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SOME EXISTENCE AND UNIQUENESS THEOREMS ON ORDERED METRIC SPACES VIA GENERALIZED DISTANCES UNDER NEW CONTROL FUNCTIONS

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Abstract. The purpose of this paper is to prove some fixed point theorems in a complete metric space equipped with a partial ordering using w-distances together with the aid of an altering functions and new functions of admissible type.

1. Introduction with Preliminaries

Kada et al. [13] initiated the idea of w-distance which was primarily utilized to improve Caristi's fixed point theorem [3], Ekeland's variational principle [5] and the nonconvex minimization theorems whose details are available in Takahashi [25]. Proving existence results on fixed points on partially ordered metric spaces has been a relatively hot topic in metric fixed points theory. In [18], a noted analogue of Banach contraction principle in partially ordered metric space was proved, which also includes interesting applications to matrix equations. Ran and Reurings [18] have further weakened the usual contraction condition but merely up to monotone operators.

In an interesting article, Branciari [2] established a fixed point result for an integral-type inequality, which is a generalization of Banach contraction principle. Vijayaraju et al. [27] obtained a general principle, which paves the way to prove many fixed point theorems for pair of maps satisfying integral type contraction conditions.

A multitude of fixed and common fixed point theorems in metric and semi-metric spaces for compatible, weakly compatible and occasionally weakly compatible pair of mappings satisfying contractive conditions

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of integral type are proved in several papers (e.g. [15, 17, 27]). In the same continuation, Suzuki [23] proved that integral type contractions are Meir-Keeler contractions and further noted that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. Jachymski [12] also showed that most contractive conditions of integral type by many authors coincide with classical ones. But he coined a new contractive condition of integral type which is independent of classical ones. Recently Popa and Mocanu [17] obtained integral type contractions employing altering distance function and proved general common fixed point results for integral type contractive conditions.

In [19], Razani et al. proved a fixed point theorem for (ϕ,ψ,p)-contractive mappings on $\mathcal{X}$ [i.e. for each $x,y\in\mathcal{X}$, $\phi p(\mathcal{T}x,\mathcal{T}y)\le\psi\phi p(x,y)$], which is a new version of the main theorem in [2] employing the notion of the w-distance. In fact, he proved the following result.

Theorem 1.1. *([19]) Let p be a w-distance on a complete metric space $(\mathcal{X},d)$, ϕ be non-decreasing, continuous and $\phi(\epsilon)>0$ for each $\epsilon>0$ and ψ be non-decreasing, right continuous and $\psi(t)<t$ for all $t>0$. Suppose $\mathcal{T}$ is a (ϕ,ψ,p)- contractive map on $\mathcal{X}$, then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$. Moreover, $\lim\limits_{n\to\infty}\mathcal{T}^n x$ is a fixed point of $\mathcal{T}$ for each $x\in\mathcal{X}$.*

In [9] Hossein and Ing-Jer obtained some generalizations of certain fixed point theorems contained in Kada et al. [13], Hicks and Rhoades [8] and similar other results with respect to (ϕ,ψ,p)-contractive maps on a complete metric space. For further details, see [4, 10, 11, 20, 22].

In this paper, using the concept of w-distance, we prove the fixed point theorems in partially ordered metric spaces. Our results generalize, improve and simplify several fixed point results of existing literature.

Before presenting our results, we collect relevant definitions and results which will be needed in our subsequent discussion.

Definition 1.2. *Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X},d,\preceq)$ is called a partially ordered metric space if $(\mathcal{X},\preceq)$ is a partially ordered set and $(\mathcal{X},d)$ is a metric space.*

Definition 1.3. *Let $(\mathcal{X},\preceq)$ be a partially ordered set. Then:*
(i) elements $x,y\in\mathcal{X}$ are called comparable with respect to " $\preceq$ " if either $x\preceq y$ or $y\preceq x$;
(ii) a mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is called non-decreasing with respect to " $\preceq$ " if $x\preceq y$ implies $\mathcal{T}x\preceq\mathcal{T}y$.

Definition 1.4. *([13]) Let $(\mathcal{X}, d)$ be a metric space. Then a function $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is called a w-distance on $\mathcal{X}$ if the following conditions are satisfied:*

(i) *$p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in \mathcal{X}$,*
(ii) *for any $x \in \mathcal{X}$, $p(x, .) : \mathcal{X} \to [0, \infty)$ is lower semi-continuous, i.e., if $x \in \mathcal{X}$ and $y_n \to y$ in $\mathcal{X}$, then $p(x, y) \leq \liminf_n p(x, y_n)$,*
(iii) *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(x, z) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.*

Example 1.5. *([26]) Let $(\mathcal{X}, d)$ be a metric space and g be a continuous mapping from $\mathcal{X}$ into itself. Then a function $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by*

$$p(x, y) = \max\{d(gx, y), d(gx, gy)\} \text{ for every } x, y \in \mathcal{X}$$

is w-distance on $\mathcal{X}$.

Clearly every metric is a w-distance but not conversely, (see Examples 2 and 3, [21]).

Definition 1.6. *Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a function.*

(a) *$\mathcal{F}_{\mathcal{T}} = \{x \in \mathcal{X} | x = \mathcal{T}(x)\}$(i.e. the set of fixed points of $\mathcal{T}$).*
(b) *The function $\mathcal{T}$ is called Picard operator (briefly, PO) if there exists $x^* \in \mathcal{X}$ such that $\mathcal{F}_{\mathcal{T}} = \{x^*\}$ and $\{\mathcal{T}^n(x)\}$ converges to x^*, for all $x \in \mathcal{X}$.*
(c) *The function $\mathcal{T}$ is called orbitally $\mathcal{U}$-continuous for any $\mathcal{U} \subset \mathcal{X} \times \mathcal{X}$ if the following condition is satisfied:*
for any $x \in \mathcal{X}, \mathcal{T}^{n_i}(x) \to a \in \mathcal{X}$ as $i \to \infty$, and $(\mathcal{T}^{n_i}(x), a) \in \mathcal{U}$ for any $i \in N$, imply that $\mathcal{T}^{n_i+1}(x) \to \mathcal{T}a$ as $i \to \infty$.

Let $(\mathcal{X}, \preceq)$ be a partially ordered set. Let us denote by $\mathcal{X}_{\preceq}$ the subset of $\mathcal{X} \times \mathcal{X}$ defined by:

$$\mathcal{X}_{\preceq} = \{(x, y) \in \mathcal{X} \times \mathcal{X} \mid x \preceq y \text{ or } y \preceq x\}.$$

Definition 1.7. *A map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be orbitally continuous if $x \in \mathcal{X}$, and $\mathcal{T}^{n_i}(x) \to a \in \mathcal{X}$ as $i \to \infty$, imply that $\mathcal{T}^{n_i+1}(x) \to \mathcal{T}a$ as $i \to \infty$.*

Suppose

$$\Phi = \{\phi | \phi : [0, \infty) \to [0, \infty)\}$$

where ϕ is nondecreasing, continuous and $\phi(\epsilon) > 0$ for each $\epsilon > 0$. Moreover, let

$$\Psi = \{\psi | \psi : [0, \infty) \to [0, \infty)\}$$

where ψ is nondecreasing, right continuous and $\psi(t) < t$ for all $t > 0$. Also, let

$$\Gamma = \{\gamma | \gamma : [0, \infty) \to [0, \infty)\}$$

where γ is nondecreasing, continuous map and $\gamma(t) = 0$ if and only if $t = 0$ (cf. [14]).

Example 1.8. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be two non-negative sequences such that $\{a_n\}$ is strictly decreasing which converges to zero, and (for each $n \in \mathbb{N}$) $c_{n-1}a_n > a_{n+1}$ where $0 < c_{n-1} < 1$. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(0) = 0$, $\psi(t) = c_n t$. If $a_{n+1} \leq t < a_n$, $\psi(t) = c_0 t$ if $t \geq a_1$, then ψ is in Ψ.*

Now, we state the following two lemmas:

Lemma 1.9. *([21]) If $\psi \in \Psi$, then $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$.*

Lemma 1.10. *([21]) If $\phi \in \Phi$ $(\gamma \in \Gamma)$, $\{a_n\} \subset [0, \infty)$ and $\lim_{n\to\infty} \phi(a_n) = 0$ $(\lim_{n\to\infty} \gamma(a_n) = 0)$, then $\lim_{n\to\infty} a_n = 0$.*

The following two lemmas are crucial in the proofs of our main results.

Lemma 1.11. *([13, 24]) Let $(\mathcal{X}, d)$ be a metric space equipped with a w-distance p. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathcal{X}$ whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero. Then, the following conditions hold (for $x, y, z \in \mathcal{X}$):*

(a) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for $n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,*
(b) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for $n \in N$, then $\lim_{n\to\infty} d(y_n, z) = 0$,*
(c) *if $p(x_n, x_m) \leq \alpha_n$ for $n, m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence,*
(d) *if $p(y, x_n) \leq \alpha_n$ for $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.12. *([13]) Let p be a w-distance on metric space $(\mathcal{X}, d)$ and $\{x_n\}$ be a sequence in $\mathcal{X}$ such that for each $\epsilon > 0$ there exists $N_\epsilon \in N$ such that $m > n > N_\epsilon$ implies $p(x_n, x_m) < \epsilon$ (or $\lim_{m,n} p(x_n, x_m) = 0$). Then $\{x_n\}$ is a Cauchy sequence.*

Definition 1.13. *([1]) A function $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I if it satisfies the following:
for $y, z \in \mathbb{R}^+$, $g(1, z) \leq g(y, z)$.*

Example 1.14. *Define* $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by*
(1) $g(y,z) = (z+l)^y, l > 1$;
(2) $g(y,z) = (y+l)^z, l > 1$;
(3) $g(y,z) = yz$;
(4) $g(y,z) = (\frac{1+y}{2})z$;
(5) $g(y,z) = y^k z$;
(6) $g(y,z) = z$;
(7) $g(y,z) = \frac{1+2y}{3}z$;
(8) $g(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z$;
(9) $g(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z,\ l > 1$,
for all $y, z \in \mathbb{R}^+$. *Then* g *is a function of subclass of type* I.

Definition 1.15. *(*[1]*) Let* $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be a mapping. We say that the pair* (f, g) *is an upper class of type* I *if* g *is a function of subclass of type I and*
1. for $0 \leq s \leq 1,\ t \in \mathbb{R}^+ \Longrightarrow f(s,t) \leq f(1,t)$;
2. for $z, s, t \in \mathbb{R}^+$ *if* $g(1,z) \leq f(s,t) \Longrightarrow z \leq st$.

Example 1.16. *Define* $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *and* $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *by*
(1) $g(y,z) = (z+l)^y, l > 1, f(s,t) = st + l$;
(2) $g(y,z) = (y+l)^z, l > 1, f(s,t) = (1+l)^{st}$;
(3) $g(y,z) = yz, f(s,t) = st$;
(4) $g(y,z) = (\frac{1+y}{2})z, f(s,t) = st$;
(5) $g(y,z) = y^k z, f(s,t) = t$;
(6) $g(y,z) = \frac{1+2y}{3}z, f(s,t) = st$;
(7) $g(y,z) = (\frac{1+y}{2})z, f(s,t) = st$;
(8) $g(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z, f(s,t) = st$;
(9) $g(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z, l > 1, f(s,t) = (1+l)^{st}$,
for all $y, z, s, t \in \mathbb{R}^+$. *Then the pair* (f, g) *is an upper class of type* I.

Definition 1.17. *(*[1]*) A function* $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a function of subclass of type* II *if it is continuous and satisfies the following:*
for $x, y, z \in \mathbb{R}^+$, $h(1,1,z) \leq h(x,y,z)$.

Example 1.18. *Define* $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by*
(1) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(2) $h(x,y,z) = (xy+l)^z, l > 1$;

(3) $h(x,y,z) = xyz$;
(4) $h(x,y,z) = (\frac{x+y}{2})z$;
(5) $h(x,y,z) = xyz$;
(6) $h(x,y,z) = xz$;
(7) $h(x,y,z) = \frac{x+xy+y}{3}z$;
(8) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1})z$;
(9) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1} + l)^z, l > 1$;
(10) $h(x,y,z) = z$,
for all $x,y,z \in \mathbb{R}^+$. *Then* h *is a function of subclass of type* II.

Definition 1.19. *(*[1]*) Let* $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be a mapping. We say that the pair* (f,h) *is a upper class of type* II *if* h *is a function of subclass of type II and*
1. for $0 \leq s \leq 1,\ t \in \mathbb{R}^+ \Longrightarrow f(s,t) \leq f(1,t)$;
2. for $z,s,t \in \mathbb{R}^+$ *if* $h(1,1,z) \leq f(s,t) \Longrightarrow z \leq st$.

Example 1.20. *Define* $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *and* $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *by*
(1) $h(x,y,z) = (z+l)^{xy}, l > 1, f(s,t) = st + l$;
(2) $h(x,y,z) = (xy+l)^z, l > 1, f(s,t) = (1+l)^{st}$;
(3) $h(x,y,z) = xyz, f(s,t) = st$;
(4) $h(x,y,z) = (\frac{x+y}{2})z, f(s,t) = st$;
(5) $h(x,y,z) = xyz, f(s,t) = t$;
(6) $h(x,y,z) = \frac{x+xy+y}{3}z, f(s,t) = st$;
(7) $h(x,y,z) = (\frac{x+y}{2})z, f(s,t) = st$;
(8) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1})z, f(s,t) = st$;
(9) $h(x,y,z) = (\frac{\sum_{i=0}^{n} x^{n-i}y^i}{n+1} + l)^z, l > 1, f(s,t) = (1+l)^{st}$;
(10) $h(x,y,z) = z, f(s,t) = st$,
for all $x,y,z,s,t \in \mathbb{R}^+$. *Then the pair* (f,h) *is an upper class of type* II.

Definition 1.21. *Let* $\mathcal{X}$ *be a non-empty set and* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *a mapping. Let* $F \subseteq \mathcal{X}$. *Then* F *is said to be invariant under* $\mathcal{S}$ *if* $\mathcal{S}x \in F$ *for every* $x \in F$.

Admissible mappings have been defined recently by Samet et al. [22] and employed quite often in order to generalize the results on various contractions. We state next the definitions of α-F-admissible mapping, α-admissible mapping and triangular α-admissible mappings.

Definition 1.22. *Let $\mathcal{X}$ be a non-empty set and $S : \mathcal{X} \to \mathcal{X}$ a mapping. Let $F \subseteq \mathcal{X}$ and $\alpha : F \times F \to \mathbb{R}^+$. We say that $\mathcal{S}$ is an α-F-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(\mathcal{S}x, \mathcal{S}y) \geq 1, x, y \in F$.*

Note: If in Definition 1.22, $F = \mathcal{X}$, then we say that $\mathcal{S}$ is an α-admissible mapping, see ([22]).

Definition 1.23. *([6]) Let $\mathcal{S}$ be a self-mapping on $\mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ a function. Then $\mathcal{S}$ is a triangular α-admissible mapping if for $x, y, z \in \mathcal{X}$*
(i) $\alpha(x, y) \geq 1 \Longrightarrow \alpha(\mathcal{S}x, \mathcal{S}y) \geq 1$,
(ii) $\alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \Longrightarrow \alpha(x, y) \geq 1$.

Lemma 1.24. *([6]) Let $\mathcal{S}$ be a self-mapping on X and $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ a function. Then $\mathcal{S}$ is a triangular α-admissible mapping if there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, \mathcal{S}x_0) \geq 1$, then we have $\alpha(\mathcal{S}^n x_0, \mathcal{S}^m x_0) \geq 1,\ m > n$.*

Definition 1.25. *Let $\mathcal{S}$ be a self-mapping on $\mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ a function. We say that $\mathcal{S}$ is an α-regular function if for $\{x_n\} \subseteq \mathcal{X}$ with $\{x_n\} \longrightarrow x\ (\in \mathcal{X}), \alpha(x_n, x_{n+1}) \geq 1 \Longrightarrow \alpha(x_n, x) \geq 1$.*

2. On (ϕ, ψ, p)-Contractive Maps

Now, our main result is as follows:

Theorem 2.1. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist $\psi \in \Psi$, $\phi \in \Phi$, a pair (f, h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ such that*

$$h(\alpha(x, \mathcal{S}x), \alpha(y, \mathcal{S}y), \phi(p(\mathcal{S}x, \mathcal{S}y))) \leq f(\beta(x, y), \psi\phi(p(x, y))),$$

for all $(x, y) \in \mathcal{X}_{\preceq}$, where

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_{\preceq}$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k} x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k} x_0, x^*) \in \mathcal{X}_{\preceq}$ for any $k \in N$.*

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x) = 0$.

Proof. If $x_0 = \mathcal{S}x_0$ for some $x_0 \in \mathcal{X}$, then there is nothing to prove. Otherwise, let there be $x_0 \in \mathcal{X}$ such that $x_0 \neq \mathcal{S}x_0$, and $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$. Owing to monotonocity of $\mathcal{S}$, we can write $(\mathcal{S}x_0, \mathcal{S}^2x_0) \in \mathcal{X}_{\preceq}, \alpha(\mathcal{S}x_0, \mathcal{S}^2x_0) \geq 1$. Continuing this process inductively, we obtain

$$(\mathcal{S}^n x_0, \mathcal{S}^m x_0) \in \mathcal{X}_{\preceq},$$

for any $n, m \in N$.
Also, we have

$$\alpha(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) \geq 1.$$

Now, we proceed to show that

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0. \tag{1}$$

By using condition (b) and properties of ϕ, ψ, we get

$$\begin{aligned}
& h(1, 1, \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) \\
\leq \ & h(\alpha(\mathcal{S}^{n-1}x_0, \mathcal{S}^n x_0), \alpha(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0), \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) \\
\leq \ & f(\beta(\mathcal{S}^{n-1}x_0, \mathcal{S}^n x_0), \psi\phi(p(\mathcal{S}^{n-1}x_0, \mathcal{S}^n x_0))) \\
\leq \ & f(1, \psi\phi(p(\mathcal{S}^{n-1}x_0, \mathcal{S}^n x_0))).
\end{aligned}$$

Using Definition 1.19, we have

$$\begin{aligned}
\phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) &\leq \psi\phi(p(\mathcal{S}^{n-1}x_0, \mathcal{S}^n x_0) \\
&\leq \psi^2\phi(p(\mathcal{S}^{n-2}x_0, \mathcal{S}^{n-1}x_0)) \\
&\leq ... \\
&\leq \psi^{n-m}\phi(p(\mathcal{S}^m x_0, \mathcal{S}^{m+1}x_0)) \\
&\leq ... \\
&\leq \psi^n\phi(p(x_0, \mathcal{S}x_0)). \qquad (2)
\end{aligned}$$

Now, on using Lemma 1.9, $\lim\limits_{n\to\infty} \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) = 0$, which due to Lemma 1.10 gives rise

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0,$$

so that (1) is established.
Similarly, we can show

$$\lim_{n\to\infty} p(\mathcal{S}^{n+1} x_0, \mathcal{S}^n x_0) = 0. \tag{3}$$

Next, we proceed to show

$$\lim_{n,m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = 0. \tag{4}$$

Suppose (4) is untrue. Then we can find a $\delta > 0$ with sequences $\{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty}$ such that

$$p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{m_k} x_0) \geq \delta, \qquad \text{for all} \qquad k \in \{1, 2, 3, \cdots\}, \tag{5}$$

wherein $m_k > n_k$. By (1) there exists $k_0 \in \mathbb{N}$, such that $n_k > k_0$ implies

$$p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{n_k+1} x_0) < \delta. \tag{6}$$

Notice that in view of (5) and (6) $m_k \neq n_{k+1}$, we can assume that m_k is a minimum index such that (5) holds so that

$$p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{r} x_0) < \delta, \qquad \text{for} \qquad r \in \{n_{k+1}, n_{k+2}, \cdots, m_k - 1\}. \tag{7}$$

Now (1), (5) and (7) imply

$$\begin{aligned} o < \delta &\leq p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{m_k} x_0) \\ &\leq p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{m_k-1} x_0) + p(\mathcal{S}^{m_k-1} x_0, \mathcal{S}^{m_k} x_0) \\ &< \delta + p(\mathcal{S}^{m_k-1} x_0, \mathcal{S}^{m_k} x_0), \end{aligned}$$

so that

$$\lim_{k\to\infty} p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{m_k} x_0) = \delta. \tag{8}$$

If $\epsilon = \limsup_k p(\mathcal{S}^{n_k+1} x_0, \mathcal{S}^{m_k+1} x_0) \geq \delta$, then there exists $\{k_r\}_{r=1}^{\infty}$ such that

$$\lim_{r\to\infty} p(\mathcal{S}^{n_{k_r}+1} x_0, \mathcal{S}^{m_{k_r}+1} x_0) = \epsilon \geq \delta.$$

Since ϕ is continuous and nondecreasing and also $(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0) \in \mathcal{X}_{\preceq}$, by using condition (b) and (8), one gets

$$\begin{aligned} & h(1, 1, \phi(p(\mathcal{S}^{n_{k_r}+1} x_0, \mathcal{S}^{m_{k_r}+1} x_0))) \\ \leq\ & h(\alpha(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{n_{k_r}+1} x_0), \alpha(\mathcal{S}^{m_{k_r}} x_0, \mathcal{S}^{m_{k_r}+1} x_0), \\ & \phi(p(\mathcal{S}^{n_{k_r}+1} x_0, \mathcal{S}^{m_{k_r}+1} x_0)) \\ \leq\ & f(\beta(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0), \psi\phi(p(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0))) \\ \leq\ & f(1, \psi\phi(p(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0))) \\ \Longrightarrow\ & \phi(p(\mathcal{S}^{n_{k_r}+1} x_0, \mathcal{S}^{m_{k_r}+1} x_0)) \leq \psi\phi(p(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0)), \end{aligned}$$

so,

$$\phi(\delta) \leq \phi(\epsilon) = \lim_{r\to\infty} \phi(p(\mathcal{S}^{n_{k_r}+1} x_0, \mathcal{S}^{m_{k_r}+1} x_0)) \leq \psi\phi(\delta).$$

Notice that

$$\phi p(\mathcal{S}^{n_{k_r}} x_0, \mathcal{S}^{m_{k_r}} x_0) \to \phi(\delta)^{+},$$

and ψ is right continuous, therefore $\phi(\delta) = 0$. This is a contradiction and
$$\limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) < \delta,$$
so we have
$$\begin{aligned}\delta &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{m_k}x_0)\\ &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) + p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) + p(\mathcal{S}^{m_k+1}x_0, \mathcal{S}^{m_k}x_0).\end{aligned}$$
On using (1), we have
$$\begin{aligned}\delta &\leq \lim_{k\to\infty} p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) + \limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0)\\ &\quad + \lim_{k\to\infty} p(\mathcal{S}^{m_k+1}x_0, \mathcal{S}^{m_k}x_0)\\ &= \limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) < \delta.\end{aligned}$$
which is a contradiction. Thus, (4) is proved.

Owing to Lemma 1.12, $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete metric space, there exists $x^* \in \mathcal{X}$ such that $\mathcal{S}^n x_0 \to x^*$ as $n \to \infty$.

Now, we show that x^* is a fixed point of $\mathcal{S}$. If (c) holds, then $\mathcal{S}^{n+1}x_0 \to \mathcal{S}x^*$ (as $n \to \infty$). By lower semi-continuity of $p(\mathcal{S}^n x_0, \ .)$, we have
$$\begin{aligned}p(\mathcal{S}^n x_0, x^*) &\leq \liminf_{m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = \alpha_n \quad (say),\\ p(\mathcal{S}^n x_0, \mathcal{S}x^*) &\leq \liminf_{m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{m+1} x_0) = \beta_n \quad (say).\end{aligned}$$
By using (4), we have $\alpha_n, \beta_n \to 0$ as $n \to \infty$. Now, in view of Lemma 1.11, we conclude that
$$\mathcal{S}x^* = x^*.$$
Next, suppose that (c') holds. Since $\{\mathcal{S}^{n_k}x_0\}$ converges to x^*, $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_\preceq$ and $\mathcal{S}$ is $\mathcal{X}_\preceq-$continuous, it follows that $\{\mathcal{S}^{n_k+1}x_0\}$ converges to $\mathcal{S}x^*$. As earlier, by lower semi-continuity of $p(\mathcal{S}^n x_0, \ .)$, we conclude that $\mathcal{S}x^* = x^*$.

If $\mathcal{S}x = x$, we have
$$\begin{aligned}h(1, 1, \phi p(x, x)) &\leq h(\alpha(x, x), \alpha(x, x), \phi p(x, x))\\ &= h(\alpha(x, \mathcal{S}x), \alpha(x, \mathcal{S}x), \phi p(\mathcal{S}x, \mathcal{S}x))\\ &\leq f(\beta(x, x), \psi\phi(p(x, x)))\\ &\Longrightarrow \phi p(x, x) \leq \psi\phi(p(x, x)) < \phi p(x, x).\end{aligned}$$
This is a contradiction which amounts to say that $\phi p(x, x) = 0$, so that $p(x, x) = 0$. This completes the proof. □

Example 2.2. *Consider $\mathcal{X} = [0,1]$ which is a complete metric space under usual metric $d(x,y) = |x-y|$ (for all $x, y \in \mathcal{X}$). Define $p(x,y) = 4|x-y|$ for all $x, y \in \mathcal{X}$. Then p is a w-distance on $(\mathcal{X}, d)$. Now, consider $\mathcal{X}_{\preceq}$ as follows:*

$$\mathcal{X}_{\preceq} = \big\{(x,y) \in \mathcal{X} \times \mathcal{X} : x = y \ \text{ or } \ x, y \in \{0\} \cup \{\frac{1}{n} : n = 1,2,3,...\}\big\},$$

where " $\preceq$" be the usual ordering.
Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$, $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$, $\beta : \mathcal{X} \times \mathcal{X} \to [0,1]$ be given by

$$\mathcal{S}(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ \dfrac{1}{10n-1}, & \text{if} \quad x = \dfrac{1}{n} \\ \dfrac{1}{\sqrt{3}}, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \alpha(x,y) &= \begin{cases} 1, & \text{if} \quad x, y = 0 \\ 0, & \text{otherwise,} \end{cases} \\ \beta(x,y) &= 1. \end{aligned}$$

Obviously, $\mathcal{S}$ is a non-decreasing map. Also there is $x_0 = 0$ in $\mathcal{X}$ such that $0 = x_0 \preceq \mathcal{S}x_0 = 0$ i.e., $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}, \alpha(x_0, \mathcal{S}x_0) = 1 \geq 1$ and $\mathcal{S}$ satisfies condition (c'). We now show that $\mathcal{S}$ satisfies condition (b) for each $\phi \in \Phi$ and $\psi \in \Psi$. If $x = y$, condition (b) is satisfied, otherwise we have $\alpha(x, \mathcal{S}x) = 0$ or $\alpha(y, \mathcal{S}y) = 0$

$$\alpha(x, \mathcal{S}x)\alpha(y, \mathcal{S}y)\phi p(\mathcal{S}x, \mathcal{S}y) = 0 \leq \beta(x,y)\psi\phi(p(x,y)).$$

Hence condition (b) is satisfied.
Thus, all the conditions of Theorem 2.1 are satisfied implying thereby the existence of fixed point of the map $\mathcal{S}$ which are indeed two in number namely: $x = 0, \frac{1}{\sqrt{3}}$.

In Theorem 2.1, if $h(x,y,z) = z$, $f(s,t) = st$, $\beta(x,y) = 1$, we deduce the following theorem, see [21].

Theorem 2.3. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p and $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing mapping. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$,*
(b) *there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that,*

$$\phi(p(\mathcal{S}x,\mathcal{S}y)) \leq \psi(\phi(p(x,y)))$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') $\mathcal{S}$ *is orbitally* $\mathcal{X}_{\preceq}$*-continuous and there exists a subsequence* $\{\mathcal{S}^{n_k}x_0\}$ *of* $\{\mathcal{S}^n x_0\}$ *converging to* x^* *such that* $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ *for any* $k \in N$.

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if* $x = \mathcal{S}x$, *then* $p(x,x) = 0$.

Proof. Define $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ by $\alpha(x,y) = 1$. Then $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is an α-admissible mapping. So all the conditions of Theorem 2.1 are satisfied and hence proof is completed. □

In Theorem 2.1, if $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is a continuous map, we deduce the following corollary:

Corollary 2.4. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space equipped with a* w*-distance* p. *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a continuous, nondecreasing,* α*-admissible and* α*-regular mapping, where* $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. *Also, suppose that*

(a) *there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ *with* $\alpha(x_0, \mathcal{S}x_0) \geq 1$,

(b) *there exist* $\psi \in \Psi$, $\phi \in \Phi$, *a pair* (f,h) *of upper class of type* II *and* $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ *such that*

$$h(\alpha(x,\mathcal{S}x), \alpha(y,\mathcal{S}y), \phi(p(\mathcal{S}x,\mathcal{S}y)) \leq f(\beta(x,y), \psi\phi(p(x,y))),$$

for all $(x,y) \in \mathcal{X}_{\preceq}$. *Then* $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x) = 0$.

In Theorem 2.1, if $h(x,y,z) = xyz$, $f(s,t) = st$, $\beta(x,y) = 1$, we deduce the following corollary:

Corollary 2.5. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space equipped with a* w*-distance* p. *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a nondecreasing,* α*-admissible and* α*-regular mapping, where* $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. *Also, suppose that*

(a) *there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ *with* $\alpha(x_0, \mathcal{S}x_0) \geq 1$,

(b) *there exist* $\psi \in \Psi$ *and* $\phi \in \Phi$ *such that*

$$\alpha(x,\mathcal{S}x), \alpha(y,\mathcal{S}y), \phi(p(\mathcal{S}x,\mathcal{S}y)) \leq \psi(\phi(p(x,y))),$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_{\preceq}$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k}x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ for any $k \in N$.*

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, then $p(x,x) = 0$.

In Theorem 2.1, if $h(x,y,z) = (xy+l)^z, l > 1,\ f(s,t) = (1+l)^{st}$, we deduce the following corollary:

Corollary 2.6. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist $\psi \in \Psi$, $\phi \in \Phi$ and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ such that*

$$(\phi(p(\mathcal{S}x, \mathcal{S}y)) + l)^{\alpha(x,\mathcal{S}x),\alpha(y,\mathcal{S}y)} \leq \beta(x,y)\psi(\phi(p(x,y))) + l,\ l > 1,$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, where

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or,*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_{\preceq}$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k}x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ for any $k \in N$.*

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, then $p(x,x) = 0$.

In Theorem 2.3, if $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is a continuous map, we deduce the following corollary:

Corollary 2.7. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p and $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing and continuous mapping. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$,*
(b) *there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that,*

$$\phi(p(\mathcal{S}x, \mathcal{S}y)) \leq \psi(\phi(p(x,y)))$$

for all $(x,y) \in \mathcal{X}_{\preceq}$. Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if $x = \mathcal{S}x$, then $p(x,x) = 0$.

In Theorem 2.1, setting $\phi = \mathcal{I}$, the identity mapping, we deduce the following corollary:

Corollary 2.8. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_\preceq$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exists $\psi \in \Psi$, a pair (f, h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ such that*

$$h(\alpha(x, \mathcal{S}x), \alpha(y, \mathcal{S}y), p(\mathcal{S}x, \mathcal{S}y)) \leq f(\beta(x, y), \psi(p(x, y))),$$

for all $(x, y) \in \mathcal{X}_\preceq$, where

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_\preceq$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k}x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_\preceq$ for any $k \in N$.*

Then $\mathcal{F}_\mathcal{S} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x, x) \geq 1$, then $p(x, x) = 0$.

Choosing $\phi = \mathcal{I}$, the identity mapping and $\psi(t) = \alpha t$ (for all $t \in [0, \infty)$ and $\alpha \in [0, 1)$) in Theorem 2.1, we deduce the following corollary:

Corollary 2.9. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_\preceq$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist a pair (f, h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ such that*

$$h(\alpha(x, \mathcal{S}x), \alpha(y, \mathcal{S}y), p(\mathcal{S}x, \mathcal{S}y)) \leq f(\beta(x, y), \alpha p(x, y)),$$

for all $(x, y) \in \mathcal{X}_\preceq$, where $\alpha \in [0, 1)$,

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or,*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_\preceq$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k}x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_\preceq$ for any $k \in N$.*

Then $\mathcal{F}_\mathcal{S} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x, x) \geq 1$, then $p(x, x) = 0$.

Suppose, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-integrable mapping which is summable and $\int_0^\epsilon \omega(\xi) d\xi > 0$ for each $\epsilon > 0$. Now, in Theorem 2.3, set

$\phi(t) = \int_0^t \omega(\xi)d\xi$ and $\psi(t) = \alpha t$, where $\phi \in \Phi$ and $\psi \in \Psi$ and $\alpha \in [0,1)$. Hence, we can derive the following corollary as a special case:

Corollary 2.10. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p and $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be non-decreasing mapping. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$,*
(b) *for all $(x,y) \in \mathcal{X}_{\preceq}$,*

$$\int_0^{p(\mathcal{S}x,\mathcal{S}y)} \omega(\xi)d\xi \leq \alpha \int_0^{p(x,y)} \omega(\xi)d\xi$$

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_{\preceq}$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k}x_0\}$ of $\{\mathcal{S}^n x_0\}$ which converges to x^* such that $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ for any $k \in N$.*

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if $x = \mathcal{S}x$, then $p(x,x) = 0$.

We also prove the following theorem:

Theorem 2.11. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}, \alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist $\psi \in \Psi$, $\phi \in \Phi$, a pair (f,h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ such that*

$$h(\alpha(x,\mathcal{S}x), \alpha(y,\mathcal{S}y), \phi(p(\mathcal{S}x,\mathcal{S}y))) \leq f(\beta(x,y), \psi\phi(p(x,y))),$$

for all $(x,y) \in \mathcal{X}_{\preceq}$ and
(c") *for every $y \in \mathcal{X}$ with $y \neq \mathcal{S}y$,*

$$\inf\{p(x,y) + p(x,\mathcal{S}x) : x \in \mathcal{X}\} > 0.$$

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, then $p(x,x) = 0$.

Proof. Observe that the sequence $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence (in view of the proof of Theorem 2.1) and so there exists a point x^* in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}^n x_0 = x^*$. Since $\lim_{m,n\to\infty} d(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = 0$, therefore for each

$\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $p(\mathcal{S}^{N_\epsilon}x_0, \mathcal{S}^n x_0) < \epsilon$. Since $\lim\limits_{n\to\infty} \mathcal{S}^n x_0 = x^*$ and $p(x, .)$ is lower semi continuous, therefore

$$p(\mathcal{S}^{N_\epsilon}x_0, x^*) \leq \liminf_n p(\mathcal{S}^{N_\epsilon}x_0, \mathcal{S}^n x_0) \leq \epsilon.$$

Therefore $p(\mathcal{S}^{N_\epsilon}, x^*) \leq \epsilon$. Set $\epsilon = \frac{1}{k}, N_\epsilon = n_k$ so that

$$\lim_{k\to\infty} p(\mathcal{S}^{n_k}x_0, x^*) = 0.$$

Now, assume that $x^* \neq \mathcal{S}x^*$. Then due to hypothesis (c"), we have

$$\begin{aligned}0 &< \inf\{p(x, x^*) + p(x, \mathcal{S}x) : x \in \mathcal{X}\} \\ &\leq \inf\{p(\mathcal{S}^{n_k}x_0, x^*) + p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) : n \in \mathbb{N}\} \to 0,\end{aligned}$$

as $n \to \infty$. This is a contradiction. Hence $x^* = \mathcal{S}x^*$.
If $\mathcal{S}x = x$, we have

$$\begin{aligned} h(1, 1, \phi p(x, x)) &\leq h(\alpha(x, x), \alpha(x, x), \phi p(x, x)) \\ &= h(\alpha(x, \mathcal{S}x), \alpha(x, \mathcal{S}x), \phi p(\mathcal{S}x, \mathcal{S}x)) \\ &\leq f(\beta(x, x), \psi\phi(p(x, x))) \end{aligned}$$
$$\implies \phi p(x, x) \leq \psi\phi(p(x, x)) < \phi p(x, x).$$

This is a contradiction which amounts to say that $\phi p(x, x) = 0$, so that $p(x, x) = 0$. This completes the proof. □

Corollary 2.12. *Replacing condition (c''), by the hypothesis (c) or (c') of Corollary 2.8 (also of Corollary 2.9 or Corollary 2.10) the fixed point of $\mathcal{S}$ continues to exists.*

In what follows, we give a sufficient condition for the uniqueness of fixed point in Theorem 2.1 which runs as follows:
(A): for every $x, y \in \mathcal{X}$, there exists a lower bound or an upper bound z with $\alpha(x, z) \geq 1,\ \alpha(y, z) \geq 1$ (or $\alpha(z, x) \geq 1,\ \alpha(z, y) \geq 1$.
In [16], it is proved that condition (A) is equivalent to the following one:
(B): for every $x, y \in \mathcal{X}$, there exists $z = c(x, y) \in \mathcal{X}$ for which $(x, z) \in \mathcal{X}_\preceq,\ \alpha(x, z) \geq 1$ and $(y, z) \in \mathcal{X}_\preceq,\ \alpha(y, z) \geq 1$.

Theorem 2.13. *With the addition of condition (B) to the hypotheses of Theorem 2.1 (or Theorem 2.11), the fixed point of $\mathcal{S}$ turns out to be unique. Moreover*

$$\lim_{n\to\infty} \mathcal{S}^n(x) = x^*,$$

for every $x \in \mathcal{X}$ provided $x^* \in \mathcal{F}_\mathcal{S}$, i.e., map $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is a Picard operator.

Proof. Following the proof of Theorem 2.1, $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Suppose, there exist two fixed points x^* and y^* of $\mathcal{S}$ in $\mathcal{X}$. We prove that

$$p(y^*, x^*) = 0. \tag{9}$$

We distinguish two cases:
Case 1: If $(y^*, x^*) \in \mathcal{X}_{\preceq}$. Suppose that $p(y^*, x^*) > 0$, then by using condition (b) and property of ψ we get

$$\begin{aligned} h(1, 1, \phi p(y^*, x^*)) &\leq h(\alpha(y^*, y^*), \alpha(x^*, x^*), \phi p(y^*, x^*)) \\ &= h(\alpha(y^*, \mathcal{S}y^*), \alpha(x^*, \mathcal{S}x^*), \phi p(\mathcal{S}y^*, \mathcal{S}x^*)) \\ &\leq f(\beta(y^*, x^*), \psi\phi\big(p(y^*, x^*)\big)) \end{aligned}$$

$$\Longrightarrow \phi p(y^*, x^*) \leq \beta(y^*, x^*)\psi\phi\big(p(y^*, x^*)\big) \leq \psi\phi\big(p(y^*, x^*)\big) < \phi p(y^*, x^*),$$

which is a contradiction. Therefore we have (9).
Also, in view of Theorem 2.1, we have

$$p(y^*, y^*) = 0. \tag{10}$$

On using (9), (10) and Lemma 1.11, we have $y^* = x^*$, i.e., the fixed point of $\mathcal{S}$ is unique.
Case 2: If $(x^*, y^*) \notin \mathcal{X}_{\preceq}$, then owing to condition (B), there exists $z \in \mathcal{X}$ such that $(x^*, z) \in \mathcal{X}_{\preceq}$,$\alpha(x^*, z) \geq 1$ and $(y^*, z) \in \mathcal{X}_{\preceq}, \alpha(y^*, z) \geq 1$. As $(z, x^*) \in \mathcal{X}_{\preceq}$ and $(y^*, z) \in \mathcal{X}_{\preceq}$, proceeding on the lines of proof of Theorem 2.1, we can prove

$$\lim_{n\to\infty} p(\mathcal{S}^n z, x^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(\mathcal{S}^n z, y^*) = 0.$$

By using Lemma 1.11, we infer that $y^* = x^*$, i.e., the fixed point of $\mathcal{S}$ is unique.
Now, we prove

$$\lim_{n\to\infty} \mathcal{S}^n(x) = x^*,$$

for every $x \in \mathcal{X}$ provided $x^* \in \mathcal{F}_{\mathcal{S}}$ (so $\alpha(x^*, x^*) \geq 1$). Let $x \in \mathcal{X}$ and $(x_0, x) \in \mathcal{X}_{\preceq}$. Proceeding on the lines of proof of Theorem 2.1, we can prove $\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^n x) = 0$, and owing to $x^* \in \mathcal{F}_{\mathcal{S}}$ and p is a w-distance (lower semi-continuous), then $\lim_{n\to\infty} p(\mathcal{S}^n x_0, x^*) = 0$, by Lemma 1.11, we get

$$\lim_{n\to\infty} \mathcal{S}^n x = x^*.$$

Suppose $x \in \mathcal{X}$ and $(x_0, x) \notin \mathcal{X}_{\preceq}$. Owing to condition (B), there exists some z in $\mathcal{X}$ such that $(x_0, z) \in \mathcal{X}_{\preceq}, \alpha(x_0, z) \geq 1$ and $(x, z) \in \mathcal{X}_{\preceq}, \alpha(x, z) \geq 1$.
Since $(x_0, z) \in \mathcal{X}_{\preceq}, \alpha(x_0, z) \geq 1$ and $(x, z) \in \mathcal{X}_{\preceq}, \alpha(x, z) \geq 1$ by using condition (b) (proceeding on the lines of proof of Theorem 2.1) we can

prove $\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^n z) = 0$ and $\lim_{n\to\infty} p(\mathcal{S}^n z, \mathcal{S}^n x) = 0$.
By triangular inequality, we can write

$$p(\mathcal{S}^n x_0, \mathcal{S}^n x) \leq p(\mathcal{S}^n x_0, \mathcal{S}^n z) + p(\mathcal{S}^n z, \mathcal{S}^n x).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^n x) = 0$, and also p is a w-distance (lower semi-continuous) we have $\lim_{n\to\infty} p(\mathcal{S}^n x_0, x^*) = 0$, which due to Lemma 1.11 implies

$$\lim_{n\to\infty} \mathcal{S}^n x = x^*.$$

This completes the proof. □

Corollary 2.14. *With the addition of condition (B) to the hypotheses of Corollary 2.4 (or Corollaries 2.8, 2.9, 2.10 and 2.12) the fixed point of $\mathcal{S}$ turns out to be unique. Moreover*

$$\lim_{n\to\infty} \mathcal{S}^n(x) = x^*,$$

for every $x \in \mathcal{X}$ provided $x^* \in \mathcal{F}_{\mathcal{S}}$, i.e., the map $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is a Picard operator.

Theorem 2.15. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, triangular α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist $\psi \in \Psi$, $\phi \in \Phi$, a pair (f, h) of upper class of type I and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ such that*

$$h(\alpha(x, y), \phi(p(\mathcal{S}x, \mathcal{S}y))) \leq f(\beta(x, y), \psi\phi(p(x, y))),$$

for all $(x, y) \in \mathcal{X}_{\preceq}$, where

(c) *either $\mathcal{S}$ is orbitally continuous at x_0 or*
(c') *$\mathcal{S}$ is orbitally $\mathcal{X}_{\preceq}$-continuous and there exists a subsequence $\{\mathcal{S}^{n_k} x_0\}$ of $\{\mathcal{S}^n x_0\}$ converging to x^* such that $(\mathcal{S}^{n_k} x_0, x^*) \in \mathcal{X}_{\preceq}$ for any $k \in N$.*

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_{\mathcal{S}}$, $\alpha(x, x) \geq 1$, then $p(x, x) = 0$.

Proof. If $x_0 = \mathcal{S}x_0$ for some $x_0 \in \mathcal{X}$, then there is nothing to prove. Otherwise, let there be $x_0 \in \mathcal{X}$ such that $x_0 \neq \mathcal{S}x_0$, and

$(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$. Owing to monotonocity of $\mathcal{S}$, we can write $(\mathcal{S}x_0, \mathcal{S}^2x_0) \in \mathcal{X}_{\preceq}, \alpha(\mathcal{S}x_0, \mathcal{S}^2x_0) \geq 1$. Continuing this process inductively, we obtain

$$(\mathcal{S}^n x_0, \mathcal{S}^m x_0) \in \mathcal{X}_{\preceq},$$

for any $n, m \in N$.
Also due to Lemma 1.24, we have

$$\alpha(\mathcal{S}^n x_0, \mathcal{S}^m x_0) \geq 1, m > n.$$

Now, we proceed to show that

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0. \tag{11}$$

On using condition (b) and properties of ϕ and ψ, we get

$$\begin{aligned} h(1, \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) &\leq h(\alpha(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0), \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) \\ &\leq f(\beta(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0), \psi\phi(p(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0))) \\ &\leq f(1, \psi\phi(p(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0))). \end{aligned}$$

Using Definition 1.19, we have

$$\begin{aligned} \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) &\leq \psi\phi(p(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0) \\ &\leq \psi^2\phi(p(\mathcal{S}^{n-2} x_0, \mathcal{S}^{n-1} x_0)) \\ &\leq ... \\ &\leq \psi^{n-m}\phi(p(\mathcal{S}^m x_0, \mathcal{S}^{m+1} x_0)) \\ &\leq ... \\ &\leq \psi^n\phi(p(x_0, \mathcal{S}x_0)). \end{aligned} \tag{12}$$

Now, on using Lemma 1.9, $\lim\limits_{n\to\infty} \phi(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) = 0$, which due to Lemma 1.10 gives rise

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0,$$

so that (11) is established.
Similarly, we can show

$$\lim_{n\to\infty} p(\mathcal{S}^{n+1} x_0, \mathcal{S}^n x_0) = 0. \tag{13}$$

Next, we proceed to show

$$\lim_{n,m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = 0. \tag{14}$$

Suppose (14) is untrue. Then we can find a $\delta > 0$ with sequences $\{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty}$ such that

$$p(\mathcal{S}^{n_k} x_0, \mathcal{S}^{m_k} x_0) \geq \delta, \quad \text{for all} \quad k \in \{1, 2, 3, \cdots\}, \tag{15}$$

wherein $m_k > n_k$. By (11) there exists $k_0 \in \mathbb{N}$, such that $n_k > k_0$ implies

$$p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) < \delta. \tag{16}$$

Notice that in view of (15) and (16) $m_k \neq n_{k+1}$, we can assume that m_k is a minimum index such that (15) holds so that

$$p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{r}x_0) < \delta, \qquad \text{for} \qquad r \in \{n_{k+1}, n_{k+2}, \cdots, m_k - 1\}. \tag{17}$$

Now (11), (15) and (17) imply

$$\begin{aligned} o < \delta &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{m_k}x_0) \\ &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{m_k-1}x_0) + p(\mathcal{S}^{m_k-1}x_0, \mathcal{S}^{m_k}x_0) \\ &< \delta + p(\mathcal{S}^{m_k-1}x_0, \mathcal{S}^{m_k}x_0), \end{aligned}$$

so that

$$\lim_{k\to\infty} p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{m_k}x_0) = \delta. \tag{18}$$

If $\epsilon = \limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) \geq \delta$, then there exists $\{k_r\}_{r=1}^{\infty}$ such that

$$\lim_{r\to\infty} p(\mathcal{S}^{n_{k_r}+1}x_0, \mathcal{S}^{m_{k_r}+1}x_0) = \epsilon \geq \delta.$$

Since ϕ is continuous and nondecreasing and also $(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0) \in \mathcal{X}_{\preceq}$, by using condition (b) and (18), one gets

$$\begin{aligned} & h(1, \phi(p(\mathcal{S}^{n_{k_r}+1}x_0, \mathcal{S}^{m_{k_r}+1}x_0))) \\ \leq\ & h(\alpha(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0), \phi(p(\mathcal{S}^{n_{k_r}+1}x_0, \mathcal{S}^{m_{k_r}+1}x_0)) \\ \leq\ & f(\beta(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0), \psi\phi(p(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0))) \\ \leq\ & f(1, \psi\phi(p(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0))) \\ \Longrightarrow\ & \phi(p(\mathcal{S}^{n_{k_r}+1}x_0, \mathcal{S}^{m_{k_r}+1}x_0)) \leq \psi\phi(p(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0)), \end{aligned}$$

so,

$$\phi(\delta) \leq \phi(\epsilon) = \lim_{r\to\infty} \phi(p(\mathcal{S}^{n_{k_r}+1}x_0, \mathcal{S}^{m_{k_r}+1}x_0)) \leq \psi\phi(\delta).$$

Notice that

$$\phi p(\mathcal{S}^{n_{k_r}}x_0, \mathcal{S}^{m_{k_r}}x_0) \to \phi(\delta)^+,$$

and ψ is right continuous, therefore $\phi(\delta) = 0$. This is a contradiction and

$$\limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) < \delta,$$

so we have

$$\begin{aligned} \delta &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{m_k}x_0) \\ &\leq p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) + p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) + p(\mathcal{S}^{m_k+1}x_0, \mathcal{S}^{m_k}x_0). \end{aligned}$$

On using (11), we have

$$\begin{aligned}\delta &\leq \lim_{k\to\infty} p(\mathcal{S}^{n_k}x_0, \mathcal{S}^{n_k+1}x_0) + \limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0)\\ &\quad + \lim_{k\to\infty} p(\mathcal{S}^{m_k+1}x_0, \mathcal{S}^{m_k}x_0)\\ &= \limsup_k p(\mathcal{S}^{n_k+1}x_0, \mathcal{S}^{m_k+1}x_0) < \delta.\end{aligned}$$

which is a contradiction. Thus, (14) is proved.

Owing to Lemma 1.12, $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete metric space, there exists $x^* \in \mathcal{X}$ such that $\mathcal{S}^n x_0 \to x^*$ as $n \to \infty$.

Now, we show that x^* is a fixed point of $\mathcal{S}$. If (c) holds, then $\mathcal{S}^{n+1}x_0 \to \mathcal{S}x^*$ (as $n \to \infty$). By lower semi-continuity of $p(\mathcal{S}^n x_0, \ .)$, we have

$$\begin{aligned}p(\mathcal{S}^n x_0, x^*) &\leq \liminf_{m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = \alpha_n \quad (say),\\ p(\mathcal{S}^n x_0, \mathcal{S}x^*) &\leq \liminf_{m\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{m+1} x_0) = \beta_n \quad (say).\end{aligned}$$

By using (12), we have $\alpha_n, \beta_n \to 0$ as $n \to \infty$. Now, in view of Lemma 1.11, we conclude that

$$\mathcal{S}x^* = x^*.$$

Next, suppose that (c') holds. Since $\{\mathcal{S}^{n_k}x_0\}$ converges to x^*, $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ and $\mathcal{S}$ is $\mathcal{X}_{\preceq}$-continuous, it follows that $\{\mathcal{S}^{n_k+1}x_0\}$ converges to $\mathcal{S}x^*$. As earlier, by lower semi-continuity of $p(\mathcal{S}^n x_0, \ .)$, we conclude that $\mathcal{S}x^* = x^*$.

If $\mathcal{S}x = x$, we have

$$\begin{aligned}h(1, \phi p(x,x)) &\leq h(\alpha(x,x), \phi p(x,x)) = h(1, \alpha(x,x), \phi p(\mathcal{S}x, \mathcal{S}x))\\ &\leq f(\beta(x,x), \psi\phi(p(x,x)))\\ &\Longrightarrow \phi p(x,x) \leq \psi\phi(p(x,x)) < \phi p(x,x).\end{aligned}$$

This is a contradiction which amounts to say that $\phi p(x,x) = 0$, so that $p(x,x) = 0$. This completes the proof. □

In Theorem 2.15, if $h(y,z) = yz$, $f(s,t) = st$, we deduce the following corollary:

Corollary 2.16. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, triangular α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*

(b) *there exist* $\psi \in \Psi,\ \phi \in \Phi$, *and* $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ *such that*

$$\alpha(x,y)\phi(p(\mathcal{S}x,\mathcal{S}y))) \leq \beta(x,y)\psi\phi(p(x,y))),$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') $\mathcal{S}$ *is orbitally* $\mathcal{X}_{\preceq}$*-continuous and there exists a subsequence* $\{\mathcal{S}^{n_k}x_0\}$ *of* $\{\mathcal{S}^n x_0\}$ *converging to* x^* *such that* $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ *for any* $k \in N$.

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x) = 0$.

In Theorem 2.15, if $h(y,z) = (y+l)^z, l > 1,\ f(s,t) = (1+l)^{st}$, we deduce the following corollary:

Corollary 2.17. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space equipped with a* w*-distance* p. *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a nondecreasing, triangular* α*-admissible and* α*-regular mapping, where* $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. *Also, suppose that*

(a) *there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ *with* $\alpha(x_0, \mathcal{S}x_0) \geq 1$,

(b) *there exist* $\psi \in \Psi,\ \phi \in \Phi$, *and* $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ *such that*

$$(\alpha(x,y)+l)^{\phi(p(\mathcal{S}x,\mathcal{S}y))} \leq (1+l)^{\beta(x,y)\psi\phi(p(x,y)))},$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') $\mathcal{S}$ *is orbitally* $\mathcal{X}_{\preceq}$*-continuous and there exists a subsequence* $\{\mathcal{S}^{n_k}x_0\}$ *of* $\{\mathcal{S}^n x_0\}$ *converging to* x^* *such that* $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ *for any* $k \in N$.

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x) = 0$.

In Theorem 2.15, if $h(y,z) = (z+l)^y, l > 1,\ f(s,t) = st+l$, we deduce the following corollary:

Corollary 2.18. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete partially ordered metric space equipped with a* w*-distance* p. *Let* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ *be a nondecreasing, triangular* α*-admissible and* α*-regular mapping, where* $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. *Also, suppose that*

(a) *there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ *with* $\alpha(x_0, \mathcal{S}x_0) \geq 1$,

(b) *there exist* $\psi \in \Psi,\ \phi \in \Phi$, *and* $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ *such that*

$$(\phi(p(\mathcal{S}x,\mathcal{S}y))+l)^{\alpha(x,y)} \leq \beta(x,y)\psi\phi(p(x,y)))+l,$$

for all $(x,y) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') $\mathcal{S}$ *is orbitally* $\mathcal{X}_{\preceq}$*-continuous and there exists a subsequence* $\{\mathcal{S}^{n_k}x_0\}$ *of* $\{\mathcal{S}^n x_0\}$ *converging to* x^* *such that* $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ *for any* $k \in N$.

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x)=0$.

3. On (γ,ψ,p)-Contractive Maps

In this section, we prove some results in partially ordered metric space with (γ,ψ,p)-contractive Maps. In section 2, we considered the condition of nondecreasing for function $\mathcal{S}$, but in this section we prove theorems by replacing the condition of nondecreasing to monotonocity for function S.

Theorem 3.1. *Let* $(\mathcal{X},d,\preceq)$ *be a complete partially ordered metric space equipped with a w-distance* p. *Let* $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ *be a nondecreasing,* α*-admissible and* α*-regular mapping, where* $\alpha:\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$. *Also, suppose that*

(a) *there exists* $x_0 \in \mathcal{X}$ *such that* $(x_0,\mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ *with* $\alpha(x_0,\mathcal{S}x_0) \geq 1$,

(b) *there exist* $\psi \in \Psi$, $\gamma \in \Gamma$, *a pair* (f,h) *of upper class of type* II *and* $\beta:\mathcal{X}\times\mathcal{X}\longrightarrow[0,1]$ *such that,*

$$h(\alpha(x,\mathcal{S}x),\alpha(\mathcal{S}x,\mathcal{S}^2x),\gamma(p(\mathcal{S}x,\mathcal{S}^2x))) \leq f(\beta(x,\mathcal{S}x),\psi\gamma(p(x,\mathcal{S}x)))$$

for all $(x,\mathcal{S}x) \in \mathcal{X}_{\preceq}$, *where*

(c) *either* $\mathcal{S}$ *is orbitally continuous at* x_0 *or*

(c') $\mathcal{S}$ *is orbitally* $\mathcal{X}_{\preceq}$*-continuous and there exists a subsequence* $\{\mathcal{S}^{n_k}x_0\}$ *of* $\{\mathcal{S}^n x_0\}$ *converging to* x^* *such that* $(\mathcal{S}^{n_k}x_0, x^*) \in \mathcal{X}_{\preceq}$ *for any* $k \in N$.

Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. *Moreover if for each* $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, *then* $p(x,x)=0$.

Proof. If $x_0=\mathcal{S}x_0$ for some $x_0 \in \mathcal{X}$, then there is nothing to prove. Otherwise, let there be $x_0 \in \mathcal{X}$ such that $x_0 \neq \mathcal{S}x_0$, and $(x_0,\mathcal{S}x_0) \in \mathcal{X}_{\preceq}$.

Owing to monotonocity of $\mathcal{S}$, we can write $(\mathcal{S}x_0, \mathcal{S}^2 x_0) \in \mathcal{X}_{\preceq}$. Continuing this process inductively, we obtain

$$(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) \in \mathcal{X}_{\preceq},$$

for any $n, m \in N$. Now, we proceed to show that

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0. \tag{19}$$

On using condition (b) and properties of γ and ψ, we get

$$\begin{aligned} & h(1, 1, \gamma(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0))) \\ \leq \ & h(\alpha(x, \mathcal{S}x), \alpha(\mathcal{S}x, \mathcal{S}^2 x), \gamma(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0))) \\ \leq \ & f(\beta(x, \mathcal{S}x), \psi\gamma(p(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0))), \end{aligned}$$

which implies that,

$$\begin{aligned} \gamma(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0)) & \leq \psi\gamma(p(\mathcal{S}^{n-1} x_0, \mathcal{S}^n x_0)) \\ & \leq \psi^2\gamma(p(\mathcal{S}^{n-2} x_0, \mathcal{S}^{n-1} x_0)) \\ & \leq \ ... \\ & \leq \psi^{n-m}\gamma(p(\mathcal{S}^m x_0, \mathcal{S}^{m+1} x_0)) \\ & \leq \ ... \\ & \leq \psi^n\gamma(p(x_0, \mathcal{S}x_0)). \end{aligned} \tag{20}$$

By using Lemma 1.9, we have $\lim_{n\to\infty} \gamma p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0$, so that by Lemma 1.10, we have

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) = 0,$$

which establishes (19).
Similarly, we can show

$$\lim_{n\to\infty} p(\mathcal{S}^{n+1} x_0, \mathcal{S}^n x_0) = 0. \tag{21}$$

Now, we proceed to show that $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence. By triangle inequality, continuity of γ and (19), we have

$$\gamma p(\mathcal{S}^n x_0, \mathcal{S}^{n+2} x_0) \leq \gamma(p(\mathcal{S}^n x_0, \mathcal{S}^{n+1} x_0) + p(\mathcal{S}^{n+1} x_0, \mathcal{S}^{n+2} x_0)) \to 0,$$

as $n \to \infty$ so that $\lim_{n\to\infty} \gamma p(\mathcal{S}^n x_0, \mathcal{S}^{n+2} x_0) = 0$ which amounts to say that

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+2} x_0) = 0.$$

By induction, for any $k > 0$, we have

$$\lim_{n\to\infty} p(\mathcal{S}^n x_0, \mathcal{S}^{n+k} x_0) = 0.$$

So, by Lemma 1.11, $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence and due to completeness of $\mathcal{X}$, there exists $x^* \in \mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}^n x_0 = x^*$.

If (c) or (c') holds, then proceeding on the lines of the proof of Theorem 2.1, we can show that

$$\mathcal{S}x^* = x^*.$$

If $\mathcal{S}x = x$, we have

$$\begin{aligned} h(1,1,\gamma p(x,x)) &\leq h(\alpha(x,x),\alpha(x,x),\gamma p(x,x)) \\ &= h(\alpha(x,\mathcal{S}x),\alpha(\mathcal{S}x,\mathcal{S}^2x),\gamma p(\mathcal{S}x,\mathcal{S}^2x)) \\ &\leq f(\beta(x,\mathcal{S}x),\psi\gamma(p(x,\mathcal{S}x))) \end{aligned}$$

$$\Longrightarrow \gamma p(x,x) \leq \psi\gamma(p(x,\mathcal{S}x)) < \gamma p(x,x),$$

which is a contradiction so that $\gamma p(x,x) = 0$, implying thereby $p(x,x) = 0$. This completes the proof. □

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping, where $\alpha : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$. Also, suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_\preceq$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*
(b) *there exist $\psi \in \Psi$, $\gamma \in \Gamma$, a pair (f,h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,1]$ such that,*

$$h(\alpha(x,\mathcal{S}x),\alpha(\mathcal{S}x,\mathcal{S}^2x),\gamma(p(\mathcal{S}x,\mathcal{S}^2x))) \leq f(\beta(x,\mathcal{S}x),\psi\gamma(p(x,\mathcal{S}x)))$$

for all $(x,\mathcal{S}x) \in \mathcal{X}_\preceq$,
(c”) *and for every $y \in \mathcal{X}$ with $y \neq \mathcal{S}y$,*

$$\inf\{p(x,y) + p(x,\mathcal{S}x) : x \in \mathcal{X}\} > 0.$$

Then $\mathcal{F}_\mathcal{S} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x,x) \geq 1$, then $p(x,x) = 0$.

Proof. Proceeding on the lines of the proof of Theorem 3.1, the sequence $\{\mathcal{S}^n x_0\}$ is a Cauchy sequence and so there exists a point x^* in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}^n x_0 = x^*$. Since $\lim_{m,n\to\infty} d(\mathcal{S}^n x_0, \mathcal{S}^m x_0) = 0$, therefore for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $p(\mathcal{S}^{N_\epsilon} x_0, \mathcal{S}^n x_0) < \epsilon$. As $\lim_{n\to\infty} \mathcal{S}^n x_0 = x^*$ and $p(x,.)$ is lower semi continuous, therefore

$$p(\mathcal{S}^{N_\epsilon} x_0, x^*) \leq \liminf_n p(\mathcal{S}^{N_\epsilon} x_0, \mathcal{S}^n x_0) \leq \epsilon.$$

Therefore $p(\mathcal{S}^{N_\epsilon}x_0, x^*) \leq \epsilon$. Setting $\epsilon = \frac{1}{k}, N_\epsilon = n_k$, we have

$$\lim_{k\to\infty} p(\mathcal{S}^{n_k}x_0, x^*) = 0.$$

Now, assume that $x^* \neq \mathcal{S}x^*$. Then by hypothesis (c"), we have

$$\begin{aligned} 0 &< \inf\{p(x, x^*) + p(x, \mathcal{S}x) : x \in \mathcal{X}\} \\ &\leq \inf\{p(\mathcal{S}^n x_0, x^*) + p(\mathcal{S}^n x_0, \mathcal{S}^{n+1}x_0) : n \in \mathbb{N}\} \to 0, \end{aligned}$$

as $n \to \infty$. This is a contradiction so that $x^* = \mathcal{S}x^*$.
If $\mathcal{S}x = x$, we have

$$\begin{aligned} h(1, 1, \gamma p(x, x)) &\leq h(\alpha(x, x), \alpha(x, x), \gamma p(x, x)) \\ &= h(\alpha(x, \mathcal{S}x), \alpha(\mathcal{S}x, \mathcal{S}^2x), \gamma p(\mathcal{S}x, \mathcal{S}^2x)) \\ &\leq f(\beta(x, \mathcal{S}x), \psi\gamma(p(x, \mathcal{S}x))) \end{aligned}$$

$$\Longrightarrow \gamma p(x, x) \leq \psi\gamma(p(x, \mathcal{S}x)) < \gamma p(x, x),$$

which is a contradiction so that $\gamma p(x, x) = 0$, implying thereby $p(x, x) = 0$. This completes the proof. □

Theorem 3.3. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_\preceq$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*

(b_1) *there exist $\gamma \in \Gamma$, a pair (f, h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ and $k \in [0, \frac{1}{2})$ such that,*

$$\begin{aligned} &h(\alpha(x, \mathcal{S}x), \alpha(\mathcal{S}x, \mathcal{S}^2x), \gamma(p(\mathcal{S}x, \mathcal{S}y))) \\ &\qquad\leq f(\beta(x, \mathcal{S}x), k\{\gamma p(x, \mathcal{S}x) + \gamma p(y, \mathcal{S}y)\}) \end{aligned}$$

for all $(x, y) \in \mathcal{X}_\preceq$,

(c") *for every $y \in \mathcal{X}$ with $y \neq \mathcal{S}y$,*

$$\inf\{p(x, y) + p(x, \mathcal{S}x) : x \in \mathcal{X}\} > 0.$$

Then $\mathcal{F}_\mathcal{S} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x, x) \geq 1$, then $p(x, x) = 0$.

Proof. For $x \in \mathcal{X}$, set $y = \mathcal{S}x$ and $\alpha = \frac{k}{1-k}$. Then we have $(x, \mathcal{S}x) \in \mathcal{X}_\preceq$ and $\alpha \in [0, 1)$. On using condition (b_1), we get

$$\gamma(p(\mathcal{S}x, \mathcal{S}^2x)) \leq k\{\gamma p(x, \mathcal{S}x) + \gamma p(\mathcal{S}x, \mathcal{S}^2x)\},$$

or

$$\gamma(p(\mathcal{S}x, \mathcal{S}^2x)) \leq \alpha\gamma p(x, \mathcal{S}x).$$

Therefore, by choosing $\psi(t) = \alpha t$, all the conditions of Theorem 3.2 are satisfied ensuring the conclusions of the theorem. $\square$

The set of all subadditive functions γ in Γ is denoted by Γ'.

Theorem 3.4. *Let $(\mathcal{X}, d, \preceq)$ be a complete partially ordered metric space equipped with a w-distance p. Let $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ be a nondecreasing, α-admissible and α-regular mapping. Suppose that*

(a) *there exists $x_0 \in \mathcal{X}$ such that $(x_0, \mathcal{S}x_0) \in \mathcal{X}_{\preceq}$ with $\alpha(x_0, \mathcal{S}x_0) \geq 1$,*

(b_2) *there exist $\psi \in \Psi$, $\gamma \in \Gamma'$, a pair (f, h) of upper class of type II and $\beta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, 1]$ and $k \in [0, \frac{1}{2})$ such that,*

$$h(\alpha(x, \mathcal{S}x), \alpha(\mathcal{S}x, \mathcal{S}^2x), \gamma(p(\mathcal{S}x, \mathcal{S}^2x))) \leq f(\beta(x, \mathcal{S}x), k\gamma(p(x, \mathcal{S}^2x)))$$

for all $(x, \mathcal{S}x) \in \mathcal{X}_{\preceq}$ and

$$\inf\{p(x, y) + p(x, \mathcal{S}x) : x \in \mathcal{X}\} > 0$$

for every $y \in \mathcal{X}$ with $y \neq \mathcal{S}y$. Then $\mathcal{F}_{\mathcal{S}} \neq \emptyset$. Moreover if for each $x \in \mathcal{F}_S$, $\alpha(x, x) \geq 1$, then $p(x, x) = 0$.

Proof. Set $\alpha = \frac{k}{1-k}$, then $\alpha \in [0, 1)$. On using condition (b_2) (as $\gamma \in \Gamma'$), we have

$$\begin{aligned} \gamma(p(\mathcal{S}x, \mathcal{S}^2x)) \leq k\gamma(p(x, \mathcal{S}^2x)) &\leq k\gamma(p(x, \mathcal{S}x) + p(\mathcal{S}x, \mathcal{S}^2x)) \\ &\leq k\gamma p(x, \mathcal{S}x) + k\gamma p(\mathcal{S}x, \mathcal{S}^2x). \end{aligned}$$

Thus, $\gamma(p(\mathcal{S}x, \mathcal{S}^2x)) \leq \alpha\gamma(p(x, \mathcal{S}x))$.
Therefore, by choosing $\psi(t) = \alpha t$, all the conditions of Theorem 3.2 are satisfied ensuring the conclusions of the theorem. $\square$

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