

REGULARITY OF 3D NAVIER-STOKES EQUATIONS WITH SPECTRAL DECOMPOSITION

HYOSUK JEONG

Abstract. In this paper, we consider the global existence of strong solutions to the incompressible Navier-Stokes equations on the cubic domain in \mathbf{R}^3 . While the global existence for arbitrary data remains as an important open problem, we here provide with some new observations on this matter. We in particular prove the global existence result when Ω is a cubic domain and initial and forcing functions are some linear combination of functions of at most two variables and the like by decomposing the spectral basis differently.

1. Introduction

We consider the initial boundary value problem of the incompressible Navier-Stokes equations,

$$(1) \quad \frac{du}{dt} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f,$$

$$(2) \quad \nabla \cdot u = 0,$$

over a rectangular box Ω with periodic boundary conditions. Here u denotes the velocity of a homogeneous, viscous incompressible fluid, f is the density of force per unit volume, p denotes the pressure, and ν is the kinematic viscosity. We require that the forcing function f and the initial data u_0 satisfy

$$(3) \quad \int_{\Omega} f dx = \int_{\Omega} u_0 dx = 0.$$

By the classical results of Leray and Hopf ([11], [6]), there exists a global weak solution of the Navier-Stokes equations. But global strong solutions have until recently only been guaranteed for small data. See,

Received June 29, 2016. Accepted July 4, 2016.

2010 Mathematics Subject Classification. 35Q30, 35K51.

Key words and phrases. Navier-Stokes equations, global existence, strong solution.

for example, [3], [5], [13], [16] and the references therein. While the global existence for arbitrary data remains as an important open problem, we here provide with some new observations on this matter.

In this paper, we split the ambient Hilbert space H into V_N and W_N orthogonally by using the Fourier modes, where any function in V_N is a finite linear combination of functions and each typical term has at most first N Fourier modes for at least one variable and an element in W_N corresponds to the higher frequency mode. Here N denotes the number of mode used to generate the space V_N . In contrast to the Bubnov-Galerkin approximation ([13]), V_N is an infinite dimensional closed subspace. Moreover it contains all the functions (essentially of two variables) obtained after averaging in every direction and preserves two spacial dimensional properties while the first eigenvalue of the Stokes operator restricted on W_N is of order N^2 . Thus we may apply the argument used for thin domains by taking N as a new variable instead of ϵ .

2. N-S equations with spectral decomposition

We reformulate (1) and (2) in the standard nonlinear evolutionary equation on a suitable Hilbert space H ,

$$(4) \quad \frac{du}{dt} + \nu Au + B(u, u) = \mathbb{P}f,$$

where \mathbb{P} is the Leray projection of $L^2(\Omega)$ onto its range H , the space of divergence-free and average-free vector fields and $Au = -\mathbb{P}\Delta u$ is the Stokes operator with domain

$$D(A) = H^2(\Omega) \cap V, \quad V = H^1(\Omega) \cap H,$$

where $H^2(\Omega)$ and $H^1(\Omega)$ denote the usual Sobolev spaces. $D(A)$ and V are equipped with norms $\|Au\|$ and $\|A^{1/2}u\| = \|\nabla u\|$, (see for example [3], [13], and [16]). We also define the bilinear form $B(u, v) = \mathbb{P}(u \cdot \nabla)v$ and the trilinear form $b(u, v, w)$ by

$$b(u, v, w) = \langle B(u, v), w \rangle = \int_{\Omega} B(u, v) \cdot w dx.$$

We will be interested in solutions of (4) with the initial data u_0 satisfying

$$(5) \quad u_0 \in V.$$

We also assume that the forcing function $f = f(t)$ satisfies

$$(6) \quad f(t) \in L^\infty((0, \infty), H) \cap L^2((0, \infty), H).$$

Under these assumption made on u_0 and $f(t)$, there exists a local strong solution $u \in L^\infty((0, T), V) \cap L^2((0, T), D(A))$ of (4). Thus in order to extend solutions globally it suffices to control $\|\nabla u(t)\|$ uniformly on the interval of local existence, (see for example [3], [13], and [16]).

The space H can be described in terms of the Fourier series expansion for the function $u \in L^2(\Omega)$. For $k = (k_1, k_2, k_3)$ in the integer lattice \mathbf{Z}^3 , the Fourier series expansion for $u \in L^2(\Omega)$ is given by

$$(7) \quad u(x) = \sum_{k \in \mathbf{Z}^3} c^k e^{2\pi i k \cdot x},$$

where $c^k \in \mathbf{C}^3, \overline{c^k} = c^{-k}$, and

$$c^k = \int_{\Omega} u(x) e^{2\pi i k \cdot x} dx, \quad k \in \mathbf{Z}^3.$$

Consequently, one has $u \in H$ if and only if $c^0 = 0, \sum_{k \in \mathbf{Z}^3} |c^k|^2 < \infty$, and

$$(8) \quad k_1 c_1^k + k_2 c_2^k + k_3 c_3^k = 0 \quad \text{for all } k \in \mathbf{Z}^3.$$

Similarly one has $u \in V$ if and only if $u \in H$ and

$$\sum_{k \in \mathbf{Z}^3} |k|^2 |c^k|^2 < \infty,$$

where $|k|^2 = k_1^2 + k_2^2 + k_3^2$.

For any nonnegative integer N , the space H has the natural orthogonal decomposition

$$(9) \quad H = V_N \oplus W_N.$$

In fact, for u given by (7), $u = v + w$ and $v \in V_N$ if and only if

$$v = v(x) = \sum c^k e^{2\pi i k \cdot x}$$

and the summation is taken over the $k = (k_1, k_2, k_3)$'s such that $|k_i| \leq N$ for some $i = 1, 2, 3$ and $w \in W_N$ if and only if

$$w = w(x) = \sum c^k e^{2\pi i k \cdot x}$$

and the summation is taken over the $k = (k_1, k_2, k_3)$'s such that $|k_i| > N$ for all $i = 1, 2, 3$.

We will denote by P_N and Q_N the orthogonal projection from H onto V_N and W_N respectively. We apply the projections P_N and Q_N to

the equation (4) with $v = P_N u$ and $w = Q_N \bar{u}$. Then, the v -equation satisfied by $v = P_N u$ is

$$(10) \quad \frac{dv}{dt} + \nu Av + P_N B(v + w, v + w) = P_N f$$

and the w -equation satisfied by $w = Q_N \bar{u}$ is

$$(11) \quad \frac{dw}{dt} + \nu Aw + Q_N B(v + w, v + w) = Q_N f.$$

In comparison with thin domain case, $w(t) \equiv 0$ is not an invariant subspace. Nevertheless by taking $w = 0$ in (10), we may find an approximation to 3D Navier-Stokes equations:

$$\frac{dv}{dt} + \nu Av + P_N B(v, v) = P_N f.$$

It features 2D Navier-Stokes equations and the global existence of strong solutions is guaranteed by the estimate (17) below.

It is easy to observe that the Pöincare inequality holds for $w \in W_N \cap V$:

$$(12) \quad \|w\|^2 \leq \frac{1}{12\pi^2(N+1)^2} \|A^{1/2} w\|^2.$$

The essential ingredient of our theory is the Sobolev imbedding:

$$(13) \quad \|u\|_{L^6(\Omega)}^6 \leq d_1 \Pi_{i=1}^3 \left(\|u\|^2 + \left\| \frac{\partial u}{\partial x_i} \right\|^2 \right),$$

see Lemma 5.9 of [1]. For $v \in V_N$, (13) reads as follows since each term of v has at most N Fourier mode for at least one variable:

Lemma 2.1. For $v \in V_N, \forall p \geq 2$

$$(14) \quad \|v\|_p \leq CN \|v\|^{2/p} \|A^{1/2} v\|^{1-2/p}.$$

Proof. For $v \in V_N$, the summation is taken over the $k = (k_1, k_2, k_3)'$'s such that $|k_i| \leq N$ for some $i = 1, 2, 3$. Thus we have

$$\begin{aligned} v &= v_1 + v_2 + v_3 \\ &\equiv \sum_{|k_1| \leq N} c^k e^{2\pi i k \cdot x} + \sum_{|k_1| > N, |k_2| \leq N} c^k e^{2\pi i k \cdot x} + \sum_{|k_1| > N, |k_2| > N, |k_3| \leq N} c^k e^{2\pi i k \cdot x} \end{aligned}$$

Where

$$v_1 = \sum_{|k_1| \leq N} e^{2\pi i k_1 x_1} \sum_{\hat{k}=(k_2, k_3)} c^{(k_1, \hat{k})} e^{2\pi i \hat{k} \cdot \hat{x}}, \quad \hat{x} = (x_2, x_3).$$

Now,

$$\begin{aligned} \|v_1\|_p &\leq \sum_{|k_1| \leq N} \left\| \sum_{\hat{k}=(k_2, k_3)} c^{(k_1, \hat{k})} e^{2\pi i \hat{k} \cdot \hat{x}} \right\|_p \\ &\leq \sum_{|k_1| \leq N} C \left\| \sum_{\hat{k}=(k_2, k_3)} c^{(k_1, \hat{k})} e^{2\pi i \hat{k} \cdot \hat{x}} \right\|_p^{\frac{2}{p}} \|\nabla \left(\sum_{\hat{k}=(k_2, k_3)} c^{(k_1, \hat{k})} e^{2\pi i \hat{k} \cdot \hat{x}} \right)\|^{1-\frac{2}{p}} \\ &= \sum_{|k_1| \leq N} C \left(\sum_{\hat{k}=(k_2, k_3)} |c^{(k_1, \hat{k})}|^2 \right)^{\frac{1}{2} \frac{2}{p}} \left(\sum_{\hat{k}=(k_2, k_3)} 4\pi^2 |\hat{k}|^2 |c^{(k_1, \hat{k})}|^2 \right)^{\frac{1}{2} (1-\frac{2}{p})} \\ &\leq CN \|v_1\|_p^{\frac{2}{p}} \|A^{1/2} v_1\|^{1-\frac{2}{p}} \end{aligned}$$

Similar inequalities hold for v_2, v_3 and thus (14). □

With the use of this inequality, we find for $v \in V_N$

$$(15) \quad \|v\|_4 \leq d_2 N^{\frac{1}{4}} \|v\|^{1/2} \|A^{1/2} v\|^{1/2}.$$

Moreover for $w \in W_N$, one can see from (12) that

$$(16) \quad \|w\|_4 \leq d_3 N^{-1/4} \|A^{1/2} w\|.$$

In turn, the auxiliary estimates regarding the trilinear form $b(v, v, Av)$ can be derived. If $v = Pv$, then one has

$$(17) \quad |b(v, v, Av)| \leq d_4 N^{1/2} \|v\|^{1/2} \|A^{1/2} v\| \|Av\|^{3/2}.$$

Since the first eigenvalue of A restricted on W_N grows as N , we may apply the previous arguments developed for thin domains. But here we can not take advantage of smallness of w as before because v and w are linked somehow and $w(t) \equiv 0$ is not a solution anymore even if we assume $Qf = 0$. Instead we here apply a small data argument to prove the next main result:

Theorem 2.2. *The Navier-Stokes equations has a global strong solution whenever $u_0 = v_0 + w_0 \in V_N \oplus W_N$ and $f(t)$ satisfy*

$$(18) \quad \|\nabla v_0\| \leq k(\nu) N^{1/4}, \quad \|\nabla w_0\| \leq k(\nu) N^{1/4},$$

$$(19) \quad \int_0^\infty \frac{1}{\nu N} \|f\|^2 dt \leq k(\nu)$$

for some positive constant $k(\nu)$ depending only on ν .

3. The proof of Theorem 2.2

In this section, We give the proof of main result, Theorem 2.2. We next present estimates for the trilinear form $b(\cdot, \cdot, \cdot)$ and its proof depend heavily on the interpolation inequalities ([1]):

$$(20) \quad \|u\|_\infty = \|u\|_{L^\infty(\Omega)} \leq k \|A^{1/2}u\|^{1/2} \|Au\|^{1/2},$$

$$(21) \quad \|A^{1/2}u\| \leq \|u\|^{1/2} \|Au\|^{1/2}.$$

Here and hereafter, we denote by k universal constants (not depending on N and ν).

Lemma 3.1. *For $v, w \in D(A)$ and $u \in H$, the following inequalities hold:*

$$(22) \quad |b(v, v, u)| \leq kN^{1/2} \|v\|^{1/2} \|A^{1/2}v\| \|Av\|^{1/2} \|u\|,$$

$$(23) \quad |b(v, w, u)| \leq k \|A^{1/2}v\|^{1/2} \|Av\|^{1/2} \|A^{1/2}w\| \|u\|,$$

$$(24) \quad |b(w, v, u)| \leq k \|A^{1/2}v\| \|A^{1/2}w\|^{1/2} \|Aw\|^{1/2} \|u\|.$$

Moreover, given $w \in W_N \cap D(A)$ and $u \in H$, we have

$$(25) \quad |b(w, w, u)| \leq kN^{-1/2} \|A^{1/2}w\| \|Aw\| \|u\|.$$

Proof. For (22), we use the Hölder’s inequality and (15) to get

$$\begin{aligned} |b(v, v, u)| &\leq \|v\|_4 \|A^{1/2}v\|_4 \|u\| \\ &\leq k_1 N^{1/4} \|v\|^{1/2} \|A^{1/2}v\|^{1/2} k_2 N^{1/4} \|A^{1/2}v\|^{1/2} \|Av\|^{1/2} \|u\| \\ &\leq kN^{1/2} \|v\|^{1/2} \|A^{1/2}v\| \|Av\|^{1/2} \|u\|. \end{aligned}$$

(23) and (24) are easily derived from (20).

With the use of (22), we have

$$|b(w, w, u)| \leq kN^{1/2} \|w\|^{1/2} \|A^{1/2}w\| \|Aw\|^{1/2} \|u\|$$

and then, using (12) and (21), we finally get (25). □

Lemma 3.2. *For $v, w \in D(A)$, the following inequalities hold:*

$$(26) \quad |b(v, v, \nabla w)| \leq CN^2 \|v\|^{2/3} \|\nabla v\|^{10/3} + \frac{1}{3} \|A^{3/4}w\|^2,$$

$$(27) \quad |b(w, w, \nabla w)| \leq C_0 \|A^{1/4}w\| \|A^{3/4}w\|^2,$$

$$(28) \quad |b(v, w, \nabla w)| \leq CN^4 \|v\|^{4/3} \|\nabla v\|^{8/3} \|A^{1/4}w\|^2 + \frac{1}{3} \|A^{3/4}w\|^2,$$

$$(29) \quad |b(w, v, \nabla w)| \leq C \|A^{1/4}w\|^2 \|\nabla v\|^4 + \frac{1}{3} \|A^{3/4}w\|^2.$$

Proof. The proof is accomplished by applying the Hölder’s inequality, the Sobolev imbedding, the Young’s inequality and the Poincaré inequality. Exactly by the same argument as in the previous lemma. \square

Multiply $A^{1/2}w$ to the w -equation (11)(For simplicity, we will consider $\nu = 1$ and $\mathbb{P}f = 0$.) and integrate to obtain

$$(30) \quad \frac{1}{2} \frac{d}{dt} \|A^{1/4}w\|^2 + \|A^{3/4}w\|^2 + (B(u, u), A^{1/2}w) = 0.$$

From Lemma 3.2 and the Poincaré inequality,

$$\begin{aligned} \frac{d}{dt} \|A^{1/4}w\|^2 + (1 - 2C_0\|A^{1/4}w\|)\|A^{3/4}w\|^2 \\ \leq C(1 + N^4)\|A^{1/2}v\|^4\|A^{1/4}w\|^2 + CN^2\|A^{1/2}v\|^4. \end{aligned}$$

Let us assume that

$$1 - 2C_0\|A^{1/4}w(t)\| < 0 \quad \text{for } 0 \leq t < T.$$

Then by the Grönwall inequality

$$\begin{aligned} \|A^{1/4}w(t)\|^2 \leq \exp\left(C(1 + N^4) \int_0^T \|A^{1/2}v(\tau)\|^4 d\tau\right) \\ \left[\|A^{1/4}w(0)\|^2 + CN^2 \int_0^T \|A^{1/2}v(\tau)\|^4 d\tau \right]. \end{aligned}$$

As a consequence we come to a conclusion:

Theorem 3.3. *For some constants k_1, k_2 , there exists such that whenever*

$$(1 + N^4) \int_0^T \|A^{1/2}v(\tau)\|^4 d\tau \leq k_1, \quad \|A^{1/4}w(0)\| \leq k_2$$

a global strong solution for the Navier-Stokes equations (4) exists.

We shall write differential inequalities for $\|A^{1/2}v\|^2$ and $\|A^{1/2}w\|^2$ at the same time and derive an estimate for the suitable sum of them. First of all, taking the scalar product of the v -equation (10) with Av , we obtain

$$\frac{1}{2} \frac{d}{dt} \|A^{1/2}v\|^2 + \nu \|Av\|^2 + b(u, u, Av) = \langle P_N f, Av \rangle \leq \|P_N f\| \|Av\|.$$

From the linearity

$$b(u, u, Av) = b(v, v, Av) + b(v, w, Av) + b(w, v, Av) + b(w, w, Av)$$

and by Lemma 3.1, the Hölder inequality and (12)

$$(31) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{1/2}v\|^2 + \frac{\nu}{2} \|Av\|^2 &\leq kN^2 \|v\|^2 \|A^{1/2}v\|^4 + \frac{1}{\nu} \|P_N f\|^2 \\ &+ \frac{k}{\nu N} \|Aw\|^2 \left(\|A^{1/2}v\|^2 + \|A^{1/2}w\|^2 + \frac{1}{N} \|A^{1/2}v\|^2 \|A^{1/2}w\|^2 \right). \end{aligned}$$

Similarly taking the scalar product of the w -equation (11) with Aw , we have

$$\frac{1}{2} \frac{d}{dt} \|A^{1/2}w\|^2 + \nu \|Aw\|^2 + b(u, u, Aw) = \langle Q_N f, Aw \rangle.$$

We then obtain

$$(32) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{1/2}w\|^2 + \frac{\nu}{2} \|Aw\|^2 &\leq kN^2 \|v\|^2 \|A^{1/2}v\|^4 + \frac{1}{\nu} \|Q_N f\|^2 + \frac{\nu}{2} \|Av\|^2 \\ &+ \frac{k}{N^2} \|A^{1/2}v\|^2 \|A^{1/2}w\|^2 \|Av\|^2 \\ &+ \frac{k}{\sqrt{N}} \left(\|A^{1/2}v\| + \|A^{1/2}w\| \right) \|Aw\|^2. \end{aligned}$$

Let us denote

$$G^2 = \frac{1}{N} \|A^{1/2}v\|^2 + \frac{1}{N} \|A^{1/2}w\|^2.$$

Then (31) and (32) become

$$(33) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} G^2 + \left(\frac{\nu}{2} - kG^4 \right) \frac{\|Av\|^2}{N} + \left(\frac{\nu}{2} - kG - \frac{k}{\nu} G^2 - \frac{k}{\nu} G^4 \right) \frac{\|Aw\|^2}{N} \\ \leq kN^2 \|v\|^2 \|A^{1/2}v\|^2 G^2 + \frac{1}{\nu N} \|f\|^2. \end{aligned}$$

Let $G = G(k)$ be the first positive zero for one of

$$\frac{\nu}{2} - kG^4 = 0, \quad \frac{\nu}{2} - kG - \frac{k}{\nu} G^2 - \frac{k}{\nu} G^4 = 0.$$

If we take $G(0) < G(k)$, from the local existence of solution, there would be the first time $t = r > 0$ such that $u(t)$ is the unique strong solution for $0 \leq t \leq r$, $0 \leq G(t) < G(k)$ and $G(r) = G(k)$. For $0 \leq t \leq r$,

$$\frac{1}{2} \frac{d}{dt} G^2 \leq kN^2 \|v\|^2 \|A^{1/2}v\|^2 G^2 + \frac{1}{\nu N} \|f\|^2.$$

Then integrating, we would have

$$\begin{aligned} G(T)^2 &\leq \exp\left(2kN^2 \int_0^T \|v\|^2 \|A^{1/2}v\|^2 ds\right) \left[G(0)^2 + \int_0^T \frac{2}{\nu N} \|f\|^2 dt\right] \\ &\leq \exp\left(2kN^2(\|u_0\|^2 + \frac{1}{\nu\lambda_1} \int_0^T \|f(s)\|^2 ds)\right) \left[G(0)^2 + \int_0^T \frac{2}{\nu N} \|f(s)\|^2 ds\right]. \end{aligned}$$

Redefining $k(\nu)$ if necessary, this leads a contradiction to the assumptions. Therefore $G(t) < k(\nu)$ for all $t > 0$ and the solution becomes globally regular, which completes all the proofs.

References

- [1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] J. D. Avrin, *Large-eigenvalue global existence and regularity results for the Navier-Stokes equations*, J. Differential Equations **127** (1996), 365-390.
- [3] P. Constantin and C. Foias, *Navier-Stokes equations*, University of Chicago Press, Chicago, 1988.
- [4] Lawrence C. Evans, *Partial Differential Equations*, American Mathematical Society, Rhode Island, 1998.
- [5] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal. **16** (1964), 269-315.
- [6] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213-231.
- [7] D. Iftimie and G. Raugel, *Some results on the Navier-Stokes equations in thin 3D domains*, J. Differential Equations. **169** (2001), 281-331.
- [8] D. Iftimie, G. Raugel, and G. R. Sell, *Navier-Stokes equations in thin 3D domains with Navier boundary conditions*, Indiana Univ. Math. J. **56(3)** (2007), 1083-1156.
- [9] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch, Rational Mech. Anal. **16** (1964), 269-315.
- [10] I. Kukavica and M. Ziane, *Regularity of the Navier-Stokes equation in a thin periodic domain with large data*, Discrete and continuous dynamical system **18(1)** (2006), 67-86.
- [11] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193-248.
- [12] G. Raugel and G. R. Sell, *Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc. **6** (1993), 503-568.
- [13] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Math. Sciences 143, Springer, Berlin, 2002.
- [14] G. Ponce, R. Racke, T.C. Sideris and E.S. Titi, *Global Stability of Large Solutions to the 3D Navier-Stokes Equations*, Commun. Math. Phys. **159** (1993), 329-341.
- [15] R. Temam and M. Ziane, *Navier-Stokes equations in three-dimensional thin domains with various boundary conditions*, Adv. Differential Equations. **1** (1996), 499-546.

- [16] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, CBMS Regional Conference Series, No. 66, SIAM, Philadelphia, 1995.

Hyosuk Jeong
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Korea.
E-mail: yozosukyoo@hanmail.net