaces (SIS) when its generator is given by the sinc function. SIS generated by an arbitrary generator has also been thoroughly explored by many engineers and mathematicians [6, 16]. General SIS associated with FrFT is investigated in [1] where the authors use extended notion of Zak transform and Poisson summation formula to achieve sampling theorem. In [13], FrFT is also used to obtain a useful sampling expansion.

In this paper we present a new approach to reach a sampling theorem in SIS associated with FrFT domain. Adapting the idea of [4] we make use of a bounded invertible mapping from a certain subspace of $L^2(\mathbb{R})$ onto the resulting SIS associated with FrFT domain. As will be seen, the mapping allows us to benefit from existing sampling theory for signals in SIS so that sufficient conditions for various sampling scheme to guarantee perfect reconstruction of a signal in the chirp-modulated SIS can be achieved.

This paper is organized as follows. In Section 2 we define the notation and terminology needed throughout the paper. In Section 3 we provide conditions under which $V_\theta(\phi)$ is generated by some Riesz or frame generator and becomes a reproducing kernel Hilbert space. In Section 4 we obtain conditions under which a sampling expansion holds on $V_\theta(\phi)$. Concluding remarks are given in Section 5.

2. Preliminary

We borrow notations from [1] unless otherwise specified.

A sequence $\{\phi_n : n \in \mathbb{Z}\}$ of vectors in a separable Hilbert space $\mathcal{H}$ is

- a frame of $\mathcal{H}$ with bounds (A, B) if there are constants $B \geq A > 0$ such that
$$A\|\phi\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle \phi, \phi_n\rangle|^2 \leq B\|\phi\|^2,\ \phi \in \mathcal{H};$$
- a Riesz basis of $\mathcal{H}$ with bounds (A, B) if it is complete in $\mathcal{H}$ and there are constants $B \geq A > 0$ such that
$$A\|\mathbf{c}\|^2 \leq \|\sum_{n\in\mathbb{Z}} c(n)\phi_n\|^2 \leq B\|\mathbf{c}\|^2,\ \mathbf{c} = \{c(n)\}_{n\in\mathbb{Z}} \in \ell^2,$$
where $\|\mathbf{c}\|^2 := \sum_{n\in\mathbb{Z}} |c(n)|^2$.

A Hilbert space $\mathcal{H}$ consisting of complex valued functions on a set E is called a reproducing kernel Hilbert space (RKHS) if there is a function $q(s, t)$ on $E \times E$, called the reproducing kernel of $\mathcal{H}$, satisfying

- $q(\cdot, t) \in \mathcal{H}$ for each t in E,

- $\langle f(s), q(s,t)\rangle = f(t),\ f \in \mathcal{H}$.

In an RKHS $\mathcal{H}$ any norm converging sequence also converges uniformly on any subset of E, on which $\|q(\cdot,t)\|^2_{\mathcal{H}} = q(t,t)$ is bounded. The reproducing kernel of $\mathcal{H}$ is unique and can be obtained by $q(s,t) = \sum_{n\in\mathbb{Z}} e_n(s)\overline{e_n(t)}$ where $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{H}$ [5].

For $\phi(t) \in L^2(\mathbb{R})$ let

$$(3)\qquad C_\phi(t) := \sum_{n\in\mathbb{Z}} |\phi(t-n)|^2 \text{ and } G_{\phi,\theta}(\xi) := \sum_{n\in\mathbb{Z}} |\hat{\phi}_\theta(\xi + \frac{2\pi n}{|b|})|^2.$$

Note that $\|\phi(t)\|^2_{L^2(\mathbb{R})} = \|C_\phi(t)\|_{L^1[0,1]} = \|G_{\phi,\theta}(\xi)\|_{L^1[0,2\pi/|b|]}$.

Bhandari and Zayed [1] studied a chirp-modulated SIS $V_\theta(\phi)$ as a sampling space where

$$(4)\qquad V_\theta(\phi) := \mathit{closure}\,\{c\,e^{-iat^2}\sum_{n\in\mathbb{Z}} d(n)\overrightarrow{\phi}(t-n) : \{d(n)\}_n \in \ell^2(\mathbb{Z})\}$$

and introduced a sampling expansion on $V_\theta(\phi)$. Note here that $V_\theta(\phi)$ is no longer (integer) shift-invariant and $V_{\frac{\pi}{2}}(\phi)$ corresponds to the shift-invariant space $V(\phi) := \mathit{closure}\,\{\sum_{n\in\mathbb{Z}} d(n)\phi(t-n) : \{d(n)\}_n \in \ell^2(\mathbb{Z})\}$.

3. $V_\theta(\phi)$ as an RKHS

Suppose that $\theta \neq n\pi,\ n \in \mathbb{Z}$. It is not difficult to see that if $G_{\phi,\theta}(\xi) \leq B$ a.e. on $[0, 2\pi/|b|]$ then $V_\theta(\phi)$, defined by (4), is a closed subspace of $L^2(\mathbb{R})$. In the following we assume this condition so that $V_\theta(\phi)$ is well-defined in $L^2(\mathbb{R})$. Then $V_\theta(\phi)$ can be rewritten as

$$V_\theta(\phi) = \overline{\text{span}}\{c\,e^{-iat^2}\overrightarrow{\phi}(t-n) : n \in \mathbb{Z}\}.$$

The aim of Section 3 is to establish $V_\theta(\phi)$ as an RKHS. In [7], Kim and Kwon showed that a shift-invariant space $V(\phi)$ becomes an RKHS depending on behavior of $C_\phi(t)$. Their result, however, cannot be directly applied to $V_\theta(\phi)$ since, as mentioned earlier, $V_\theta(\phi)$ is not shift-invariant.

We first give conditions under which a sequence $\{c\,e^{-iat^2}\overrightarrow{\phi}(t-n) : n \in \mathbb{Z}\}$ is a frame, a Riesz basis or orthonormal basis of $V_\theta(\phi)$.

Proposition 3.1. *Let $\phi(t) \in L^2(\mathbb{R})$ and $B \geq A > 0$. Then $\{c\,e^{-iat^2}\overrightarrow{\phi}(t-n) : n \in \mathbb{Z}\}$ is*

(a) *a frame of $V_\theta(\phi)$ with bound (A, B) if and only if*

$$A \leq G_{\phi,\theta}(\xi) \leq B \text{ a.e. on } [0, 2\pi/|b|] \cap \operatorname{supp} G_{\phi,\theta}(\xi);$$

(b) *a Riesz basis of* $V_\theta(\phi)$ *with bound* (A, B) *if and only if*

$$A \le G_{\phi,\theta}(\xi) \le B \text{ a.e. on } [0, 2\pi/|b|];$$

(c) *an orthonormal basis of* $V_\theta(\phi)$ *if and only if* $G_{\phi,\theta}(\xi) = 1$ *a.e. on* $[0, 2\pi/|b|]$.

Proof. For $\phi(t) \in L^2(\mathbb{R})$ let T_θ be an operator from $\ell^2(\mathbb{Z})$ to $V_\theta(\phi)$ defined by

(5)
$$T_\theta(\mathbf{d}) = (\mathbf{d} *_\theta \phi)(t) := c\, e^{-iat^2} \sum_{n\in\mathbb{Z}} d(n)\vec{\phi}(t-n), \;\; \mathbf{d} = \{d(n)\}_n \in \ell^2(\mathbb{Z}).$$

Then

$$\begin{aligned} \|T_\theta(\mathbf{d})\|^2_{L^2(\mathbb{R})} &= \|\mathbf{d} *_\theta \phi\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}_\theta[\mathbf{d} *_\theta \phi]\|^2_{L^2(\mathbb{R})} \\ &= \|\hat{\mathbf{d}}_\theta(\xi)\hat{\phi}_\theta(\xi)e^{-ia\xi^2}\|^2_{L^2(\mathbb{R})} = \int_0^{2\pi/|b|} |\hat{\mathbf{d}}_\theta(\xi)|^2 G_{\phi,\theta}(\xi)d\xi \end{aligned}$$

The third equality in the above holds due to Theorem 1 of [19]. For the rest of the proof we refer to the proof of Theorem 7.1.7 of [3].

□

One can easily see that if $V_\theta(\phi)$ is an RKHS then $\phi(t)$ is well-defined everywhere on $\mathbb{R}$ and $C_\phi(t) < \infty$ for $t \in \mathbb{R}$. Conversely, we have:

Proposition 3.2. *Let* T_θ *be the operator from* $\ell^2(\mathbb{Z})$ *to* $V_\theta(\phi)$ *defined by* (5). *Denote by* $N(T_\theta)^\perp$ *the orthogonal complement of the null space* $N(T_\theta)$. *Assume that* $\phi(t) \in L^2(\mathbb{R})$ *is well-defined pointwise (in Lebesgue sense) everywhere on* $\mathbb{R}$ *and* $C_\phi(t) < \infty$ *for* $t \in \mathbb{R}$.

(a) *If* $\{c\, e^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *is a frame of* $V_\theta(\phi)$ *then* $V^p_\theta(\phi) := \{(\mathbf{d} *_\theta \phi)(t) : \mathbf{d} \in N(T_\theta)^\perp\}$ *is an RKHS in which any* $f(t) = (\mathbf{d} *_\theta \phi)(t)$ *is the pointwise limit of* $c\, e^{-iat^2} \sum_{n\in\mathbb{Z}} d(n)\vec{\phi}(t-n)$.

(b) *If* $\{c\, e^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *is a Riesz basis of* $V_\theta(\phi)$ *then* $V_\theta(\phi)$ *is an RKHS in which any* $f(t) = (\mathbf{d} *_\theta \phi)(t)$ *is the pointwise limit of* $c\, e^{-iat^2} \sum_{n\in\mathbb{Z}} d(n)\vec{\phi}(t-n)$.

(c) *If* $\{c\, e^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *is a frame of* $V_\theta(\phi)$, $\vec{\phi}(t)$ *is continuous on* $\mathbb{R}$ *and* $\sup_{\mathbb{R}} C_\phi(t) < \infty$ *then* $V_\theta(\phi)$ *is an RKHS in which any* $f(t) = (\mathbf{d} *_\theta \phi)(t)$ *is the pointwise limit of* $c\, e^{-iat^2} \sum_{n\in\mathbb{Z}} d(n)\vec{\phi}(t-n)$. *In this case the pointwise convergence is uniform, so* $V_\theta(\phi) \subset C(\mathbb{R}) \cup L^2(\mathbb{R})$.

Proof. $\{c\,e^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ is a frame, a Riesz basis, or an orthonormal basis of $V_\theta(\phi)$ if and only if $\{\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ is a frame, a Riesz basis, or an orthonormal basis of $V(\vec{\phi})$, respectively. By Proposition 2.3 of [7] it suffices to show that (i) the point evaluation operator $l_t(\cdot) : V_\theta(\phi) \mapsto \mathbb{C}$, defined by $l_t(f) := f(t)$ for $t \in \mathbb{R}$, is bounded, and (ii) $\mathbf{d} \in N(T_\theta)$ if and only if $\vec{\mathbf{d}} \in N(\tilde{T})$ where $\tilde{T}$ is a pre-frame operator of $\{c\,e^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$.

Since $V_\theta(\phi)$ is an isomorphic image of $V(\vec{\phi})$ being an RKHS, (i) follows. (ii) is apparent by definition.

□

4. Uniform sampling

Uniform sampling refers to reconstructing a given continuous signal by its uniformly spaced discrete samples. We achieve sampling expansion which describe such sampling and reconstruction scheme for signals in $V_\theta(\phi)$.

In what follows we always assume that

- $\theta \neq n\pi$, $n \in \mathbb{Z}$;
- $\phi(t)$ is well-defined everywhere (in Lebesgue sense) on $\mathbb{R}$;
- there exist constants $B \geq A > 0$ such that $A \leq |G_{\phi,\theta}(\xi)| \leq B$ a.e. on $[0, 2\pi/|b|]$;
- $C_\phi(t) < \infty$ for $t \in \mathbb{R}$.

Then $V_\theta(\phi)$ becomes an RKHS and $\{ce^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ is a Riesz basis of $V_\theta(\phi)$ by Propositions 3.1 and 3.2.

Remark 4.1. *In* [1]*, the authors claim that the formula (26) of* [1] *is the necessary and sufficient condition for* $\{\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *to be a Riesz basis of* $V_\theta(\phi)$*. However it is the one for* $\{\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *to be a Riesz basis of* $V(\vec{\phi})$*, but not of* $V_\theta(\phi)$*. In general* $\{\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *is not even complete in* $V_\theta(\phi)$*. For instance, let* $\phi(t) = \chi_{[0,1)}(t)$ *where* $\chi_{[0,1)}(t) := 1$ *for* $t \in [0,1)$ *and* 0 *otherwise. Then* $ce^{-iat^2}\vec{\phi}(t) = c\,\chi_{[0,1)}(t) \in V_\theta(\phi)$ *and* $\langle ce^{-iat^2}\vec{\phi}(t), \vec{\phi}(t-n)\rangle_{L^2(\mathbb{R})} = \langle c\,\chi_{[0,1)}(t), e^{ia(t-n)^2}\chi_{[0,1)}(t-n)\rangle_{L^2(\mathbb{R})} = 0$ *for* $n \in \mathbb{Z}\backslash\{0\}$*. In order that* $\{\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ *is complete in* $V_\theta(\phi)$*, there should be some constant* M *such that* $c\,\chi_{[0,1)}(t) = Me^{iat^2}\chi_{[0,1)}(t)$ *for* $t \in \mathbb{R}$*, which is only true if* $a = a(\theta) = 0$*, or equivalently* $\theta = \frac{\pi}{2} + n\pi$ *for* $n \in \mathbb{Z}$*.*

We now consider the bounded linear operator $\mathcal{J}_\theta$ from $L^2[0, 2\pi/|b|]$ into $V_\theta(\phi)$, defined by

$$(\mathcal{J}_\theta F)(t) := \langle F, \overline{c\, e^{iat^2} Z_{\phi,\theta}(t,\xi)} \rangle_{L^2[0,2\pi/|b|]}, \tag{6}$$

where

$$Z_{\phi,\theta}(t,\xi) := \sum_{n\in\mathbb{Z}} \vec{\phi}(t-n) e^{ian^2} \overline{K_\theta(n,\xi)} \tag{7}$$

is the fractional Zak transform of $\phi(t)$ w.r.t. θ (cf.[1]).

Proposition 4.2. *$\mathcal{J}_\theta$ is an isomorphism from $L^2[0, 2\pi/|b|]$ onto $V_\theta(\phi)$.*

Proof. Note that $\{K_\theta(n,\xi) : n \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2[0, 2\pi/|b|]$. $\mathcal{J}_\theta$ maps the orthonormal basis $\{K_\theta(n,\xi) : n \in \mathbb{Z}\}$ of $L^2[0, 2\pi/|b|]$ to the Riesz basis $\{ce^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{Z}\}$ of $V_\theta(\phi)$, so it is bijective. Hence it suffices to show that $\mathcal{J}_\theta$ is bounded. This follows since $\|(\mathcal{J}_\theta F)(t)\|^2_{L^2(\mathbb{R})} = \|\sum_{n\in\mathbb{Z}} d(n) ce^{-iat^2}\vec{\phi}(t-n)\|^2_{L^2(\mathbb{R})} \le B\|\mathbf{d}\|^2 = B\|F\|^2_{L^2[0,2\pi/|b|]}$ where $\mathbf{d} = \{d(n) := e^{ian^2}\langle F, K_\theta(n,\cdot)\rangle_{L^2[0,2\pi/|b|]} : n \in \mathbb{Z}\}$ and B denotes the upper bound of the Riesz basis $\{ce^{-iat^2}\vec{\phi}(t-n) : n \in \mathbb{R}\}$ of $V_\theta(\phi)$. $\square$

Notice that $\mathcal{J}_\theta$ has the following properties: For $F(\xi) \in L^2[0, 2\pi/|b|]$,

$$\mathcal{F}_\theta[\mathcal{J}_\theta F](\xi) = e^{-ia\xi^2} F(\xi)\hat{\phi}_\theta(\xi); \tag{8}$$

$$e^{iat^2}\mathcal{J}_\theta[F(\cdot)e^{-ibk\cdot}](t) = e^{ia(t-k)^2}\mathcal{J}_\theta F(t-k). \tag{9}$$

We are to derive a sampling expansion on $V_\theta(\phi)$ of the following form:

$$f(t) = \sum_{n\in\mathbb{R}} f(\sigma+n) S_n(t), \;\; f \in V_\theta(\phi) \tag{10}$$

where $0 \le \sigma < 1$ and $\{S_n(t) : n \in \mathbb{Z}\}$ is a Riesz basis or a frame of $V_\theta(\phi)$.

Since $V_\theta(\phi)$ is an RKHS, $f(t)$ is well-defined pointwise (as a Lebesgue point) on $\mathbb{R}$. Thus we have from (6) that for $F(\xi) \in L^2[0, 2\pi/|b|]$,

$$f(\sigma+n) = \langle F(\xi), \overline{c\, Z_{\phi,\theta}(\sigma,\xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} \rangle_{L^2[0,2\pi/|b|]}, \;\; n \in \mathbb{Z} \tag{11}$$

where $f(t) := (\mathcal{J}_\theta F)(t)$.

Lemma 4.3. *Let $g(\xi) \in L^2[0, 2\pi/|b|]$ and $\{\rho(n)\}_n$ be a sequence of complex numbers satisfying $|\rho(n)| = 1$, $n \in \mathbb{Z}$. Then the following are equivalent:*

(a) *$\{g(\xi)\rho(n)e^{-ibn\xi} : n \in \mathbb{Z}\}$ is a frame of $L^2[0, 2\pi/|b|]$ with bounds (A, B).*

(b) $\{g(\xi)\rho(n)e^{-ibn\xi} : n \in \mathbb{Z}\}$ *is a Riesz basis of* $L^2[0,2\pi/|b|]$ *with bounds* (A,B).

(c) *There exist constants* $B \geq A > 0$ *such that* $A \leq \frac{2\pi}{|b|}|g(\xi)|^2 \leq B$ *a.e. on* $[0,2\pi/|b|]$.

Proof. Note that $\{g(\xi)\rho(n)e^{-ibn\xi} : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $L^2[0,2\pi/|b|]$ if and only if $\{g(\xi)e^{-ibn\xi} : n \in \mathbb{Z}\}$ is a frame or a Riesz basis of $L^2[0,2\pi/|b|]$, respectively. For $F(\xi) \in L^2[0,2\pi/|b|]$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\langle F(\xi), g(\xi)e^{-ibn\xi}\rangle_{L^2[0,2\pi/|b|]}|^2 &= \sum_{n\in\mathbb{Z}} |\langle F(\xi)\overline{g(\xi)}, e^{-ibn\xi}\rangle_{L^2[0,2\pi/|b|]}|^2 \\ &= (2\pi/|b|)\|F(\xi)\overline{g(\xi)}\|^2_{L^2[0,2\pi/|b|]}\end{aligned}$$

for which (a) is equivalent to (c). Since $\{g(\xi)e^{-ibn\xi} : n \in \mathbb{Z}\}$ is ω-independent, by Theorem 5.5.4 of [3], (a) is also equivalent to (b). □

Theorem 4.4. *Let* $0 \leq \sigma < 1$, *and let* $C_\phi(t)$ *be given by* (3). *The following are equivalent:*

(a) *There is a frame* $\{S_n(t) : n \in \mathbb{Z}\}$ *of* $V_\theta(\phi)$ *such that*

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n)S_n(t), \ f \in V_\theta(\phi). \tag{12}$$

(b) *There is a Riesz basis* $\{S_n(t) : n \in \mathbb{Z}\}$ *of* $V_\theta(\phi)$ *such that*

$$f(t) = \sum_{n\in\mathbb{Z}} f(\sigma+n)S_n(t), \ f \in V_\theta(\phi). \tag{13}$$

(c) *There are constants* $\beta \geq \alpha > 0$ *such that*

$$\alpha \leq |Z_{\phi,\theta}(\sigma,\xi)| \leq \beta, \quad \text{a.e. on } [0,2\pi/|b|] \tag{14}$$

where $Z_{\phi,\theta}(\cdot,\cdot)$ *is the fractional Zak transform of* ϕ, *defined by* (7).

In this case (12) *and* (13) *converge in* $L^2(\mathbb{R})$ *and uniformly on a subset of* $\mathbb{R}$ *on which* $C_\phi(t)$ *is bounded, and*

$$S_n(t) = e^{-iat^2}e^{ia(\sigma+n)t^2}e^{ia(t-n)^2}S(t-n) \tag{15}$$

where $S(t)$ *is in* $V_\theta(\phi)$ *such that*

$$\hat{S}_\theta(\xi) = \frac{\hat{\phi}_\theta(\xi)}{ce^{ia\xi^2}Z_{\phi,\theta}(\sigma,\xi)} \quad \text{a.e. on } \mathbb{R}. \tag{16}$$

In addition, $S_n(\sigma+k) = \delta_{n,k}$ *for* $n,k \in \mathbb{Z}$.

Proof. Recall from (11) that for $F(\xi) \in L^2[0, 2\pi/|b|]$ we have

$$f(\sigma + n) = \langle F(\xi), \overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} \rangle_{L^2[0,2\pi/|b|]}, \ n \in \mathbb{Z}$$

where $f(t) := (\mathcal{J}_\theta F)(t)$. Assume (c). Let $g(\xi) = \overline{cZ_{\phi,\theta}(\sigma, \xi)}$ and $\rho(n) = e^{ia(\sigma+n)^2}$. Then by Lemma 4.3, $\{\overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$ is a frame (or a Riesz basis) of $L^2[0, 2\pi/|b|]$. It is easy to check that for some $h_\theta(\xi) \in L^2[0, 2\pi/|b|]$, $\{h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$ is the dual of $\{\overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$. Thus we obtain a frame (or a Riesz basis) expansion of $F(\xi)$ in $L^2[0, 2\pi/|b|]$ as

$$F(\xi) = \sum_{n \in \mathbb{Z}} f(\sigma + n) h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi}. \tag{17}$$

Applying $\mathcal{J}_\theta$ to (17) implies

$$f(t) = \sum_{n \in \mathbb{Z}} f(\sigma + n) \mathcal{J}_\theta [h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi}](t), \ f(t) \in V_\theta(\phi),$$

which proves (a) (or (b)) by setting $S_n(t) = \mathcal{J}_\theta [h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi}](t)$.

Now assume (b). Applying $\mathcal{J}_\theta^{-1}$ to the Riesz basis expansion (13) we have a Riesz basis expansion in $L^2[0, 2\pi/|b|]$:

$$\begin{aligned} F(\xi) &= \sum_{n \in \mathbb{Z}} f(\sigma + n) \mathcal{J}_\theta^{-1}[S_n](\xi) \\ &= \sum_{n \in \mathbb{Z}} \langle F(\xi), \overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} \rangle_{L^2[0,2\pi/|b|]} \mathcal{J}_\theta^{-1}[S_n](\xi) \end{aligned}$$

where $F := \mathcal{J}_\theta^{-1}[f]$. Since the dual of a Riesz basis is also a Riesz basis, $\{\overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$ is a Riesz basis of $L^2[0, 2\pi/|b|]$. Setting $g(\xi) = \overline{cZ_{\phi,\theta}(\sigma, \xi)}$ and $\rho(n) = e^{ia(\sigma+n)^2}$, we have (c) by Lemma 4.3.

Recall that for some $h_\theta(\xi) \in L^2[0, 2\pi/|b|]$

$$S_n(t) = \mathcal{J}_\theta [h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi}](t).$$

Then (15) follows by the property (9). Since $\{\overline{cZ_{\phi,\theta}(\sigma, \xi)} e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$ and $\{h_\theta(\xi) e^{ia(\sigma+n)^2} e^{-ibn\xi} : n \in \mathbb{Z}\}$ are bi-orthonormal pair in $L^2[0, 2\pi/|b|]$ we have (16) by the property (8).

The Riesz basis expansion (13) with $f(t) = S_n(t)$ and the uniqueness of coefficients of a basis expansion implies $S_n(\sigma + k) = \delta_{n,k}$ for $n, k \in \mathbb{Z}$ where δ is Kronecker's delta function.

Finally the convergence mode follows since $V_\theta(\phi)$ is an RKHS in which any L^2-convergent sequence converges uniformly where $\|q_\theta(t, \cdot)\|_{L^2(\mathbb{R})} =$

$q_\theta(t,t) = C_\phi(t)$ is bounded. Here, $q_\theta(\cdot,\cdot)$ denotes the reproducing kernel of $V_\theta(\phi)$. □

It is easy to see that the conditions given in Theorem 4.4 is also equivalent to

(d) There are constants $\beta \geq \alpha > 0$ such that

$$\alpha \|f\|^2_{L^2(\mathbb{R})} \leq \sum_{n\in\mathbb{Z}} |f(\sigma+n)|^2 \leq \beta \|f\|^2_{L^2(\mathbb{R})}, \; f \in V_\theta(\phi).$$

5. Concluding remarks

We address a uniform sampling problem in SIS associated with FrFT domain. The method used in the paper allows us to benefit from existing sampling theory for signals in SIS so that sufficient conditions for various sampling scheme to guarantee perfect reconstruction of a signal in the chirp-modulated SIS can be achieved. Generalized sampling problem in the space will be discussed in the future work.

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