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ON SOME GEOMETRIC PROPERTIES OF QUADRIC SURFACES IN EUCLIDEAN SPACE

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Abstract. This paper is concerned with the classifications of quadric surfaces of first and second kinds in Euclidean 3-space satisfying the Jacobi condition with respect to their curvatures, the Gaussian curvature K, the mean curvature H, second mean curvature H_{II} and second Gaussian curvature K_{II}. Also, we study the zero and non-zero constant curvatures of these surfaces. Furthermore, we investigated the (A, B)-Weingarten, (A, B)-linear Weingarten as well as some special $\left(C^2, K\right)$ and $\left(C^2, K\sqrt{K}\right)$-nonlinear Weingarten quadric surfaces in $\mathbf{E}^3$, where $A \neq B$, $A,\ B \in \{K, H, H_{II}, K_{II}\}$ and $C \in \{H, H_{II}, K_{II}\}$. Finally, some important new lemmas are presented.

1. Introduction

Weingarten surfaces are surfaces whose Gaussian and mean curvatures satisfy a functional relationship (of class C^0 at least). The class of Weingarten surfaces contains already mentioned surfaces of constant curvatures K or H. Furthermore, a C^r-surface, $r > 3$, is Weingarten if and only if $K_u\, H_v - K_v\, H_u = 0$. On the other hand, let A and B be smooth functions on a surface $M(u, v)$ in Euclidean 3-space $\mathbf{E}^3$. The Jacobi function $\Phi(A, B)$ formed with A and B is defined by:

$$\Phi(A, B) = \det \begin{pmatrix} A_u & A_v \\ B_u & B_v \end{pmatrix},$$

where $A_u = \dfrac{\partial A}{\partial u}$ and $A_v = \dfrac{\partial A}{\partial v}$.

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For the pair (A, B) of curvatures K, H, H_{II} and K_{II} of M in $\mathbf{E}^3$, if M satisfies $\Phi(A, B) = 0$, $a\,A + b\,B = c$, $a\,C^2 + b\,K = c$ and $a\,C^2 + b\,K\,\sqrt{K} = c$, then we call (A, B)-Weingarten surface (*W-surface*), (A, B)-linear Weingarten surface (*LW-surface*), $\left(C^2, K\right)$-the first type of nonlinear Weingarten surface (*FNW-surface*) and $\left(C^2, K\,\sqrt{K}\right)$-the second type of nonlinear Weingarten surface (*SNW-surface*), respectively, where a, b, $c \in \mathbb{R}$, $(a, b, c) \neq (0, 0, 0)$.

The classification of the Weingarten surfaces in Euclidean space is almost completely open today. These surfaces were introduced by Weingarten [1, 2] in the context of the problem of finding all surfaces isometric to a given surface of revolution. Applications of Weingarten surfaces on computer aided design and shape investigation can seen in [3].

The authors in [4, 5] have investigated ruled Weingarten surfaces and ruled linear Weingarten surfaces in $\mathbf{E}^3$. Besides, a classification of ruled Weingarten surfaces and ruled linear Weingarten surfaces in a Minkowski 3-space $\mathbf{E}^3_1$ is given in [6, 7, 8]. Munteanu and Nistor [9] studied polynomial translation linear Weingarten surfaces in Euclidean 3-space. Also, Lopez [10, 11] studied cyclic linear Weingarten surface in Euclidean 3-space. In [12] Lopez classified all parabolic linear Weingarten surfaces in hyperbolic 3-space. Ro and Yoon [13] studied a tube of Weingarten types in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, mean curvature and second Gaussian curvature. Recently, Kim and Yoon [14] classified quadric surfaces in Euclidean 3-space in terms of the Gaussian curvature and the mean curvature. In addition to, Yoon and Jun [15] classified non-degenerate quadric surfaces in Euclidean 3-space in terms of the isometric immersion and the Gauss map. Furthermore in ([16, 17]), Weingarten timelike tube surfaces around spacelike and timelike curves were studied in Minkowski 3-space $\mathbf{E}^3_1$.

A quadratic surface intersects every plane in a (proper or degenerate) conic section. Moreover, the cone consisting of all tangents from a fixed point to a Quadratic surface cuts every plane in a conic section, and the points of contact of this cone with the surface form a conic section [18]. There are 17-standard form types of quadric surfaces. Examples of quadratic surfaces include the cone, cylinder, ellipsoid, elliptic cone, elliptic cylinder, elliptic hyperboloid, elliptic paraboloid, hyperbolic cylinder, hyperbolic paraboloid, sphere and spheroid, etc.

In this paper, some kinds of the quadric surfaces in 3-dimensional Euclidean space satisfying the Jacobi condition with respect to their curvatures have been studied. Also, we study the zero and non-zero constant curvatures of quadric surfaces. Furthermore, we investigated the (A,B)-Weingarten, (A,B)-linear Weingarten, $\left(C^2, K\right)$-first type of nonlinear Weingarten and $\left(C^2, K\sqrt{K}\right)$-second type of nonlinear Weingarten quadric surfaces in $\mathbf{E}^3$.

2. Fundamental concepts

Let $\mathbf{E}^3$ be a Euclidean 3-space with the scalar product given by $\langle .,. \rangle = dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbf{E}^3$. Let $M : \Psi = \Psi(u,v)$ be a surface in Euclidean 3-space. The unit normal vector field of M can be defined by:

$$N = \frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|}, \quad \Psi_u = \frac{\partial \Psi}{\partial u}, \quad \Psi_v = \frac{\partial \Psi}{\partial v}, \tag{2.1}$$

where $\wedge$ stands the vector product of $\mathbf{E}^3$. The first fundamental form I of the surface M is

$$I = \langle d\Psi, d\Psi \rangle = E\,du^2 + 2F\,dudv + G\,dv^2, \tag{2.2}$$

with coefficients

$$E = \langle \Psi_u, \Psi_u \rangle, \quad F = \langle \Psi_u, \Psi_v \rangle, \quad G = \langle \Psi_v, \Psi_v \rangle.$$

The second fundamental form of the surface M is given by

$$II = -\langle dN, d\Psi \rangle = e\,du^2 + 2f\,dudv + g\,dv^2. \tag{2.3}$$

From which the components the second fundamental form e, f and g are expressed as

$$e = \langle \Psi_{uu}, N \rangle, \quad f = \langle \Psi_{uv}, N \rangle, \quad g = \langle \Psi_{vv}, N \rangle.$$

Under this parametrization of the surface M, the Gaussian curvature$K$ and the mean curvature H have the classical expressions, respectively [19]

$$K = \frac{e\,g - f^2}{EG - F^2}, \tag{2.4}$$

$$H = \frac{E\,g + G\,e - 2F\,f}{2\,(EG - F^2)}. \tag{2.5}$$

The mean curvature H_{II} of non-degenerate second fundamental form in a Euclidean 3-space $\mathbf{E}^3$ is defined by [20]

$$H_{II} = H + \frac{1}{2}\Delta_{II}\big(\ln\sqrt{|\,K\,|}\big), \tag{2.6}$$

where H and K are the mean curvature and the Gaussian curvatures respectively, and Δ_{II} denotes the Laplacian operator of non-degenerate second fundamental form, that is,

$$\Delta_{II} = \frac{1}{\sqrt{|\,h\,|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i} \Big[\sqrt{|\,h\,|} h^{ij} \frac{\partial}{\partial u^j} \Big],$$

where $e = h_{11},\ f = h_{12},\ g = h_{22},\ h = \det(h_{ij}),\ (h^{ij}) = (h_{ij})^{-1}$ and $\{u_i\}$ is rectangular coordinate system in $\mathbf{E}^3$. The curvature H_{II} is said to be the second mean curvature of a surface M in a Euclidean 3-space.

The second Gaussian curvature K_{II} of a surface is defined by (cf.[21])

$$\mathbf{K}_{II} = \frac{1}{(eg-f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix} \right\}. \tag{2.7}$$

Now, we can write the following important definition [22]:

Definition 2.1. *(1): A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.*

(2): A regular surface for which the mean curvature vanishes identically is minimal surface.

(3): A non developable surface is called II-flat if the second Gaussian curvature vanishes identically.

(4): A non developable surface is called II-minimal if the second mean curvature vanishes identically.

Now, we summarize the definition of a quadric surface as

A subset M of Euclidean 3-space $\mathbf{E}^3$ is called a quadric surface if it is the set of points (x_1, x_2, x_3) satisfying the following equation of the second degree:

$$\sum_{i,j=0}^{3} a_{ij}\ x_i\ x_j + \sum_{i=0}^{3} b_i\ x_i + c = 0, \tag{2.8}$$

or equivalently

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c = 0,$$

where a_{ij}, b_i, c are all real numbers.

By applying a coordinate transformation in $\mathbf{E}^3$ if necessary, M is either ruled surface, or one of the following two kinds [23].

The first kind:

$$x_3^2 - a\, x_1^2 - b\, x_2^2 = c, \quad a\,b\,c \neq 0, \tag{2.9}$$

The second kind:

$$x_3 = \frac{a}{2}x_1^2 + \frac{b}{2}x_2^2, \;\; a > 0, \;\; b > 0. \tag{2.10}$$

3. Quadric surfaces - the first kind

Let $M : \Psi = \Psi(u,v)$ be a quadric surface of the first kind in $\mathbf{E}^3$ and satisfy the equation (2.9). Then M can be written as

$$\Psi(u,v) = \Big(u, v, \sqrt{a\,u^2 + b\,v^2 + c}\Big). \tag{3.1}$$

By (2.4) and (2.5), the direct computation of (3.1) gives the Gaussian curvature K and the mean curvature H as

$$K = \frac{a\,b\,c}{\omega^2}, \tag{3.2}$$

$$H = \frac{(a+b)c + a\,b\big[(a+1)u^2 + (b+1)v^2\big]}{2\omega^{3/2}}. \tag{3.3}$$

On the other hand, the second curvatures of the surface M as in (2.6) and (2.7) are given by

(3.4)
$$K_{II} = \frac{1}{2c\,\omega^{3/2}}\Big[a\,c\,(a+1)(b-a-1)u^2 - b\,c\,(b+1)(b-a+1)v^2 \\ -2a\,b\,(a+1)(b+1)u^2v^2 - a^2(a+1)^2u^4 - b^2(b+1)^2v^4 + (a+b)c^2\Big],$$

(3.5)

$$H_{II}=\frac{1}{2c\,\omega^{3/2}}\Big[a\,c\,(a+1)(2a-b-6)u^2-b\,c\,(b+1)(a-2b+6)v^2 \\ -4a\,b\,(a+1)(b+1)u^2v^2-2a^2(a+1)^2u^4-2b^2(b+1)^2v^4-(a+b+4)c^2\Big],$$

where

$$\omega=c+a(a+1)u^2+b(b+1)v^2.$$

The above equations with the condition $a\ b\ c\neq 0$ enable us to give the following important lemma:

lemma 3.1. *Let M be a quadric surface of the first kind. Then the following are satisfied:*

(1): *M is non-developable surface.*
(2): *M is non-minimal surface.*
(3): *M is non-II-minimal surface.*
(4): *M is non-II-flat surface.*

3.1. Weingarten property of quadric surfaces of the first kind

We now differentiate K, H, K_{II} and H_{II} with respect to u and v, the results are, respectively

$$\text{(3.6)}\qquad \begin{cases} K_u=-\dfrac{4a^2b\,c\,(a+1)u}{\omega^3}, \\ K_v=-\dfrac{4a\,b^2c\,(b+1)v}{\omega^3}, \end{cases}$$

(3.7)

$$\begin{cases} H_u=-\dfrac{a(a+1)}{2\omega^{5/2}}\Big[(3a+b)c+a\,b\,(a+1)u^2+b\,(b+1)(3a-2b)v^2\Big]u, \\ H_v=-\dfrac{b(b+1)}{2\omega^{5/2}}\Big[(a+3b)c-a(a+1)(2a-3b)u^2+a\,b\,(b+1)v^2\Big]v, \end{cases}$$

(3.8)

$$\begin{cases} (K_{II})_u=-\dfrac{a(a+1)}{2c\,\omega^{5/2}}\Big[(5a+b+2)c^2-a(a+1)(a-b-3)c\,u^2 \\ \qquad\qquad +b(b+1)\big(5(a-b)+3\big)c\,v^2+(\omega-c)^2\Big]u, \\ (K_{II})_v=-\dfrac{b(b+1)}{2c\,\omega^{5/2}}\Big[(a+5b+2)c^2-a(a+1)\big(5(a-b)-3\big)c\,u^2 \\ \qquad\qquad +b(b+1)(a-b+3)c\,v^2+(\omega-c)^2\Big]v, \end{cases}$$

and
(3.9)

$$\begin{cases} (H_{II})_u = -\dfrac{a(a+1)}{2c\,\omega^{5/2}}\Big[a(a+1)(2a+2-b)c\,u^2 + b(b+1)(8b-7a+2)c\,v^2 \\ \qquad +4a\,b\,(a+1)(b+1)u^2v^2 + 2a^2(1+a)^2u^4 + 2b^2(1+b)^2v^4 - (7a+b)c^2\Big]u, \\ (H_{II})_v = -\dfrac{b(b+1)}{2c\,\omega^{5/2}}\Big[a(a+1)(8a+2-7b)c\,u^2 + b(b+1)(-a+2b+2)c\,v^2 \\ \qquad +4a\,b\,(a+1)(b+1)u^2v^2 + 2a^2(a+1)^2u^4 + 2b^2(b+1)^2v^4 - (a+7b)c^2\Big]v. \end{cases}$$

The Weingarten property for the surface (3.1) can be studied within some computations of the following Jacobi functions:
(3.10)

$$\begin{cases} \Phi\big(H,K\big) = \dfrac{4a^2b^2(a+1)(b+1)(a-b)c\,u\,v}{\omega^{9/2}}, \\ \Phi\big(K_{II},K\big) = \dfrac{8a^2b^2(a+1)(b+1)(a-b)c\,u\,v}{\omega^{9/2}}, \\ \Phi\big(H_{II},K\big) = \dfrac{12a^2b^2(a+1)(b+1)(b-a)c\,u\,v}{\omega^{9/2}}, \\ \Phi\big(K_{II},H\big) = \dfrac{a\,b\,(a+1)(b+1)(a-b)\big[(a+b-2)c - a(a+1)u^2 - b(b+1)v^2\big]u\,v}{2c\,\omega^3}, \\ \Phi\big(H_{II},H\big) = \dfrac{a\,b\,(a+1)(b+1)(b-a)\big[(a+b)c + a(a+1)u^2 + b(b+1)v^2\big]u\,v}{c\,\omega^3}, \\ \Phi\big(K_{II},H_{II}\big) = \dfrac{a\,b\,(a+1)(b+1)(a-b)\Big((a+b+6)c + 7\big[a(a+1)u^2 + b(b+1)v^2\big]\Big)u\,v}{2c\,\omega^3}. \end{cases}$$

It follows that all above Jacobi functions vanish if and only if one of the following conditions are satisfied: $a=-1$, $b=-1$ or $a=b$. Therefore, we have proved the following Lemma:

lemma 3.2. *If M is a quadric surface of the first kind in the Euclidean 3-space, then, the following are equivalent:*

(1): *M is a Weingarten surface of type $\Phi(H,K)=0$.*
(2): *M is a Weingarten surface of type $\Phi(K_{II},K)=0$.*
(3): *M is a Weingarten surface of type $\Phi(H_{II},K)=0$.*
(4): *M is a Weingarten surface of type $\Phi(K_{II},H)=0$.*
(5): *M is a Weingarten surface of type $\Phi(H_{II},H)=0$.*
(6): *M is a Weingarten surface of type $\Phi(K_{II},H_{II})=0$.*
(7): *M is an open part of one of an ellipsoid and hyperboloid:*

$$\Psi(u,v) = \Big(u, v, \sqrt{c-u^2+b\,v^2}\Big), \tag{3.11}$$

$$\Psi(u,v) = \Big(u, v, \sqrt{c+a\,u^2-v^2}\Big), \tag{3.12}$$

$$\Psi(u,v)=\Big(u,v,\sqrt{c+a(u^2+v^2)}\Big). \tag{3.13}$$

3.2. Quadric surfaces - the first kind with non-zero constant curvatures

In this section, we will study the quadric surface of the first kind (3.1) when it has non-zero constant curvatures.

First we examine quadric surfaces of the first kind having a non-zero constant Gaussian curvature. In this case, the form (3.2) can be written as

$$0=K\,\omega^2-a\,b\,c=\sum_{i,j=0}^{2}\Omega_{ij}\,u^{2i}\,v^{2j}$$

where the non-zero coefficients Ω_{ij} are

$$\begin{cases}\Omega_{00}=c(c\,K-a\,b), & \Omega_{01}=2b\,c\,(b+1)K,\\ \Omega_{10}=2a\,c\,(a+1)K, & \Omega_{11}=2a\,b\,(a+1)(b+1)K,\\ \Omega_{02}=b^2(b+1)^2K, & \Omega_{20}=a^2(a+1)^2K.\end{cases} \tag{3.14}$$

From (3.14), we can notice that the Gaussian curvature is constant if and only if $a=b=-1$. It becomes $K=\frac{1}{c}$.

By a similar discussion as above we can also do straightforward computations for constant mean,second Gaussian and second mean curvatures. Thus, we have the following lemma:

lemma 3.3. *Let M be a quadric surface of the first kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are equivalent:*

(1): *M has non-zero constant Gauss curvature $K=\frac{1}{c}$.*

(2): *M has non-zero constant mean curvature $H=-\frac{1}{\sqrt{c}}$.*

(3): *M has non-zero constant second mean curvature $H_{II}=-\frac{1}{\sqrt{c}}$.*

(4): *M has non-zero constant second Gaussian curvature $K_{II}=-\frac{1}{\sqrt{c}}$.*

(5): *M takes the parametrization $\Psi(u,v)=\Big(u,v,\sqrt{c-u^2-v^2}\Big)$ which is a sphere.*

3.3. Linear and nonlinear Weingarten quadric surfaces of the first kind

From now on, we will discuss the following three cases of the surface (3.1) related to the linear as well as nonlinear Weingarten property in the form $A+c_1\,B=c_2$, where $(A,B)\in\{(H_{II},H),(K_{II},H),(K_{II},H_{II})\}$.

Firstly, we consider the linear equation

$$H_{II} + c_1 H = c_2. \tag{3.15}$$

Therefore, substituting the Eqs. (3.3) and (3.5) into Eq. (3.15), one can obtain

$$4c^2\omega^3\Big[\big(H_{II} + c_1 H\big)^2 - c_2^2\Big] = \sum_{i,j=0}^{4} \Omega_{ij}\, u^{2i}\, v^{2j} = 0.$$

Here, the coefficients Ω_{ij} are

(3.16)

$$\begin{cases}
\Omega_{00} = c^4\left(\big[4-(a+b)(c_1-1)\big]^2 - 4\,c\,c_2^2\right),\\
\Omega_{01} = 2b(b+1)c^3\Big[2(b-3)\big(b(c_1-1)-4\big) + a^2(c_1-1)^2\\
\qquad\qquad +a(c_1-1)\big(b(c_1+1)-10\big) - 6cc_2^2\Big],\\
\Omega_{10} = 2a(a+1)c^3\Big[2(a-3)\big(a(c_1-1)-4\big) + b^2(c_1-1)^2\\
\qquad\qquad +b(c_1-1)\big(a(c_1+1)-10\big) - 6cc_2^2\Big],\\
\Omega_{11} = 2a\,b\,c^2(a+1)(b+1)\Big[2\big[26+(a+b)(1+5c_1)+(a^2+b^2)(c_1-1)\big]\\
\qquad\qquad +ab(5-2c_1+c_1^2)\Big],\\
\Omega_{02} = b^2c^2(b+1)^2\Big[4\big[13+b(b-5-c_1)+a(b-4)(c_1-1)-3cc_2^2\big]\\
\qquad\qquad +a^2(c_1-1)^2\Big],\\
\Omega_{20} = a^2c^2(a+1)^2\Big[4\big[13+a(a-5-c_1)+b(a-4)(c_1-1)-3cc_2^2\big]\\
\qquad\qquad +b^2(c_1-1)^2\Big],\\
\Omega_{12} = 4c\,b^2(b+1)^2a(a+1)\big[2(9-ac_1)-b(3+c_1)-3cc_2^2\big],\\
\Omega_{21} = 4c\,a^2(a+1)^2b(b+1)\big[2(9-bc_1)-a(3+c_1)-3cc_2^2\big],\\
\Omega_{22} = 24a^2b^2(a+1)^2(b+1)^2,\\
\Omega_{03} = 4c\,b^3(b+1)^3\big[2(3-b)-a(c_1-1)-cc_2^2\big],\\
\Omega_{30} = 4c\,a^3(a+1)^3\big[2(3-a)-b(c_1-1)-cc_2^2\big],\\
\Omega_{31} = 16a^3(a+1)^3b(1+b), \qquad \Omega_{13} = 16b^3(b+1)^3a(1+a),\\
\Omega_{04} = 4b^4(b+1)^4, \qquad\qquad \Omega_{40} = 4a^4(a+1)^4.
\end{cases}$$

It follows that, when $a = b = -1$, all the coefficients Ω_{ij} are equal zero. In this case, we have $c_2 = -\Big(\dfrac{1+c_1}{\sqrt{c}}\Big)$.

Similarly, we can discuss the another two cases and we can obtain the following lemma:

lemma 3.4. *Let M be a quadric surface of the first kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are equivalent:*

(1): *M is linear Weingarten surface satisfying $H_{II}+c_1\,H = -\Big(\dfrac{1+c_1}{\sqrt{c}}\Big)$.*

(2): *M is linear Weingarten surface satisfiying $K_{II}+c_1\,H = -\Big(\dfrac{1+c_1}{\sqrt{c}}\Big)$.*

(3): *M is linear Weingarten surface satisfying $K_{II}+c_1\,H_{II} = -\Big(\dfrac{1+c_1}{\sqrt{c}}\Big)$.*

(4): *The parametrization of M takes the form*

$$\Psi(u,v) = \Big(u, v, \sqrt{c-u^2-v^2}\Big)$$

which is a sphere.

Secondly, we will interest here with discussion of the two types (the first-the second) of nonlinear Weingarten property parallel to the study that considered in the above, so, we will investigate the following types:

The first type: $C^2 + c_1\,K = c_2,\ C \in \{H, H_{II}, K_{II}\}$.

We begin with the form

$$H^2 + c_1\,K = c_2. \tag{3.17}$$

Using (3.2) and (3.3), then Eq. (3.17) can be written

$$4\omega^3\Big[H^2 + c_1\,K - c_2\Big] = \sum_{i,j=0}^{3} \Omega_{ij}\,u^{2i}\,v^{2j} = 0,$$

for some non-zero constants Ω_{ij} given by
(3.18)

$$\begin{cases} \Omega_{00} = \big[(a+b)^2 + 4(a\,b\,c_1 + c_2)\big]c^2, \\ \Omega_{01} = 2b\,c\,(b+1)\Big[\big[a + (2c_1+1)b\big]a - 6c\,c_2\Big], \\ \Omega_{10} = 2a\,c\,(a+1)\Big[\big[(2c_1+1)a + b\big]b - 6c\,c_2\Big], \\ \Omega_{11} = 2a\,b\,(a+1)(b+1)(a\,b - 12c\,c_2), \\ \Omega_{02} = b^2(b+1)^2(a^2 - 12c\,c_2), \qquad \Omega_{20} = a^2(a+1)^2(b^2 - 12c\,c_2), \\ \Omega_{12} = -12a\,b^2\,(a+1)(b+1)^2\,c_2), \qquad \Omega_{21} = -12b\,a^2\,(b+1)(a+1)^2\,c_2), \\ \Omega_{03} = -4b^3(b+1)^3c_2, \qquad \Omega_{30} = -4a^3(a+1)^3c_2. \end{cases}$$

For satisfying Eq. (3.17), the constants a and b must be equal -1, and we get $c_2 = \dfrac{1+c_1}{c}$.

By the above construction one can do the other two cases $K_{II}^2 + c_1\,K = c_2$, $H_{II}^2 + c_1\,K = c_2$ and the following lemma can be summarized.

lemma 3.5. *Let M be a quadric surface of the first kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are equivalent:*

(1): *M is nonlinear Weingarten surface of the first type satisfying some equations $H^2 + c_1\,K = \dfrac{1+c_1}{c}$.*

(2): *M is nonlinear Weingarten surface of the first type satisfying some equations $K_{II}^2 + c_1\,K = \dfrac{1+c_1}{c}$.*

(3): *M is nonlinear Weingarten surface of the first type satisfying some equations $H_{II}^2 + c_1\,K = \dfrac{1+c_1}{c}$.*

(4): *The parametrization of M is a sphere:*

$$\Psi(u,v) = \Big(u, v, \sqrt{c - u^2 - v^2}\Big).$$

The second type: $C^2 + c_1\,K\,\sqrt{K} = c_2, \quad C \in \{H, H_{II}, K_{II}\}$.

In an analogous way, let now give the relation:

$$H^2 + c_1\,K\,\sqrt{K} = c_2. \tag{3.19}$$

By the aid of the formulas (3.2) and (3.3), Eq. (3.19) becomes

$$4\omega^3\Big[H^2+c_1\,K\,\sqrt{K}-c_2\Big]=\sum_{i,j=0}^{3}\Omega_{ij}\,u^{2i}\,v^{2j}=0,$$

with the coefficients Ω_{ij} defined by

(3.20)
$$\begin{cases}\Omega_{00}=(a+b)^2c^2+4(a\,b\,c)^{\frac{3}{2}}c_1-4c^3c_2,\\ \Omega_{01}=2b\,c\,(b+1)\Big[a(a+b)-6c\,c_2\Big],\\ \Omega_{10}=2a\,c\,(a+1)\Big[\big[b(a+b)-6c\,c_2\Big],\\ \Omega_{11}=2a\,b\,(a+1)(b+1)(a\,b-12c\,c_2),\\ \Omega_{02}=b^2(b+1)^2(a^2-12c\,c_2), \qquad \Omega_{20}=a^2(a+1)^2(b^2-12c\,c_2),\\ \Omega_{12}=-12ab^2(a+1)(b+1)^2c_2, \qquad \Omega_{21}=-12a^2b(a+1)^2(b+1)c_2,\\ \Omega_{03}=-4b^3(b+1)^3c_2, \qquad \Omega_{30}=-4a^3(a+1)^3c_2.\end{cases}$$

If $a=b=-1$ (i.e., the all coefficients Ω_{ij} are equal zero), then Eq. (3.19) is satisfied for $c_2=\dfrac{\sqrt{c}+c_1}{c\sqrt{c}}$.

Another two cases can be discussed similar to the above case and we can get the following lemma:

lemma 3.6. *Let M be a quadric surface of the first kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are equivalent:*

(1): *M is nonlinear Weingarten surface of the second type satisfying some equations $H^2+c_1\,K\sqrt{K}=\dfrac{\sqrt{c}+c_1}{c\sqrt{c}}$.*

(2): *M is nonlinear Weingarten surface of the second type satisfying some equations $K_{II}^2+c_1\,K\,\sqrt{K}=\dfrac{\sqrt{c}+c_1}{c\sqrt{c}}$.*

(3): *M is nonlinear Weingarten surface of the second type satisfying some equations $H_{II}^2+c_1\,K\,\sqrt{K}=\dfrac{\sqrt{c}+c_1}{c\sqrt{c}}$.*

(4): *M represents a sphere which can be written as $\Psi(u,v)=\Big(u,v,\sqrt{c-u^2-v^2}\Big)$.*

4. Quadric surfaces - the second kind

Consider M a quadric surface of the second kind in $\mathbf{E}^3$ and satisfy the equation (2.10). Thus, the surface M can be written as

$$r(u,v) = \Big(u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2\Big). \tag{4.1}$$

Following to similar steps that considered in the study of the surface (3.1), the straightforward computation on the surface (4.1), gives the Gaussian curvature K, the mean curvature H, the second Gaussian curvature K_{II} and the second mean curvature H_{II}, respectively:

$$K = \frac{a\,b}{\Delta^2}, \tag{4.2}$$

$$H = \frac{(a+b) + a\,b\Big[a\,u^2 + b\,v^2\Big]}{2\Delta^{\frac{3}{2}}}, \tag{4.3}$$

$$K_{II} = \frac{(a+b) - (a-b)\Big[a^2u^2 - b^2v^2\Big]}{2\Delta^{\frac{3}{2}}}, \tag{4.4}$$

$$H_{II} = \frac{(2a-b)a^2u^2 + (2b-a)b^2v^2 - (a+b)}{2\Delta^{\frac{3}{2}}}, \tag{4.5}$$

where $\Delta = 1 + a^2u^2 + b^2v^2$.

Considering the obtained quantities (4.2), (4.3), (4.4) and (4.5), we have the following:

lemma 4.1. *Let M be a quadric surface of the second kind. Then, the following are satisfied:*

(1): *M is non developable surface.*
(2): *M is non minimal surface.*
(3): *M is non II-minimal surface.*
(4): *M is non II-flat surface.*

4.1. Weingarten quadric surfaces - the second kind

In this section, we discuss Weingarten property of the surface (4.1) through the differentiation of their curvatures respect to the parameters u and v. In this case , we have

$$\tag{4.6} \begin{cases} K_u = -\dfrac{4a^3 b\, u}{\Delta^3}, \\ K_v = -\dfrac{4a\, b^3 v}{\Delta^3}, \end{cases}$$

$$\tag{4.7} \begin{cases} H_u = -\dfrac{a^2}{2\Delta^{\frac{5}{2}}}\big[(3a+b) + a^2 b\, u^2 + (3a-2b) b^2 v^2\big] u, \\ H_v = -\dfrac{b^2}{2\Delta^{\frac{5}{2}}}\big[(a+3b) - (2a-3b) a^2 u^2 + a\, b^2 v^2\big] v, \end{cases}$$

$$\tag{4.8} \begin{cases} (K_{II})_u = \dfrac{a^2}{2\Delta^{\frac{5}{2}}}\big[(a-b) a^2 u^2 - 5(a-b) b^2 v^2 - (5a+b)\big] u, \\ (K_{II})_v = \dfrac{b^2}{2\Delta^{\frac{5}{2}}}\big[5(a-b) a^2 u^2 - (a-b) b^2 v^2 - (a+5b)\big] v, \end{cases}$$

and

$$\tag{4.9} \begin{cases} (H_{II})_u = \dfrac{a^2}{2\Delta^{\frac{5}{2}}}\big[(7a+b) - (2a-b) a^2 u^2 + (7a-8b) b^2 v^2\big] u, \\ (H_{II})_v = \dfrac{b^2}{2\Delta^{\frac{5}{2}}}\big[(a+7b) - (8a-7b) a^2 u^2 + (a-2b) b^2 v^2\big] v. \end{cases}$$

Considering the above equations, it easily seen that

$$\tag{4.10} \begin{cases} \Phi\big(H, K\big) = \dfrac{4a^3 b^3 (b-a)\, u\, v}{\Delta^{9/2}}, \\ \Phi\big(K_{II}, K\big) = \dfrac{8a^3 b^3 (b-a)\, u\, v}{\Delta^{9/2}}, \\ \Phi\big(H_{II}, K\big) = \dfrac{12a^3 b^3 (b-a)\, u\, v}{\Delta^{9/2}}, \\ \Phi\big(K_{II}, H\big) = \dfrac{a^2 b^2 (b^2-a^2)\, u\, v}{2\Delta^3}, \\ \Phi\big(H_{II}, H\big) = \dfrac{a^2 b^2 (a^2-b^2)\, u\, v}{\Delta^3}, \\ \Phi\big(K_{II}, H_{II}\big) = \dfrac{a^2 b^2 (b^2-a^2)\, u\, v}{2\Delta^3}. \end{cases}$$

According to the mentioned calculations, all the jacobi functions given in (4.10) are equal zero if and only if the constants a and b are equal. Thus, we formulate the following

lemma 4.2. *Let M be a quadric surface of the second kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are equivalent:*

(1): *M is a Weingarten surface of type $\Phi(H, K) = 0$.*

(2): *M is a Weingarten surface of type* $\Phi(K_{II}, K) = 0$.
(3): *M is a Weingarten surface of type* $\Phi(H_{II}, K) = 0$.
(4): *M is a Weingarten surface of type* $\Phi(K_{II}, H) = 0$.
(5): *M is a Weingarten surface of type* $\Phi(H_{II}, H) = 0$.
(6): *M is a Weingarten surface of type* $\Phi(K_{II}, H_{II}) = 0$.
(7): *M is an elliptic paraboloid:* $\Psi(u, v) = \Big(u, v, \frac{a}{2}(u^2 + v^2)\Big)$.

4.2. Quadric surfaces - the second kind with non-zero constant curvatures

Here, depending on Eqs. (4.2)-(4.5), we can discuss the non-zero constant curvatures property of the quadric surface (4.1) as the following

If the Gaussian curvature K is non-zero constant, we can write (4.2) in the form:

$$0 = K\,\Delta^2 - a\,b = \sum_{i,j=0}^{2} \Omega_{ij}\, u^{2i}\, v^{2j}$$

where the non-zero coefficients Ω_{ij} are

$$\left\{ \begin{array}{ll} \Omega_{00} = K - a\,b, & \Omega_{01} = 2b^2\,K, \\ \Omega_{10} = 2a^2\,K, & \Omega_{11} = 2a^2\,b^2\,K, \\ \Omega_{02} = b^4\,K, & \Omega_{20} = a^4\,K, \end{array} \right. \tag{4.11}$$

From the equation (4.11), the Gaussian curvature is constant if and only if $a = b = 0$ which is contradiction.

Similarly, by the straightforward computations and discussion as above for the remainder three cases of *constant mean*, *second mean* and *second Gaussian curvatures*, we can obtain the following lemma:

lemma 4.3. *Let M be a quadric surface of the second kind in Euclidean 3-space $\mathbf{E}^3$. Then, the following are satisfied:* ***(1):*** *M has no constant Gaussian curvature.* ***(2):*** *M has no constant mean curvature.* ***(3):*** *M has no constant second Gaussian curvature.* ***(4):*** *M has no constant second mean curvature.*

4.3. Linear and nonlinear Weingarten quadric surfaces of the second kind

In the following, firstly we shall study some special relations that related to the linear Weingarten property which in the form $A + c_1\,B = c_2$, where, $(A, B) \in \{(H_{II}, H), (K_{II}, H), (K_{II}, H_{II})\}$, we start as follows. Consider the following equation

$$H_{II} + c_1\,H = c_2. \tag{4.12}$$

After substituting by (4.3) and (4.5), it becomes

$$4\Delta^3\Big[\big(H_{II}+c_1\,H\big)^2-c_2^2\Big]=\sum_{i,j=0}^{3}\Omega_{ij}\,u^{2i}\,v^{2j}=0,$$

where

$$(4.13)\quad \begin{cases} \Omega_{00}=(a+b)^2(c_1-1)^2-4c_2^2,\\ \Omega_{01}=2b^2\big[2b^2(c_1-1)+a^2(c_1-1)^2+ab(c_1^2-1)-6c_2^2,\\ \Omega_{10}=2a^2\big[2a^2(c_1-1)+b^2(c_1-1)^2+ab(c_1^2-1)-6c_2^2,\\ \Omega_{11}=2a^2b^2\big[2(a^2+b^2)(c_1-1)+ab(5-2c_1+c_1^2)-12c_2^2\big],\\ \Omega_{02}=b^4\big[4ab(c_1-1)+a^2(c_1-1)^2+4(b^2-3c_2^2)\big],\\ \Omega_{20}=a^4\big[4a^2+4ab(c_1-1)+b^2(c_1-1)^2-12c_2^2\big],\\ \Omega_{21}=-12a^4b^2c_2^2,\qquad \Omega_{03}=-4b^6c_2^2,\qquad \Omega_{30}=-4a^6c_2^2. \end{cases}$$

When $\Omega_{30}=0$, we have $c_2=0$. Therefore, $\Omega_{00}=(a+b)^2(c_1-1)^2$, which gives $c_1=0$. Now, since $\Omega_{20}=4b^6=0$ and $\Omega_{02}=4a^6=0$, it gives a contradiction. Under the previous, we consider the following lemma:

lemma 4.4. *Let M be a quadric surface of the second kind in Euclidean 3-space $\mathbf{E}^3$. Then, there are no constants c_1 and c_2 such that:* ***(1):*** *M is linear Weingarten surface satisfying $H_{II}+c_1\,H=c_2$.* ***(2):*** *M is linear Weingarten surface satisfying $K_{II}+c_1\,H=c_2$.* ***(3):*** *M is linear Weingarten surface satisfying $K_{II}+c_1\,H_{II}=c_2$.*

Secondly, we discuss also for quadric surfaces of the second kind the two different types of nonlinear Weingarten property as we have done for the first kind: **The first type:** $C^2+c_1\,K=c_2,\ \ C\in\{H,H_{II},K_{II}\}$. Here, we start with the following equation

$$(4.14)\qquad H^2+c_1\,K=c_2.$$

Using (4.3) and (4.4), we can write the above equation in the form:

$$4\Delta^3\Big[H^2+c_1\,K-c_2\Big]=\sum_{i,j=0}^{3}\Omega_{ij}\,u^{2i}\,v^{2j}=0$$

with

$$(4.15)\quad \begin{cases} \Omega_{00}=(a+b)^2+2abc_1-4c_2,\\ \Omega_{01}=2b^2\big[a^2+ab(1+c_1)-6c_2\big],\\ \Omega_{10}=2a^2\big[b^2+ab(1+c_1)-6c_2\big],\\ \Omega_{11}=2a^2b^2(ab-12c_2),\\ \Omega_{02}=b^4(a^2-12c_2),\quad \Omega_{20}=a^4(b^2-12c_2),\\ \Omega_{12}=-12a^2b^6c_2,\quad \Omega_{21}=-12a^4b^2c_2,\\ \Omega_{03}=-4b^6c_2,\qquad \Omega_{30}=-4a^6c_2. \end{cases}$$

From (4.15), if $\Omega_{30} = 0$, we have $c_2 = 0$ and therefore $\Omega_{20} = a^4 b^2 = 0$. This is a contradiction. We give the following lemma

lemma 4.5. *For some constants c_1 and c_2, there are no quadric surfaces of the second kind in $\mathbf{E}^3$satisfying $C^2 + c_1 K = c_2$ with $C = \{H, K_{II}, H_{II}\}$.*

The second type: $C^2 + c_1 K \sqrt{K} = c_2, \quad C \in \{H, H_{II}, K_{II}\}$. The following equation includes the second type, we write it as follows

$$H^2 + c_1 K \sqrt{K} = c_2. \tag{4.16}$$

The use of Eqs. (4.2) and (4.3) change Eq. (4.16) to the form

$$4\omega^3 \Big[H^2 + c_1 K \sqrt{K} - c_2 \Big] = \sum_{i,j=0}^{3} \Omega_{ij}\, u^{2i}\, v^{2j} = 0$$

where its coefficients are

$$\begin{cases} \Omega_{00} = (a+b)^2 + 2abc_1\sqrt{ab} - 4c_2, \\ \Omega_{01} = 2b^2 \big[a^2 + ab - 6c_2 \big], \\ \Omega_{10} = 2a^2 \big[b^2 + ab - 6c_2 \big], \\ \Omega_{11} = 2a^2 b^2 (ab - 12c_2), \\ \Omega_{02} = b^4 (a^2 - 12c_2), \qquad \Omega_{20} = a^4 (b^2 - 12c_2), \\ \Omega_{12} = -12a^2 b^6 c_2, \qquad \Omega_{21} = -12a^4 b^2 c_2, \\ \Omega_{03} = -4b^6 c_2, \qquad \Omega_{30} = -4a^6 c_2. \end{cases} \tag{4.17}$$

If $\Omega_{30} = 0$, then we have $c_2 = 0$, thus $\Omega_{20} = a^4 b^2 = 0$. This contradicts the fact that $a > 0$ and $b > 0$. So, we present the following lemma

lemma 4.6. *For some constants c_1 and c_2, there are no quadric surfaces of the first kind in $\mathbf{E}^3$satisfying $C^2 + c_1 K \sqrt{K} = c_2$ with $C = \{H, K_{II}, H_{II}\}$.*

5. Conclusion

In this work, we have disscused the two kinds of quadric surfaces $\Psi(u,v) = \Big(u, v, \sqrt{a\,u^2 + b\,v^2 + c} \Big)$ and $r(u,v) = \Big(u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2 \Big)$ in Euclidean 3-space $\mathbf{E}^3$. These surfaces satisfying the Jacobi condition respect to their curvatures. In addition, the linear Weingarten property is investigated. Moreover, special types of nonlinear Weingarten quadric surfaces are studied. Finally, some important results are obtained.

References

[1] J. Weingarten, *Ueber eine Klasse auf einander abwickelbarer Flaachen*, J. Reine Angew. Math. **59** (1861) 382–393.

[2] J. Weingarten, *Ueber die Flachen derer Normalen eine gegebene Flache beruhren*, J. Reine Angew. Math. **62** (1863) 61–63.

[3] Van-Brunt and K. Grant, *Potential applications of Weingarten surfaces in CAGD, Part I: Weingarten surfaces and surface shape investigation*, Comput. Aided Geom. Des. **13** (1996) 569–582.

[4] W. Kühnel, *Ruled W-surfaces*, Arch. Math. **62** (1994) 475–480.

[5] D. W. Yoon, *Some properties of the helicoid as ruled surfaces*, JP J. Geom. Topology **2(2)** (2002) 141–147.

[6] F. Dillen and W. Kühnel, *Ruled Weingarten surfaces in a Minkowski 3-space*, Manuscripta Math. **98** (1999) 307–320.

[7] H. Y. Kim and D. W. Yoon, *Classification of ruled surfaces in a Minkowski 3-space* J. Geom. Phys. **49** (2004) 89–100.

[8] W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski 3-space*, PhD. thesis, Katholieke Universiteit Leuven, Belgium, (2005).

[9] M. I. Munteanu and A. I. Nistor, *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, Differential geometry, 316-320, Worled Sci. Publ., Hackensack, NJ, (2009).

[10] R. Lopez, *On linear Weingarten surfaces*, International J. Math. **19** (2008) 439–448.

[11] R. Lopez, *Special Weingarten surfaces foliated by circles*, Monatsh. Math. **154** (2008) 289–302.

[12] R. Lopez, *Parabolic Weingarten surfaces in hyperbolic space*, Publ. Math. Debrecen **74** (2009) 59–80.

[13] J. Ro and D. W.Yoon, *Tubes of Weingarten types in a Euclidean 3-space*, J. Chungcheong Math. Soc. **22 (3)** (2009) 359–366.

[14] Y. H. Kim and D. W. Yoon, *Weingarten quadric surfaces in a Euclidean 3-space*, Turk. J. Math. **35** (2011) 479–485.

[15] D. W. Yoon and J. S. Jun, *Non-degenerate quadric surfaces in Euclidean 3-space*, Int. J. Math. Anal. **6 (52)** (2012) 2555–2562.

[16] H. S. Abdel-Aziz and M. Khalifa Saad, *Weingarten timelike tube surfaces around a spacelike curve,* Int. J. of Math. Analysis, **25 (5)** (2011) 1225-1236.

[17] H. S. Abdel-Aziz and M. Khalifa Saad and Sezai Kiziltug, *Parallel Surfaces of Weingarten Type in Minkowski 3-Space,* Int. Math. Forum, **4 (7)** (2012) 2293-2302.

[18] D. A. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination,* Now York, (1990).

[19] M. do Carmo, *Differential geometry of curves and surfaces,* Prentice-Hall, Englewood Cliffs, (1976).

[20] S. Verpoort, *The Geometry of the second fundamental form: Curvature properties and variational aspects*, PhD. Thesis, Katholieke Universiteit Leuven, Belgium, (2008).

[21] C. Baikoussis and T. Koufogiorgos, *On the inner curvature of the second fundamental form of helicoidal surfaces*, Arch. Math. **68 (2)** (1997) 169–176.

[22] T. Yilmaz and K. K. Murat, *Differential Geometry of three dimensions*, Syndic of Cambridge University press, (1981).
[23] Y. B. Chen and F. Dillen, *Quadrics of finite type*, J. Geom. **38** (1990) 16–22.

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