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# THE RELATION PROPERTY BETWEEN THE DIVISOR FUNCTION AND INFINITE PRODUCT SUMS

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**Abstract.** For a complex number $q$ and a divisor function $\sigma_1(n)$ we define

$$C(q) := q\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{2n})^4,$$
$$D(q) := q^2\prod_{n=1}^{\infty}(1-q^n)^{8}(1-q^{2n})^4(1-q^{4n})^8,$$
$$L(q) := 1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

moreover we obtain the number of representations of $n \in \mathbb{N}$ as sum of 24 squares, which are possible for us to deduce

$$L(q^4)C(q) \qquad \text{and} \qquad L(q^4)D(q).$$

## 1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, integers, and complex numbers respectively. For $n \in \mathbb{N}$, $k \in \mathbb{N}\cup\{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define a divisor function and various infinite product sums :

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$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \Delta(q) := \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

$$\begin{aligned}
A(q) &:= \sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^{12},\\
B(q) &:= \sum_{n=1}^{\infty} b(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{8}(1-q^{2n})^{8},\\
C(q) &:= \sum_{n=1}^{\infty} c(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{2n})^{4},\\
D(q) &:= \sum_{n=1}^{\infty} d(n)q^n = q^2\prod_{n=1}^{\infty}(1-q^n)^{8}(1-q^{2n})^{4}(1-q^{4n})^{8},\\
F(q) &:= \sum_{n=1}^{\infty} f(n)q^n = q^5\prod_{n=1}^{\infty}(1-q^n)^{24}(1+q^n)^{32}(1+q^{2n})^{32}.
\end{aligned} \tag{1}$$

**Proposition 1.1.** *(See [8]) Let $n$ $(\in \mathbb{N})$ be an even positive integer. Then we have*

(a)

$$d(n) = \frac{1}{16}d(2n),$$

(b)

$$f(n) = \frac{1}{135856128}\left\{3\sigma_{11}(n) - 3\sigma_{11}(\frac{n}{2}) - 694\tau(n) - 22800\tau(\frac{n}{2}) \right.$$
$$\left. -14151680\tau(\frac{n}{4})\right\},$$

(c)

$$\tau(n) = -24\tau(\frac{n}{2}) - 2048\tau(\frac{n}{4}).$$

*So by Proposition 1.1 (c) we can simplify Proposition 1.1 (b) as*

$$f(n) = \frac{1}{45285376}\left\{\sigma_{11}(n) - \sigma_{11}(\frac{n}{2}) - 2048\tau(\frac{n}{2}) - 4243456\tau(\frac{n}{4})\right\}.$$

On the other hand, for all $n \in \mathbb{N}$ we have shown that

$$c(n) = d(2n) - 32d(n) \tag{2}$$

in [11, Theorem 1.1]. And with $a_1, a_2, \cdots, a_k \in \mathbb{N}$, $k \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$, we set

$$\begin{aligned}(3)\quad & R_k(a_1, a_2, \cdots, a_k; n) \\ & := \mathrm{card}\{(x_1, x_2, \cdots, x_k) \in \mathbb{Z}^k | n = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2\}.\end{aligned}$$

Briefly we write (3) as

$$R_k(1, 1, \cdots, 1; n) := R_k(n).$$

For example, Jacobi's result [6, §§40–42, p. 159–170] is that for all $n \in \mathbb{N}$

$$R_8(n) = 16\sigma_3(n) - 32\sigma_3(\frac{n}{2}) + 256\sigma_3(\frac{n}{4}) \tag{4}$$

and in [13] we can find

$$R_{16}(n) = \frac{32}{17}\left\{\sigma_7(n) - 2\sigma_7(\frac{n}{2}) + 256\sigma_7(\frac{n}{4}) + 16b(n) + 256b(\frac{n}{2})\right\}. \tag{5}$$

Here (please, see (36)) we obtain

$$\begin{aligned}R_{24}(n) = & \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(\frac{n}{2}) + \frac{65536}{691}\sigma_{11}(\frac{n}{4}) + \frac{33152}{691}\tau(n) \\ & + \frac{1525760}{691}\tau(\frac{n}{2}) + \frac{135790592}{691}\tau(\frac{n}{4}).\end{aligned}$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then the Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{6}$$

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{7}$$

$$N(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n \tag{8}$$

by [4, p. 318]. Lahiri [14, p. 149] has derived the following identities from the work of Ramanujan [15] :

$$L^2(q) = 1 - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{9}$$

$$M^2(q) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \tag{10}$$

$$M^3(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{432000}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \tag{11}$$

$$L(q)M(q) = 1 + 720\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{12}$$

$$L(q)M^2(q) = 1 + 720\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n,$$

$$N^2(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{762048}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \tag{13}$$

$$L(q)N(q) = 1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n,$$

$$M(q)N(q) = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n.$$

Especially, we refer to

$$\begin{aligned} &L(q)M(q)N(q) \\ &= 1 - \frac{1584}{5}\sum_{n=1}^{\infty} n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{228096}{3455}\sum_{n=1}^{\infty}\tau(n)q^n \end{aligned} \tag{14}$$

and

$$L^2(q)M^2(q) = 1 + 960\sum_{n=1}^{\infty} n^2\sigma_7(n)q^n - \frac{3168}{5}\sum_{n=1}^{\infty} n\sigma_9(n)q^n + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{37248}{3455}\sum_{n=1}^{\infty}\tau(n)q^n \tag{15}$$

in [7, Lemma 3.4, Lemma 3.6]. Now we require the Jacobi theta function $\varphi(q)$ is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

so that relating to (3) we can know that

$$\sum_{n=0}^{\infty} R_k(a_1, a_2, \cdots, a_k; n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\cdots\varphi(q^{a_k}). \tag{16}$$

As [1, p. 32–33] let

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \tag{17}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{18}$$

Then we can see that

$$\varphi(q) = (1 + 2p)^{3/4}\, k^{1/2} \tag{19}$$

and

$$\varphi(q^3) = (1 + 2p)^{1/4}\, k^{1/2} \tag{20}$$

in [3, (2.6),(2.7)]. In this article, using the definitions of (17) and (18) we deduce the following theorem :

**Theorem 1.2.** *Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have*

(a)

$$(1+2p)^4\,k^8 = \frac{1}{255}M^2(q^3) - \frac{2}{255}M^2(q^6) + \frac{256}{255}M^2(q^{12}) + \frac{512}{17}B(q^3) + \frac{8192}{17}B(q^6),$$

(b)

$$(1+2p)^5\,k^{10} = -\frac{1}{1023}M(q^3)N(q^3) + \frac{1024}{1023}M(q^{12})N(q^{12}) - \frac{256}{31}D(q^3) + \frac{1261568}{31}D(q^6) + \frac{1232}{31}C(q^3) + \frac{39424}{31}C(q^6),$$

(c)

$$(1+2p)^6\,k^{12} = \frac{7}{44915}M^3(q^3) - \frac{14}{44915}M^3(q^6) + \frac{28672}{44915}M^3(q^{12}) + \frac{50}{565929}N^2(q^3) - \frac{100}{565929}N^2(q^6) + \frac{204800}{565929}N^2(q^{12}) + \frac{33152}{691}\Delta(q^3) + \frac{1525760}{691}\Delta(q^6) + \frac{135790592}{691}\Delta(q^{12}).$$

And Theorem 1.2 (c) plays an important role to obtain the following theorem :

**Theorem 1.3.** *Let $q \in \mathbb{C}$ such that $|q| < 1$. Then we have*

(a)

$$L(q^4)C(q) = -\frac{17189}{185077440}M^3(q) - \frac{11524411}{925387200}M^3(q^2) - \frac{3283972}{4819725}M^3(q^4) + \frac{67169}{809713800}N^2(q) + \frac{4036757}{323885520}N^2(q^2) + \frac{3283972}{4819725}N^2(q^4) + \frac{1821829}{2142100}\Delta(q) + \frac{725740}{21421}\Delta(q^2) + \frac{3482086144}{535525}\Delta(q^4) + \frac{212992}{5}F(q) + \frac{3}{10}\sum_{n=1}^{\infty} nc(n)q^n,$$

(b)

$$
\begin{aligned}
&L(q^4)D(q)\\
&= \frac{2508569}{153984430080}M^3(q) + \frac{222565469}{192480537600}M^3(q^2) + \frac{17525879}{115673400}M^3(q^4)\\
&\quad - \frac{88056967}{5389455052800}N^2(q) - \frac{77894701}{67368188160}N^2(q^2)\\
&\quad - \frac{17525879}{115673400}N^2(q^4) - \frac{483241}{17136800}\Delta(q) - \frac{170154}{107105}\Delta(q^2)\\
&\quad - \frac{126497592}{535525}\Delta(q^4) + \frac{1024}{5}F(q) + \frac{3}{10}\sum_{n=1}^{\infty} nd(n)q^n.
\end{aligned}
$$

## 2. Preliminaries and some results for Theorem 1.2

**Proposition 2.1.** *(See [5]) For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$L(q)L(q^2) = \frac{1}{4}L^2(q) + L^2(q^2) - \frac{1}{20}M(q) - \frac{1}{5}M(q^2),$$

(b)

$$L(q)L(q^4) = \frac{1}{8}L^2(q) + 2L^2(q^4) - \frac{3}{40}M(q) + \frac{3}{20}M(q^2) - \frac{6}{5}M(q^4),$$

(c)

$$L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2),$$

(d)

$$L(q^2)M(q) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2),$$

(e)

$$M(q)M^2(q^2) = \frac{11}{78}M^3(q) + \frac{196}{39}M^3(q^2) - \frac{25}{182}N^2(q) - \frac{1100}{273}N^2(q^2),$$

(f)

$$
\begin{aligned}
M^2(q)M(q^2) = \frac{49}{156}M^3(q) + \frac{1408}{39}M^3(q^2) - \frac{275}{1092}N^2(q)\\
- \frac{3200}{91}N^2(q^2),
\end{aligned}
$$

(g)

$$M(q)N(q)L(q^2) = \frac{1}{2}L(q)M(q)N(q) - \frac{14}{65}M^3(q) - \frac{896}{65}M^3(q^2) - \frac{19}{546}N^2(q) + \frac{3968}{273}N^2(q^2),$$

(h)

$$L(q^4)M(q) = \frac{1}{4}L(q)M(q) - \frac{4}{21}N(q) + \frac{5}{28}N(q^2) + \frac{16}{21}N(q^4) + 90A(q),$$

(i)

$$L(q)N(q^4) = 4L(q^4)N(q^4) + \frac{1}{5440}M^2(q) + \frac{63}{5440}M^2(q^2) - \frac{256}{85}M^2(q^4) - \frac{819}{34}B(q) - \frac{4788}{17}B(q^2),$$

(j)

$$L(q^4)N(q) = \frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4) - \frac{4788}{17}B(q) - \frac{104832}{17}B(q^2),$$

(k)

$$L(q)M^2(q^2) = 2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q) - \frac{23040}{31}D(q),$$

(l)

$$M^2(q)L(q^2) = \frac{1}{2}L(q)M^2(q) - \frac{171}{682}M(q)N(q) + \frac{256}{341}M(q^2)N(q^2) + \frac{5760}{31}C(q) + \frac{184320}{31}D(q),$$

(m)

$$\begin{aligned} &M(q)N(q)L(q^4) \\ &= \frac{1}{4}L(q)M(q)N(q) - \frac{853}{3120}M^3(q) - \frac{3971}{130}M^3(q^2) \\ &\quad + \frac{2651392}{195}M^3(q^4) - \frac{1}{208}N^2(q) + \frac{2805}{91}N^2(q^2) \\ &\quad - \frac{1237248}{91}N^2(q^4) + 389283840F(q). \end{aligned}$$

And we obtained more simplified identities as follows in [8, (22)]

(21)

$$
\begin{aligned}
L(q)M^2(q^4) &= 4L(q^4)M^2(q^4) + \frac{1}{87296}M(q)N(q) + \frac{255}{87296}M(q^2)N(q^2) \\
&\quad - \frac{1024}{341}M(q^4)N(q^4) - \frac{23805}{992}C(q) - \frac{4905}{31}C(q^2) - \frac{18315}{62}D(q) \\
&\quad - \frac{156960}{31}D(q^2), \\
M(q)N(q^4) &= \frac{5}{21824}M(q)N(q) + \frac{315}{21824}M(q^2)N(q^2) + \frac{336}{341}M(q^4)N(q^4) \\
&\quad + \frac{59535}{248}C(q) + \frac{124740}{31}C(q^2) + \frac{62370}{31}D(q) + \frac{3991680}{31}D(q^2), \\
N(q)M(q^4) &= \frac{21}{5456}M(q)N(q) + \frac{315}{5456}M(q^2)N(q^2) + \frac{320}{341}M(q^4)N(q^4) \\
&\quad - \frac{31185}{62}C(q) - \frac{952560}{31}C(q^2) + \frac{204120}{31}D(q) - \frac{30481920}{31}D(q^2), \\
M^2(q)L(q^4) &= \frac{1}{4}L(q)M^2(q) - \frac{64}{341}M(q)N(q) + \frac{255}{1364}M(q^2)N(q^2) \\
&\quad + \frac{256}{341}M(q^4)N(q^4) + \frac{9810}{31}C(q) + \frac{1523520}{31}C(q^2) - \frac{623520}{31}D(q) \\
&\quad + \frac{48752640}{31}D(q^2)
\end{aligned}
$$

owing to N. Cheng and K. S. Williams' results in [5, Theorem 6.1].

**Proposition 2.2.** *Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then we have*

(a) *(See* [2, (3.87)]*)*

$$L(q) - 3L(q^3) = -\left(2 + 16p + 36p^2 + 16p^3 + 2p^4\right)k^2,$$

(b) *(See* [1, (3.10)]*)*

$$L(q^3) - 2L(q^6) = -\left(1 + 2p + 2p^3 + p^4\right)k^2,$$

(c) *(See* [1, (3.11)]*)*

$$L(q^6) - 2L(q^{12}) = -\left(1 + 2p - p^3 - \frac{1}{2}p^4\right)k^2.$$

Based on Proposition 2.2 we obtained the following proposition in [12, Theorem 2.1] :

**Proposition 2.3.** *Let $q \in \mathbb{C}$ satisfying $|q| < 1$. Then we have*

(a)

$$k^2 = \frac{1}{6}L(q) - L(q^2) + \frac{1}{6}L(q^3) + \frac{2}{3}L(q^4) + \frac{1}{3}L(q^6) + \frac{2}{3}L(q^{12}),$$

(b)

$$pk^2 = -\frac{1}{12}L(q) + \frac{1}{2}L(q^2) - \frac{1}{4}L(q^3) - \frac{1}{3}L(q^4) - \frac{1}{6}L(q^6) + \frac{1}{3}L(q^{12}),$$

(c)

$$p^2k^2 = -\frac{1}{6}L(q^2) + \frac{1}{3}L(q^3) - \frac{1}{6}L(q^6),$$

(d)

$$p^3k^2 = -\frac{1}{3}L(q^3) + \frac{1}{3}L(q^4) + \frac{1}{3}L(q^6) - \frac{1}{3}L(q^{12}),$$

(e)

$$p^4k^2 = -\frac{2}{3}L(q^4) + \frac{4}{3}L(q^6) - \frac{2}{3}L(q^{12}).$$

Let us define

$$C_{1,6}(q) := \sum_{n=1}^{\infty} c_{1,6}(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2,$$

$$\begin{aligned}C_{1,12}(q) &:= \sum_{n=1}^{\infty} c_{1,12}(n)q^n \\ &= \frac{10}{11}q\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n-6}) \\ &\quad + \frac{1}{11}q\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n})^6(1+q^{3n})^2(1+q^{4n}+q^{8n})^2(1-q^{6n})^6(1-q^{12n})^{-4},\end{aligned}$$

$$\begin{aligned}C_{3,4}(q) &:= \sum_{n=1}^{\infty} c_{3,4}(n)q^n \\ &= 10q^2\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n})^2(1+q^{3n})(1-q^{6n})(1+q^{4n}+q^{8n})(1-q^{12n})^2 \\ &\quad + q\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{2n})^6(1+q^{3n})^2(1+q^{4n}+q^{8n})^2(1-q^{6n})^6(1-q^{12n})^{-4}.\end{aligned}$$

Then we have

$$c_{3,4}(n) = -11c_{1,12}(n) + 12(-1)^{n+1}c_{1,6}(n). \tag{22}$$

Now in [2, p. 503] we can see that

$$25\left(L(q)-6L(q^6)\right)^2$$
$$=19M(q)-24M(q^2)-54M(q^3)+684M(q^6)+1440\sum_{n=1}^{\infty}c_{1,6}(n)q^n,$$

which expands as

$$\begin{aligned}&25\left(L^2(q)-12L(q)L(q^6)+36L^2(q^6)\right)\\&=19M(q)-24M(q^2)-54M(q^3)+684M(q^6)+1440C_{1,6}(q)\end{aligned}$$

and so

(23)
$$\begin{aligned}L(q)L(q^6)=\frac{1}{12}L^2(q)+3L^2(q^6)-\frac{19}{300}M(q)+\frac{2}{25}M(q^2)+\frac{9}{50}M(q^3)\\-\frac{57}{25}M(q^6)-\frac{24}{5}C_{1,6}(q).\end{aligned}$$

Similarly in [2, p. 505] we can find that

$$25\left(2L(q^2)-3L(q^3)\right)^2$$
$$=-6M(q)+76M(q^2)+171M(q^3)-216M(q^6)+1440\sum_{n=1}^{\infty}c_{1,6}(n)q^n$$

thus we have

(24)
$$\begin{aligned}L(q^2)L(q^3)=\frac{1}{3}L^2(q^2)+\frac{3}{4}L^2(q^3)+\frac{1}{50}M(q)-\frac{19}{75}M(q^2)-\frac{57}{100}M(q^3)\\+\frac{18}{25}M(q^6)-\frac{24}{5}C_{1,6}(q).\end{aligned}$$

Moreover [1, p. 40] implies that

$$\begin{aligned}\left(3L(q^3)-4L(q^4)\right)^2=-\frac{3}{25}M(q)-\frac{9}{25}M(q^2)+\frac{198}{25}M(q^3)+\frac{352}{25}M(q^4)\\-\frac{81}{25}M(q^6)-\frac{432}{25}M(q^{12})+\frac{144}{5}\sum_{n=1}^{\infty}c_{3,4}(n)q^n.\end{aligned}$$

Therefore by (22) the above identity shows that

(25)

$$L(q^3)L(q^4) = \frac{3}{8}L^2(q^3) + \frac{2}{3}L^2(q^4) + \frac{1}{200}M(q) + \frac{3}{200}M(q^2) - \frac{33}{100}M(q^3) - \frac{44}{75}M(q^4) + \frac{27}{200}M(q^6) + \frac{18}{25}M(q^{12}) + \frac{66}{5}C_{1,12}(q) + \frac{72}{5}C_{1,6}(-q).$$

**Proposition 2.4.** *(See [1, p. 34, (3.24), (3.26), (3.27)]) Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have*

(a)

$$M(q) = \left(1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5 + 964p^6 + 124p^7 + p^8\right) k^4,$$

(b)

$$M(q^2) = \left(1 + 4p + 64p^2 + 178p^3 + 235p^4 + 178p^5 + 64p^6 + 4p^7 + p^8\right) k^4,$$

(c)

$$M(q^3) = \left(1 + 4p + 4p^2 + 28p^3 + 70p^4 + 28p^5 + 4p^6 + 4p^7 + p^8\right) k^4,$$

(d)

$$M(q^4) = \left(1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 + \frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8\right) k^4,$$

(e)

$$M(q^6) = \left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 4p^6 + 4p^7 + p^8\right) k^4,$$

(f)

$$M(q^{12}) = \left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + \frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8\right) k^4,$$

(g)

$$N(q^3) = \left(1 + 6p + 12p^2 - 58p^3 - 297p^4 - 396p^5 - 264p^6 - 396p^7 - 297p^8 - 58p^9 + 12p^{10} + 6p^{11} + p^{12}\right) k^6,$$

(h)

$$N(q^6) = \left(1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - 12p^6 - 18p^7 - \frac{27}{2}p^8 + 5p^9 + 12p^{10} + 6p^{11} + p^{12}\right)k^6,$$

(i)

$$N(q^{12}) = \left(1 + 6p + 12p^2 + 5p^3 - \frac{27}{2}p^4 - 18p^5 - \frac{33}{8}p^6 + \frac{45}{8}p^7 + \frac{135}{32}p^8 + \frac{17}{16}p^9 + \frac{3}{16}p^{10} + \frac{3}{32}p^{11} + \frac{1}{64}p^{12}\right)k^6.$$

**Theorem 2.5.** *Let $q \in \mathbb{C}$ satisfying $|q| < 1$. Then we have*

(a)

$$p^4k^4 = -\frac{1}{450}M(q) + \frac{11}{450}M(q^2) + \frac{31}{450}M(q^3) - \frac{41}{450}M(q^6) + \frac{8}{15}C_{1,6}(q),$$

(b)

$$p^8k^4 = -\frac{8}{225}M(q^2) + \frac{88}{225}M(q^4) + \frac{248}{225}M(q^6) - \frac{328}{225}M(q^{12}) + \frac{128}{15}C_{1,6}(q^2),$$

(c)

$$k^4 = -\frac{11}{450}M(q) + \frac{91}{225}M(q^2) + \frac{41}{450}M(q^3) - \frac{88}{225}M(q^4) - \frac{121}{225}M(q^6) + \frac{328}{225}M(q^{12}) - \frac{32}{15}C_{1,6}(q) - \frac{128}{15}C_{1,6}(q^2),$$

(d)

$$p^6k^4 = -\frac{1}{900}M(q) + \frac{1}{900}M(q^2) + \frac{61}{900}M(q^3) + \frac{4}{45}M(q^4) - \frac{61}{900}M(q^6) - \frac{4}{45}M(q^{12}) - \frac{44}{15}C_{1,12}(q) - \frac{16}{5}C_{1,6}(-q) - \frac{16}{15}C_{1,6}(q^2),$$

(e)

$$p^2k^4 = -\frac{1}{150}M(q) + \frac{91}{900}M(q^2) + \frac{11}{150}M(q^3) - \frac{4}{45}M(q^4) - \frac{151}{900}M(q^6) + \frac{4}{45}M(q^{12}) + \frac{44}{15}C_{1,12}(q) + \frac{16}{5}C_{1,6}(-q) + \frac{28}{15}C_{1,6}(q) + \frac{16}{15}C_{1,6}(q^2),$$

(f)

$$p^7k^4 = \frac{1}{900}M(q) + \frac{7}{900}M(q^2) - \frac{61}{900}M(q^3) - \frac{14}{75}M(q^4) + \frac{53}{900}M(q^6) + \frac{14}{75}M(q^{12}) + \frac{44}{15}C_{1,12}(q) + \frac{16}{5}C_{1,6}(-q) - \frac{16}{15}C_{1,6}(q^2),$$

(g)

$$pk^4 = \frac{23}{1800}M(q) - \frac{91}{450}M(q^2) - \frac{143}{1800}M(q^3) + \frac{14}{75}M(q^4) + \frac{121}{450}M(q^6) - \frac{14}{75}M(q^{12}) - \frac{44}{15}C_{1,12}(q) - \frac{16}{5}C_{1,6}(-q) - \frac{4}{3}C_{1,6}(q) + \frac{16}{15}C_{1,6}(q^2),$$

(h)

$$p^5k^4 = \frac{1}{600}M(q) - \frac{19}{1800}M(q^2) - \frac{41}{600}M(q^3) - \frac{8}{225}M(q^4) + \frac{139}{1800}M(q^6) + \frac{8}{225}M(q^{12}) + \frac{22}{15}C_{1,12}(q) + \frac{8}{5}C_{1,6}(-q) - \frac{4}{15}C_{1,6}(q),$$

(i)

$$p^3k^4 = \frac{7}{1800}M(q) - \frac{91}{1800}M(q^2) - \frac{127}{1800}M(q^3) + \frac{8}{225}M(q^4) + \frac{211}{1800}M(q^6) - \frac{8}{225}M(q^{12}) - \frac{22}{15}C_{1,12}(q) - \frac{8}{5}C_{1,6}(-q) - \frac{16}{15}C_{1,6}(q).$$

*Proof.* First from Proposition 2.4 (c) and (e) we have

$$M(q^3) - M(q^6) = \left(30p^3 + 75p^4 + 30p^5\right)k^4$$

and so

$$\left(p^3+\frac{5}{2}p^4+p^5\right)k^4=\frac{1}{30}M(q^3)-\frac{1}{30}M(q^6). \tag{26}$$

Also from Proposition 2.4 (e) and (f) we obtain

$$M(q^6)-M(q^{12})=\left(\frac{15}{4}p^6+\frac{15}{4}p^7+\frac{15}{16}p^8\right)k^4$$

and so

$$\left(p^6+p^7+\frac{1}{4}p^8\right)k^4=\frac{4}{15}M(q^6)-\frac{4}{15}M(q^{12}). \tag{27}$$

Thus by (26) and (27) we can rewrite Proposition 2.4 (c) as

$$\begin{aligned}
M(q^3)&=\left(1+4p+4p^2+28p^3+70p^4+28p^5+4p^6+4p^7+p^8\right)k^4\\
&=\left\{1+4p+4p^2+28\left(p^3+\frac{5}{2}p^4+p^5\right)+4\left(p^6+p^7+\frac{1}{4}p^8\right)\right\}k^4\\
&=\left(1+4p+4p^2\right)k^4+28\left(\frac{1}{30}M(q^3)-\frac{1}{30}M(q^6)\right)\\
&\quad+4\left(\frac{4}{15}M(q^6)-\frac{4}{15}M(q^{12})\right)
\end{aligned}$$

and so

$$\left(1+4p+4p^2\right)k^4=\frac{1}{15}M(q^3)-\frac{2}{15}M(q^6)+\frac{16}{15}M(q^{12}). \tag{28}$$

Similarly by (26), (27), and (28), we change Proposition 2.4 (b) as

$$\begin{aligned}
M(q^2)&=\left(1+4p+64p^2+178p^3+235p^4+178p^5+64p^6+4p^7+p^8\right)k^4\\
&=\left\{\left(1+4p+4p^2\right)+60p^2+178\left(p^3+\frac{5}{2}p^4+p^5\right)-210p^4+60p^6\right.\\
&\quad\left.+4\left(p^6+p^7+\frac{1}{4}p^8\right)\right\}k^4\\
&=\left(\frac{1}{15}M(q^3)-\frac{2}{15}M(q^6)+\frac{16}{15}M(q^{12})\right)+60\left(p^2-\frac{7}{2}p^4+p^6\right)k^4\\
&\quad+178\left(\frac{1}{30}M(q^3)-\frac{1}{30}M(q^6)\right)+4\left(\frac{4}{15}M(q^6)-\frac{4}{15}M(q^{12})\right)
\end{aligned}$$

and so

$$\left(p^2 - \frac{7}{2}p^4 + p^6\right) k^4 = \frac{1}{60}M(q^2) - \frac{1}{10}M(q^3) + \frac{1}{12}M(q^6). \tag{29}$$

Furthermore by (26), (27), (28), and (29), we can rewrite Proposition 2.4 (a) as

$$\begin{aligned} M(q) &= \left(1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5 + 964p^6 + 124p^7 \right. \\ &\qquad \left. +p^8\right) k^4 \\ &= \left\{-30 + 31\left(1 + 4p + 4p^2\right) + 840\left(p^2 - \frac{7}{2}p^4 + p^6\right) - 120p^4 \right. \\ &\qquad \left. +2788\left(p^3 + \frac{5}{2}p^4 + p^5\right) + 124\left(p^6 + p^7 + \frac{1}{4}p^8\right) - 30p^8\right\} k^4 \\ &= -30\left(1 + 4p^4 + p^8\right) k^4 + 31\left(\frac{1}{15}M(q^3) - \frac{2}{15}M(q^6) + \frac{16}{15}M(q^{12})\right) \\ &\qquad + 840\left(\frac{1}{60}M(q^2) - \frac{1}{10}M(q^3) + \frac{1}{12}M(q^6)\right) \\ &\qquad + 2788\left(\frac{1}{30}M(q^3) - \frac{1}{30}M(q^6)\right) + 124\left(\frac{4}{15}M(q^6) - \frac{4}{15}M(q^{12})\right) \end{aligned}$$

and so

$$\left(1 + 4p^4 + p^8\right) k^4 = -\frac{1}{30}M(q) + \frac{7}{15}M(q^2) + \frac{11}{30}M(q^3) + \frac{1}{5}M(q^6). \tag{30}$$

Second by squaring Proposition 2.2 (a) we can easily know that

$$\begin{aligned} &\left(L(q) - 3L(q^3)\right)^2 \\ &= L^2(q) - 6L(q)L(q^3) + 9L^2(q^3) \\ &= 4\left(1 + 16p + 100p^2 + 304p^3 + 454p^4 + 304p^5 + 100p^6 + 16p^7 + p^8\right) k^4 \\ &= 4\left\{4\left(1 + 4p + 4p^2\right) - 3\left(1 + 4p^4 + p^8\right) + 84\left(p^2 - \frac{7}{2}p^4 + p^6\right) \right. \\ &\qquad \left. +304\left(p^3 + \frac{5}{2}p^4 + p^5\right) + 16\left(p^6 + p^7 + \frac{1}{4}p^8\right)\right\} k^4 \end{aligned}$$

and so appealing to (26), (27), (28), (29), and (30) we conclude that

$$(31) \qquad L(q)L(q^3) = \frac{3}{2}L^2(q^3) + \frac{1}{6}L^2(q) - \frac{3}{5}M(q^3) - \frac{1}{15}M(q).$$

(a) From Proposition 2.3 (c) we note that

$$\begin{aligned} p^4k^4 &= (p^2k^2)^2 \\ &= \left(-\frac{1}{6}L(q^2) + \frac{1}{3}L(q^3) - \frac{1}{6}L(q^6)\right)^2 \\ &= \frac{1}{36}L^2(q^2) + \frac{1}{9}L^2(q^3) + \frac{1}{36}L^2(q^6) - \frac{1}{9}L(q^2)L(q^3) \\ &\quad - \frac{1}{9}L(q^3)L(q^6) + \frac{1}{18}L(q^2)L(q^6) \end{aligned}$$

and so we use (9), Proposition 2.1 (a), (24), and (31).

(b) By Proposition 2.3 (e) we have

$$\begin{aligned} p^8k^4 &= (p^4k^2)^2 \\ &= \left(-\frac{2}{3}L(q^4) + \frac{4}{3}L(q^6) - \frac{2}{3}L(q^{12})\right)^2 \\ &= \frac{4}{9}L^2(q^4) + \frac{16}{9}L^2(q^6) + \frac{4}{9}L^2(q^{12}) - \frac{16}{9}L(q^4)L(q^6) \\ &\quad - \frac{16}{9}L(q^6)L(q^{12}) + \frac{8}{9}L(q^4)L(q^{12}) \end{aligned}$$

thus we refer (9), Proposition 2.1 (a), (24), and (31).

(c) It is obvious by applying Theorem 2.5 (a) and (b) into (30).

(d) By Proposition 2.3 (c) and (e) we can observe that

$$\begin{aligned} &p^6k^4 \\ &= p^2k^2 \cdot p^4k^2 \\ &= \left(-\frac{1}{6}L(q^2) + \frac{1}{3}L(q^3) - \frac{1}{6}L(q^6)\right)\left(-\frac{2}{3}L(q^4) + \frac{4}{3}L(q^6) - \frac{2}{3}L(q^{12})\right) \\ &= \frac{1}{9}L(q^2)L(q^4) - \frac{2}{9}L(q^2)L(q^6) + \frac{1}{9}L(q^2)L(q^{12}) - \frac{2}{9}L(q^3)L(q^4) \\ &\quad + \frac{4}{9}L(q^3)L(q^6) - \frac{2}{9}L(q^3)L(q^{12}) + \frac{1}{9}L(q^4)L(q^6) - \frac{2}{9}L^2(q^6) \\ &\quad + \frac{1}{9}L(q^6)L(q^{12}) \end{aligned}$$

so we apply (9), Proposition 2.1 (a), (b), (23), (24), (25), and (31).

(e) It is clear after applying Theorem 2.5 (a) and (d) into (29).

(f) It is definite by applying Theorem 2.5 (b) and (d) into (27).

(g) It is obvious by inserting Theorem 2.5 (c) and (e) into (28).

(h) From Proposition 2.4 (d) and (f) we can see that

$$M(q^4) - M(q^{12}) = \left(15p^4 + 30p^5 + \frac{15}{2}p^6 - \frac{15}{2}p^7\right) k^4$$

thus we use Theorem 2.5 (a), (d), and (f).

(i) Proof is directly obtained by applying Theorem 2.5 (a) and (h) into (26).

□

**Corollary 2.6.** *Let $q \in \mathbb{C}$ satisfying $|q| < 1$. Then we have*

(a)

$$\left(p^3 + \frac{9}{2}p^4 + 6p^5 + 2p^6\right) k^6 = -\frac{1}{126}N(q^3) + \frac{1}{126}N(q^6) + 4A(q^3),$$

(b)

$$\left(p^6 + 3p^7 + \frac{9}{4}p^8 + \frac{1}{2}p^9\right) k^6 = -\frac{1}{252}N(q^3) + \frac{1}{252}N(q^6) - 2A(q^3),$$

(c)

$$\begin{aligned}\left(p^9 + \frac{3}{2}p^{10} + \frac{3}{4}p^{11} + \frac{1}{8}p^{12}\right) k^6 = &-\frac{1}{252}N(q^3) + \frac{11}{84}N(q^6) \\ &- \frac{8}{63}N(q^{12}) - 2A(q^3),\end{aligned}$$

(d)

$$\left(1 + 6p + 12p^2 + 8p^3\right) k^6 = -\frac{1}{63}N(q^3) + \frac{64}{63}N(q^{12}) + 16A(q^3).$$

*Proof.* (a) By Proposition 2.2 (b) and (c) we have

$$\begin{aligned}L(q^3) - 4L(q^{12}) &= \left(L(q^3) - 2L(q^6)\right) + 2\left(L(q^6) - 2L(q^{12})\right) \\ &= -3k^2 - 6pk^2\end{aligned}$$

and so combining the above identity with (26) we obtain

$$\begin{aligned}
&\left(L(q^3) - 4L(q^{12})\right)\left(M(q^3) - M(q^6)\right) \\
&= L(q^3)M(q^3) - L(q^3)M(q^6) - 4L(q^{12})M(q^3) + 4L(q^{12})M(q^6) \\
&= \left(-3k^2 - 6pk^2\right) \cdot 30\left(p^3 + \frac{5}{2}p^4 + p^5\right)k^4 \\
&= -90\left(p^3 + \frac{9}{2}p^4 + 6p^5 + 2p^6\right)k^6.
\end{aligned}$$

Thus we use (12), Proposition 2.1 (c), (d), and (h).

(b) From Proposition 2.2 (b) and (26) we note that

$$\begin{aligned}
&\left(L(q^3) - 2L(q^6)\right)\left(M(q^3) - M(q^6)\right) \\
&= L(q^3)M(q^3) - L(q^3)M(q^6) - 2L(q^6)M(q^3) + 2L(q^6)M(q^6) \\
&= -\left(1 + 2p + 2p^3 + p^4\right)k^2 \cdot 30\left(p^3 + \frac{5}{2}p^4 + p^5\right)k^4 \\
&= -30\left(p^3 + \frac{9}{2}p^4 + 6p^5 + 4p^6 + 6p^7 + \frac{9}{2}p^8 + p^9\right)k^6 \\
&= -30\left(p^3 + \frac{9}{2}p^4 + 6p^5 + 2p^6\right)k^6 - 60\left(p^6 + 3p^7 + \frac{9}{4}p^8 + \frac{1}{2}p^9\right)k^6
\end{aligned}$$

and so we refer to (12), Proposition 2.1 (c), (d), and Corollary 2.6 (a).

(c) By Proposition 2.4 (h) and (i) we can observe that

$$\begin{aligned}
&64N(q^6) - 64N(q^{12}) \\
&= \left(-504p^6 - 1512p^7 - 1134p^8 + 252p^9 + 756p^{10} + 378p^{11} + 63p^{12}\right)k^6 \\
&= 63\left(-8p^6 - 24p^7 - 18p^8 + 4p^9 + 12p^{10} + 6p^{11} + p^{12}\right)k^6 \\
&= 63\left\{-8\left(p^6 + 3p^7 + \frac{9}{4}p^8 + \frac{1}{2}p^9\right)k^6\right. \\
&\qquad \left. +8\left(p^9 + \frac{3}{2}p^{10} + \frac{3}{4}p^{11} + \frac{1}{8}p^{12}\right)k^6\right\},
\end{aligned}$$

which requires (8) and Corollary 2.6 (b).

(d) First by Proposition 2.4 (h) and (i) we have

(32)
$$64N(q^{12}) - N(q^6) = \left(63 + 378p + 756p^2 + 315p^3 - \frac{1701}{2}p^4 - 1134p^5 - 252p^6 + 378p^7 + \frac{567}{2}p^8 + 63p^9\right) k^6$$

moreover, Proposition 2.4 (g) and (h) show that

$$N(q^6) - N(q^3) = \left(63p^3 + \frac{567}{2}p^4 + 378p^5 + 252p^6 + 378p^7 + \frac{567}{2}p^8 + 63p^9\right) k^6. \tag{33}$$

Second by (32) and (33) we obtain

$$\begin{aligned}
&\left(64N(q^{12}) - N(q^6)\right) - \left(N(q^6) - N(q^3)\right) \\
&= \left(63 + 378p + 756p^2 + 252p^3 - 1134p^4 - 1512p^5 - 504p^6\right) k^6 \\
&= 63\left(1 + 6p + 12p^2 + 8p^3\right) k^6 - 252\left(p^3 + \frac{9}{2}p^4 + 6p^5 + 2p^6\right) k^6
\end{aligned}$$

therefore we use (8) and Corollary 2.6 (a).

□

## 3. Proofs of Theorem 1.2 and Theorem 1.3

We introduce Proposition 3.1 and Proposition 3.2 to deduce Theorem 1.2.

**Proposition 3.1.** *For $q \in \mathbb{C}$ satisfying $|q| < 1$, we have*

(a) *(See* [8, Theorem 2.5 (g)]*)*

$$L(q)A(q) = -\sum_{n=1}^{\infty} b(n)q^n - 32\sum_{n=1}^{\infty} b(n)q^{2n} + 2\sum_{n=1}^{\infty} na(n)q^n,$$

(b) *(See* [10, Theorem 1.3 (k)]*)*

$$L(q^4)A(q) = \frac{1}{2}\sum_{n=1}^{\infty} b(n)q^n + 4\sum_{n=1}^{\infty} b(n)q^{2n} + \frac{1}{2}\sum_{n=1}^{\infty} na(n)q^n,$$

(c) *(See* [9, Theorem 1.2 (j)]*)*

$$L(q)B(q) = -16\sum_{n=1}^{\infty} d(n)q^n - \frac{1}{2}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{2}\sum_{n=1}^{\infty} nb(n)q^n,$$

(d) *(See* [9, Theorem 1.2 (k)]*)*

$$L(q^2)B(q) = 8\sum_{n=1}^{\infty} d(n)q^n + \frac{1}{4}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{4}\sum_{n=1}^{\infty} nb(n)q^n,$$

(e) *(See* [8, Theorem 2.5(h)]*)*

$$L(q)B(q^2) = 3\sum_{n=1}^{\infty} d(n)q^n - 160\sum_{n=1}^{\infty} d(n)q^{2n} - 5\sum_{n=1}^{\infty} c(n)q^{2n} + 3\sum_{n=1}^{\infty} nb(n)q^{2n},$$

(f) *(See* [10, (2.11)]*)*

$$L(q^4)B(q) = 8\sum_{n=1}^{\infty} d(n)q^n + \frac{5}{8}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{8}\sum_{n=1}^{\infty} nb(n)q^n,$$

(g) *(See* [9, Theorem 1.2 (h)]*)*

$$\begin{aligned} &L^2(q)M^2(q^2) \\ &= 1 + \frac{240}{691}\sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{65280}{691}\sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{288}{155}\sum_{n=1}^{\infty} n\sigma_9(n)q^n \\ &\quad - \frac{39168}{31}\sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + 3840\sum_{n=1}^{\infty} n^2\sigma_7(n)q^{2n} \\ &\quad + \frac{31968}{3455}\sum_{n=1}^{\infty} \tau(n)q^n + \frac{996096}{691}\sum_{n=1}^{\infty} \tau(n)q^{2n} \\ &\quad - \frac{55296}{31}\sum_{n=1}^{\infty} nd(n)q^n - \frac{1728}{31}\sum_{n=1}^{\infty} nc(n)q^n, \end{aligned}$$

(h) *(See* [9, Theorem 1.2 (e)]*)*

$$\begin{aligned} &L(q)L(q^2)M^2(q) \\ &= 1 + \frac{16368}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{49152}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} \\ &\quad - \frac{36792}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{73728}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} \\ &\quad + 480\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n - \frac{115008}{3455}\sum_{n=1}^{\infty}\tau(n)q^n \\ &\quad - \frac{34209792}{3455}\sum_{n=1}^{\infty}\tau(n)q^{2n} + \frac{221184}{31}\sum_{n=1}^{\infty}nd(n)q^n \\ &\quad + \frac{6912}{31}\sum_{n=1}^{\infty}nc(n)q^n. \end{aligned}$$

**Proposition 3.2.** *Let $n \in \mathbb{N}$. Then we have*

(a) *(See* [14]*)*

$$\begin{aligned} &\sum_{m=1}^{n-1}\sigma_3(m)\sigma_7(n-m) \\ &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n), \end{aligned}$$

(b) *(See* [5, Theorem 7.2]*)*

$$\begin{aligned} &\sum_{m<\frac{n}{2}}\sigma_3(m)\sigma_7(n-2m) \\ &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{2}) \\ &\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(\frac{n}{2}), \end{aligned}$$

(c) *(See* [5, Theorem 7.2]*)*

$$\begin{aligned} &\sum_{m<\frac{n}{2}}\sigma_7(m)\sigma_3(n-2m) \\ &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_3(n) \\ &\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(\frac{n}{2}), \end{aligned}$$

(d) *(See* [5, Theorem 7.2]*)*

$$\begin{aligned}&\sum_{m<\frac{n}{4}} \sigma_3(m)\sigma_7(n-4m)\\&= \frac{121}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{8}{10365}\sigma_{11}(\frac{n}{4})\\&\quad - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{4}) + \frac{729}{176896}\tau(n) + \frac{6003}{11056}\tau(\frac{n}{2})\\&\quad - \frac{71496}{691}\tau(\frac{n}{4}) - 1920f(n),\end{aligned}$$

(e) *(See* [10, Remark 3.1]*)*

$$\begin{aligned}&\sum_{m<\frac{n}{4}} \sigma_7(m)\sigma_3(n-4m)\\&= -\frac{7}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{17}{20730}\sigma_{11}(\frac{n}{4})\\&\quad - \frac{1}{240}\sigma_7(\frac{n}{4}) - \frac{1}{480}\sigma_3(n) + \frac{369}{176896}\tau(n) + \frac{1641}{22112}\tau(\frac{n}{2})\\&\quad + \frac{22203}{1382}\tau(\frac{n}{4}) + 120f(n),\end{aligned}$$

(f) *(See* [9, Lemma 3.1 (c)]*)*

$$\sum_{m=1}^{n-1} \sigma_3(m)b(n-m) = \frac{1}{240}\left\{\tau(n) + 256\tau(\frac{n}{2}) - b(n)\right\},$$

(g) *(See* [11, (3.5)]*)*

$$\sum_{m<\frac{n}{2}} \sigma_3(m)b(n-2m) = \frac{1}{240}\left\{\tau(n) + 16\tau(\frac{n}{2}) - b(n)\right\},$$

(h) *(See* [11, (3.7)]*)*

$$\begin{aligned}&\sum_{m<\frac{n}{2}} b(m)\sigma_3(n-2m)\\&= \frac{1}{2653440}\Big\{15\sigma_{11}(n) - 15\sigma_{11}(\frac{n}{2}) - 15\tau(n) - 20024\tau(\frac{n}{2})\\&\quad -39624704\tau(\frac{n}{4}) - 679280640f(n) - 11056b(\frac{n}{2})\Big\},\end{aligned}$$

(i) *(See* [11, (3.6)]*)*

$$\sum_{m<\frac{n}{4}} \sigma_3(m)b(n-4m)$$
$$= -\frac{1}{2653440}\Big\{15\sigma_{11}(n) - 15\sigma_{11}(\frac{n}{2}) - 11071\tau(n) - 207976\tau(\frac{n}{2}) - 84910080\tau(\frac{n}{4}) - 679280640f(n) + 11056b(n)\Big\},$$

(j) *(See* [8, (49)]*)*

$$\sum_{m=1}^{n-1} \sigma_1(m)c(n-m)$$
$$= -\frac{1}{82920}\Big\{2\sigma_{11}(n) - 2\sigma_{11}(\frac{n}{2}) - 693\tau(n) - 26256\tau(\frac{n}{2}) - 11321344\tau(\frac{n}{4}) - 90570752f(n) + 691\,(6n-5)\,d(2n) - 22112\,(6n-5)\,d(n)\Big\},$$

(k) *(See* [8, (48)]*)*

$$\sum_{m=1}^{n-1} \sigma_1(m)d(n-m)$$
$$= \frac{1}{1326720}\Big\{\sigma_{11}(n) - \sigma_{11}(\frac{n}{2}) - \tau(n) + 75320\tau(\frac{n}{2}) - 5660672\tau(\frac{n}{4}) - 45285376f(n) - 11056\,(6n-5)\,d(n)\Big\},$$

(l) *(See* [8, (47)]*)*

$$\sum_{m<\frac{n}{2}} d(2m)\sigma_1(n-2m)$$
$$= \frac{1}{2653440}\Big\{15\sigma_{11}(n) - 15\sigma_{11}(\frac{n}{2}) - 15\tau(n) + 123704\tau(\frac{n}{2}) - 45285376\tau(\frac{n}{4}) - 679280640f(n) - 707584\,(6n-5)\,d(\frac{n}{2}) - 22112\,(6n-5)\,c(\frac{n}{2})\Big\},$$

(m) *(See* [9, Theorem 1.1 (f)]*)*

$$\begin{aligned}
&\sum_{m<\frac{n}{2}} m\sigma_1(m)\sigma_7(n-2m) \\
&= \frac{1}{446400}\Big\{255n\sigma_9(n) + 768n\sigma_9(\frac{n}{2}) - 775n^2\sigma_7(n) - 465n\sigma_1(\frac{n}{2}) \\
&\quad +1240\tau(n) + 190464\tau(\frac{n}{2}) - 23040nd(n) - 720nc(n)\Big\}.
\end{aligned}$$

***Proof of Theorem 1.2.*** (a) Now Corollary 2.6 (d) implies that

(34)
$$\begin{aligned}
(1+2p)^3\,k^6 &= \left(1+6p+12p^2+8p^3\right)k^6 \\
&= -\frac{1}{63}N(q^3) + \frac{64}{63}N(q^{12}) + 16A(q^3).
\end{aligned}$$

And by Proposition 2.3 (a) and (b) we obtain

(35)
$$(1+2p)\,k^2 = -\frac{1}{3}L(q^3) + \frac{4}{3}L(q^{12}).$$

Multiplying (34) and (35) together we have

$$\begin{aligned}
&(1+2p)^4\,k^8 \\
&= (1+2p)^3\,k^6 \cdot (1+2p)\,k^2 \\
&= \left(-\frac{1}{63}N(q^3) + \frac{64}{63}N(q^{12}) + 16A(q^3)\right)\left(-\frac{1}{3}L(q^3) + \frac{4}{3}L(q^{12})\right) \\
&= \frac{1}{63\cdot 3}N(q^3)L(q^3) - \frac{4}{63\cdot 3}N(q^3)L(q^{12}) - \frac{64}{63\cdot 3}N(q^{12})L(q^3) \\
&\quad + \frac{64\cdot 4}{63\cdot 3}N(q^{12})L(q^{12}) - \frac{16}{3}A(q^3)L(q^3) + \frac{16\cdot 4}{3}A(q^3)L(q^{12})
\end{aligned}$$

thus we refer to Proposition 2.1 (i), (j), Proposition 3.1 (a), and (b).

(b) From Theorem 1.2 (a) and (35) we can know that

$$
\begin{aligned}
&(1+2p)^5\,k^{10}\\
&=(1+2p)^4\,k^8\cdot(1+2p)\,k^2\\
&=\left(\frac{1}{255}M^2(q^3)-\frac{2}{255}M^2(q^6)+\frac{256}{255}M^2(q^{12})+\frac{512}{17}B(q^3)\right.\\
&\qquad\left.+\frac{8192}{17}B(q^6)\right)\left(-\frac{1}{3}L(q^3)+\frac{4}{3}L(q^{12})\right)\\
&=-\frac{1}{255\cdot 3}M^2(q^3)L(q^3)+\frac{4}{255\cdot 3}M^2(q^3)L(q^{12})\\
&\qquad+\frac{2}{255\cdot 3}M^2(q^6)L(q^3)-\frac{8}{255\cdot 3}M^2(q^6)L(q^{12})\\
&\qquad-\frac{256}{255\cdot 3}M^2(q^{12})L(q^3)+\frac{256\cdot 4}{255\cdot 3}M^2(q^{12})L(q^{12})\\
&\qquad-\frac{512}{17\cdot 3}B(q^3)L(q^3)+\frac{512\cdot 4}{17\cdot 3}B(q^3)L(q^{12})\\
&\qquad-\frac{8192}{17\cdot 3}B(q^6)L(q^3)+\frac{8192\cdot 4}{17\cdot 3}B(q^6)L(q^{12})
\end{aligned}
$$

so we use Proposition 2.1 (k), (l), (21), Proposition 3.1 (c), (d), (e), and (f).

(c) In advance by (4) and (5) we can calculate $R_{24}(n)$ as

$$
\begin{aligned}
R_{24}(n) &= \sum_{m=0}^{n} R_{16}(m)R_8(n-m) \\
&= R_8(n) + R_{16}(n) + \sum_{m=1}^{n-1} R_{16}(m)R_8(n-m) \\
&= R_8(n) + R_{16}(n) + \sum_{m=1}^{n-1} \frac{32}{17}\left\{\sigma_7(m) - 2\sigma_7(\frac{m}{2}) + 256\sigma_7(\frac{m}{4}) + 16b(m)\right. \\
&\quad \left. +256b(\frac{m}{2})\right\}\left\{16\sigma_3(n-m) - 32\sigma_3(\frac{n-m}{2}) + 256\sigma_3(\frac{n-m}{4})\right\} \\
&= R_8(n) + R_{16}(n) \\
&\quad + \frac{32\cdot 16}{17}\sum_{m=1}^{n-1}\left\{\sigma_7(m) - 2\sigma_7(\frac{m}{2}) + 256\sigma_7(\frac{m}{4}) + 16b(m)\right. \\
&\quad \left. +256b(\frac{m}{2})\right\}\left\{\sigma_3(n-m) - 2\sigma_3(\frac{n-m}{2}) + 16\sigma_3(\frac{n-m}{4})\right\} \\
&= R_8(n) + R_{16}(n) + \frac{32\cdot 16}{17}\left\{\sum_{m=1}^{n-1}\sigma_7(m)\sigma_3(n-m)\right. \\
&\quad - 2\sum_{m=1}^{n-1}\sigma_7(m)\sigma_3(\frac{n-m}{2}) + 16\sum_{m=1}^{n-1}\sigma_7(m)\sigma_3(\frac{n-m}{4}) \\
&\quad - 2\sum_{m=1}^{n-1}\sigma_7(\frac{m}{2})\sigma_3(n-m) + 2\cdot 2\sum_{m=1}^{n-1}\sigma_7(\frac{m}{2})\sigma_3(\frac{n-m}{2}) \\
&\quad - 2\cdot 16\sum_{m=1}^{n-1}\sigma_7(\frac{m}{2})\sigma_3(\frac{n-m}{4}) + 256\sum_{m=1}^{n-1}\sigma_7(\frac{m}{4})\sigma_3(n-m) \\
&\quad - 256\cdot 2\sum_{m=1}^{n-1}\sigma_7(\frac{m}{4})\sigma_3(\frac{n-m}{2}) + 256\cdot 16\sum_{m=1}^{n-1}\sigma_7(\frac{m}{4})\sigma_3(\frac{n-m}{4}) \\
&\quad + 16\sum_{m=1}^{n-1}b(m)\sigma_3(n-m) - 16\cdot 2\sum_{m=1}^{n-1}b(m)\sigma_3(\frac{n-m}{2}) \\
&\quad + 16\cdot 16\sum_{m=1}^{n-1}b(m)\sigma_3(\frac{n-m}{4}) + 256\sum_{m=1}^{n-1}b(\frac{m}{2})\sigma_3(n-m) \\
&\quad \left. -256\cdot 2\sum_{m=1}^{n-1}b(\frac{m}{2})\sigma_3(\frac{n-m}{2}) + 256\cdot 16\sum_{m=1}^{n-1}b(\frac{m}{2})\sigma_3(\frac{n-m}{4})\right\},
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
&R_{24}(n)\\
&= R_8(n) + R_{16}(n) + \frac{32\cdot 16}{17}\left\{\sum_{m=1}^{n-1}\sigma_3(m)\sigma_7(n-m)\right.\\
&\quad - 2\sum_{m<\frac{n}{2}}\sigma_3(m)\sigma_7(N-2m) + 16\sum_{m<\frac{n}{4}}\sigma_3(m)\sigma_7(n-4m)\\
&\quad - 2\sum_{m<\frac{n}{2}}\sigma_7(m)\sigma_3(n-2m) + 2\cdot 2\sum_{m<\frac{n}{2}}\sigma_3(m)\sigma_7(\frac{n}{2}-m)\\
&\quad - 2\cdot 16\sum_{m<\frac{n}{4}}\sigma_3(m)\sigma_7(\frac{n}{2}-2m) + 256\sum_{m<\frac{n}{4}}\sigma_7(m)\sigma_3(n-4m)\\
&\quad - 256\cdot 2\sum_{m<\frac{n}{4}}\sigma_7(m)\sigma_3(\frac{n}{2}-2m) + 256\cdot 16\sum_{m<\frac{n}{4}}\sigma_3(m)\sigma_7(\frac{n}{4}-m)\\
&\quad + 16\sum_{m=1}^{n-1}b(m)\sigma_3(n-m) - 16\cdot 2\sum_{m<\frac{n}{2}}\sigma_3(m)b(n-2m)\\
&\quad + 16\cdot 16\sum_{m<\frac{n}{4}}\sigma_3(m)b(n-4m) + 256\sum_{m<\frac{n}{2}}b(m)\sigma_3(n-2m)\\
&\quad \left.-256\cdot 2\sum_{m<\frac{n}{2}}b(m)\sigma_3(\frac{n}{2}-m) + 256\cdot 16\sum_{m<\frac{n}{4}}\sigma_3(m)b(\frac{n}{2}-2m)\right\}.
\end{aligned}
$$

Thus we apply Proposition 3.2 (a), (b), (c), (d), (e), (f), (g), (h), and (i) so that we can have

$$
\begin{aligned}
R_{24}(n) = \frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(\frac{n}{2}) + \frac{65536}{691}\sigma_{11}(\frac{n}{4}) + \frac{33152}{691}\tau(n)\\
+ \frac{1525760}{691}\tau(\frac{n}{2}) + \frac{135790592}{691}\tau(\frac{n}{4}).
\end{aligned}
\tag{36}
$$

Now by (16), (19), (20), and (36) we note that

$$
(1+2p)^{18}k^{12} = \varphi^{24}(q) = \sum_{n=0}^{\infty}R_{24}(n)q^n = 1 + \sum_{n=1}^{\infty}R_{24}(n)q^n
$$

and

$$
\begin{aligned}
(1+2p)^6\,k^{12} &= \varphi^{24}(q^3) = \sum_{n=0}^{\infty} R_{24}(n)q^{3n} \\
&= 1 + \sum_{n=1}^{\infty} R_{24}(\frac{n}{3})q^n \\
&= 1 + \sum_{n=1}^{\infty} \left\{ \frac{16}{691}\sigma_{11}(\frac{n}{3}) - \frac{32}{691}\sigma_{11}(\frac{n}{6}) + \frac{65536}{691}\sigma_{11}(\frac{n}{12}) \right. \\
&\quad \left. + \frac{33152}{691}\tau(\frac{n}{3}) + \frac{1525760}{691}\tau(\frac{n}{6}) + \frac{135790592}{691}\tau(\frac{n}{12}) \right\} q^n.
\end{aligned} \tag{37}
$$

Finally owing to (11) and (13) we can deduce that

$$
\sum_{n=1}^{\infty} \sigma_{11}(n)q^n = -\frac{691}{65520} + \frac{7}{1040}M^3(q) + \frac{25}{6552}N^2(q) \tag{38}
$$

and so we apply the above identity to Eq. (37).

□

**Lemma 3.3.** *Let $q \in \mathbb{C}$ such that $|q| < 1$. Then we have*

(a)

$$
\begin{aligned}
&L^2(q^2)M^2(q) \\
&= -\frac{171}{682}L(q)M(q)N(q) + \frac{512}{341}L(q^2)M(q^2)N(q^2) + \frac{492681}{12252812}M^3(q) \\
&\quad - \frac{1102080}{3063203}M^3(q^2) + \frac{2932625}{128654526}N^2(q) - \frac{13120000}{64327263}N^2(q^2) \\
&\quad + \frac{1}{4}L^2(q)M^2(q) + \frac{1517616}{21421}\Delta(q) + \frac{52862976}{21421}\Delta(q^2) \\
&\quad + \frac{221184}{31}\sum_{n=1}^{\infty} nd(n)q^n + \frac{6912}{31}\sum_{n=1}^{\infty} nc(n)q^n,
\end{aligned}
$$

(b)

$$\begin{aligned}
&L(q)L(q^4)M^2(q^2)\\
&= \frac{1}{1364}L(q)M(q)N(q) - \frac{342}{341}L(q^2)M(q^2)N(q^2)\\
&\quad + \frac{1024}{341}L(q^4)M(q^4)N(q^4) - \frac{2634991}{245056240}M^3(q)\\
&\quad - \frac{27140447}{61264060}M^3(q^2) - \frac{674557696}{15316015}M^3(q^4)\\
&\quad + \frac{10500895}{1029236208}N^2(q) + \frac{44744365}{64327263}N^2(q^2)\\
&\quad + \frac{2688229120}{64327263}N^2(q^4) + L^2(q^2)M^2(q^2) + \frac{25812}{21421}\Delta(q)\\
&\quad + \frac{6070464}{21421}\Delta(q^2) + \frac{105725952}{21421}\Delta(q^4) - \frac{6912}{31}\sum_{n=1}^{\infty} nd(n)q^n\\
&\quad + \frac{442368}{31}\sum_{n=1}^{\infty} nd(n)q^{2n} - \frac{216}{31}\sum_{n=1}^{\infty} nc(n)q^n + \frac{13824}{31}\sum_{n=1}^{\infty} nc(n)q^{2n},
\end{aligned}$$

(c)

$$\begin{aligned}
&L(q)L(q^2)M^2(q^2)\\
&= \frac{1}{682}L(q)M(q)N(q) - \frac{342}{341}L(q^2)M(q^2)N(q^2) - \frac{2721091}{367584360}M^3(q)\\
&\quad - \frac{13345313}{45948045}M^3(q^2) + \frac{3431965}{514618104}N^2(q) + \frac{6274985}{21442421}N^2(q^2)\\
&\quad + 2L^2(q^2)M^2(q^2) + \frac{51624}{21421}\Delta(q) + \frac{6070464}{21421}\Delta(q^2)\\
&\quad - \frac{13824}{31}\sum_{n=1}^{\infty} nd(n)q^n - \frac{432}{31}\sum_{n=1}^{\infty} nc(n)q^n.
\end{aligned}$$

*Proof.* (a) From (9) and (10) we observe that

$$
\begin{aligned}
&L^2(q^2)M^2(q) = M^2(q)\cdot L^2(q^2)\\
&= \left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\\
&\qquad \times\left(1-288\sum_{m=1}^{\infty}m\sigma_1(m)q^{2m}+240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-144N\sigma_1(\frac{N}{2})+240\sigma_3(\frac{N}{2})+480\sigma_7(N)\\
&\qquad -480\cdot 288\sum_{m<\frac{N}{2}}m\sigma_1(m)\sigma_7(N-2m)\\
&\qquad +480\cdot 240\sum_{m<\frac{N}{2}}\sigma_3(m)\sigma_7(N-2m)\Bigg\}q^N
\end{aligned}
$$

and so we use Proposition 3.2 (b) and (m). Then we have

(39)

$$
\begin{aligned}
&L^2(q^2)M^2(q)\\
&= 1+\frac{4080}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n+\frac{61440}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n}-\frac{2448}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^n\\
&\quad -\frac{73728}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n}+240\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n+\frac{62256}{691}\sum_{n=1}^{\infty}\tau(n)q^n\\
&\quad +\frac{8183808}{3455}\sum_{n=1}^{\infty}\tau(n)q^{2n}+\frac{221184}{31}\sum_{n=1}^{\infty}nd(n)q^n\\
&\quad +\frac{6912}{31}\sum_{n=1}^{\infty}nc(n)q^n.
\end{aligned}
$$

Now by (14) and (38) we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\sigma_9(n)q^n \\
&= \frac{5}{1584}\Bigg(1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{228096}{3455}\sum_{n=1}^{\infty}\tau(n)q^n \\
&\qquad - L(q)M(q)N(q)\Bigg) \qquad (40)\\
&= -\frac{5}{1584}L(q)M(q)N(q) + \frac{245}{121616}M^3(q) + \frac{625}{547272}N^2(q) \\
&\qquad - \frac{144}{691}\Delta(q)
\end{aligned}$$

similarly, by (15), (38), and (40) we have

(41)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^2\sigma_7(n)q^n \\
&= \frac{1}{960}\Bigg(L^2(q)M^2(q) - 1 + \frac{3168}{5}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n \\
&\qquad -\frac{37248}{3455}\sum_{n=1}^{\infty}\tau(n)q^n\Bigg) \\
&= -\frac{1}{480}L(q)M(q)N(q) + \frac{147}{221120}M^3(q) + \frac{25}{66336}N^2(q) \\
&\qquad + \frac{1}{960}L^2(q)M^2(q) - \frac{514}{3455}\Delta(q).
\end{aligned}$$

Lastly we apply (38), (40), and (41) to Eq. (39).

(b) First by (38), (40), and (41) we can rewrite Proposition 3.1 (g) as

(42)

$$\begin{aligned}
&L^2(q)M^2(q^2)\\
&= \frac{2}{341}L(q)M(q)N(q) - \frac{1368}{341}L(q^2)M(q^2)N(q^2) - \frac{4305}{3063203}M^3(q)\\
&\quad + \frac{1970724}{3063203}M^3(q^2) - \frac{51250}{64327263}N^2(q) + \frac{23461000}{64327263}N^2(q^2)\\
&\quad + 4L^2(q^2)M^2(q^2) + \frac{206496}{21421}\Delta(q) + \frac{24281856}{21421}\Delta(q^2)\\
&\quad - \frac{55296}{31}\sum_{n=1}^{\infty} nd(n)q^n - \frac{1728}{31}\sum_{n=1}^{\infty} nc(n)q^n.
\end{aligned}$$

Second from Proposition 2.1 (b) we can know

$$\begin{aligned}
&L(q)L(q^4)M^2(q^2) = L(q)L(q^4)\cdot M^2(q^2)\\
&= \left(\frac{1}{8}L^2(q) + 2L^2(q^4) - \frac{3}{40}M(q) + \frac{3}{20}M(q^2) - \frac{6}{5}M(q^4)\right)M^2(q^2)\\
&= \frac{1}{8}L^2(q)M^2(q^2) + 2L^2(q^4)M^2(q^2) - \frac{3}{40}M(q)M^2(q^2) + \frac{3}{20}M^3(q^2)\\
&\quad - \frac{6}{5}M(q^4)M^2(q^2)
\end{aligned}$$

so that we use Proposition 2.1 (e), (f), Lemma 3.3 (a), and (42).

(c) Owing to Proposition 2.1 (a) we note that

$$\begin{aligned}
&L(q)L(q^2)M^2(q^2) = L(q)L(q^2)\cdot M^2(q^2)\\
&= \left(\frac{1}{4}L^2(q) + L^2(q^2) - \frac{1}{20}M(q) - \frac{1}{5}M(q^2)\right)M^2(q^2)\\
&= \frac{1}{4}L^2(q)M^2(q^2) + L^2(q^2)M^2(q^2) - \frac{1}{20}M(q)M^2(q^2) - \frac{1}{5}M^3(q^2)
\end{aligned}$$

therefore we refer to Proposition 2.1 (e) and (42).

□

***Proof of Theorem 1.3.*** (a) First depending on (38), (40), and (41), Proposition 3.1 (h) shows that

(43)

$$\begin{aligned}
&L(q)L(q^2)M^2(q)\\
&= -\frac{171}{682}L(q)M(q)N(q) + \frac{512}{341}L(q^2)M(q^2)N(q^2) + \frac{5376}{15316015}M^3(q)\\
&\quad - \frac{7343616}{15316015}M^3(q^2) + \frac{12800}{64327263}N^2(q) - \frac{17484800}{64327263}N^2(q^2)\\
&\quad + \frac{1}{2}L^2(q)M^2(q) - \frac{1183104}{21421}\Delta(q) - \frac{209977344}{21421}\Delta(q^2)\\
&\quad + \frac{221184}{31}\sum_{n=1}^{\infty} nd(n)q^n + \frac{6912}{31}\sum_{n=1}^{\infty} nc(n)q^n.
\end{aligned}$$

Now by Proposition 2.1 (k) let us investigate Lemma 3.3 (b) as

$$\begin{aligned}
&L(q)L(q^4)M^2(q^2) = L(q)M^2(q^2)\cdot L(q^4)\\
&= \bigg(2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q)\\
&\qquad - \frac{23040}{31}D(q)\bigg)L(q^4)\\
&= 2L(q^2)M^2(q^2)L(q^4) + \frac{1}{341}M(q)N(q)L(q^4) - \frac{342}{341}M(q^2)N(q^2)L(q^4)\\
&\qquad - \frac{720}{31}C(q)L(q^4) - \frac{23040}{31}D(q)L(q^4)
\end{aligned}$$

and so

$$
\begin{aligned}
&C(q)L(q^4) + 32D(q)L(q^4) \\
&= \frac{31}{720}\Bigg(2L(q^2)M^2(q^2)L(q^4) + \frac{1}{341}M(q)N(q)L(q^4) \\
&\qquad - \frac{342}{341}M(q^2)N(q^2)L(q^4) - L(q)L(q^4)M^2(q^2)\Bigg) \\
&= \frac{13301}{31045248}M^3(q) + \frac{635083}{25871040}M^3(q^2) + \frac{77744}{18657}M^3(q^4) \\
&\quad - \frac{477973}{1086583680}N^2(q) - \frac{1110877}{45274320}N^2(q^2) - \frac{77744}{18657}N^2(q^4) \\
&\quad - \frac{717}{13820}\Delta(q) - \frac{58588}{3455}\Delta(q^2) - \frac{730112}{691}\Delta(q^4) + 49152F(q) \\
&\quad + \frac{48}{5}\sum_{n=1}^{\infty} nd(n)q^n + \frac{3}{10}\sum_{n=1}^{\infty} nc(n)q^n \\
&:= \alpha(q),
\end{aligned}
\tag{44}
$$

where we use Proposition 2.1 (g), (m), and (43) for the fourth line. Second from Theorem 1.2 (b) and (35) we can rewrite Theorem 1.2 (c) as

(45)

$$
\begin{aligned}
&(1+2p)^6\,k^{12}\\
&=(1+2p)^5\,k^{10}\cdot(1+2p)\,k^2\\
&=\left(-\frac{1}{1023}M(q^3)N(q^3)+\frac{1024}{1023}M(q^{12})N(q^{12})-\frac{256}{31}D(q^3)\right.\\
&\qquad\left.+\frac{1261568}{31}D(q^6)+\frac{1232}{31}C(q^3)+\frac{39424}{31}C(q^6)\right)\\
&\qquad\times\left(-\frac{1}{3}L(q^3)+\frac{4}{3}L(q^{12})\right)\\
&=\frac{1}{1023\cdot 3}M(q^3)N(q^3)L(q^3)-\frac{4}{1023\cdot 3}M(q^3)N(q^3)L(q^{12})\\
&\qquad-\frac{1024}{1023\cdot 3}M(q^{12})N(q^{12})L(q^3)+\frac{1024\cdot 4}{1023\cdot 3}M(q^{12})N(q^{12})L(q^{12})\\
&\qquad+\frac{256}{31\cdot 3}D(q^3)L(q^3)-\frac{256\cdot 4}{31\cdot 3}D(q^3)L(q^{12})-\frac{1261568}{31\cdot 3}D(q^6)L(q^3)\\
&\qquad+\frac{1261568\cdot 4}{31\cdot 3}D(q^6)L(q^{12})-\frac{1232}{31\cdot 3}C(q^3)L(q^3)\\
&\qquad+\frac{1232\cdot 4}{31\cdot 3}C(q^3)L(q^{12})-\frac{39424}{31\cdot 3}C(q^6)L(q^3)\\
&\qquad+\frac{39424\cdot 4}{31\cdot 3}C(q^6)L(q^{12})\\
&=\frac{1}{1023\cdot 3}M(q^3)N(q^3)L(q^3)-\frac{4}{1023\cdot 3}M(q^3)N(q^3)L(q^{12})\\
&\qquad-\frac{1024}{1023\cdot 3}M(q^{12})N(q^{12})L(q^3)+\frac{1024\cdot 4}{1023\cdot 3}M(q^{12})N(q^{12})L(q^{12})\\
&\qquad+\frac{256}{31\cdot 3}D(q^3)L(q^3)-\frac{1232}{31\cdot 3}C(q^3)L(q^3)\\
&\qquad-\frac{39424}{31\cdot 3}\left\{C(q^6)L(q^3)+32D(q^6)L(q^3)\right\}\\
&\qquad+\frac{39424\cdot 4}{31\cdot 3}\left\{C(q^6)L(q^{12})+32D(q^6)L(q^{12})\right\}\\
&\qquad-\frac{256\cdot 4}{31\cdot 3}D(q^3)L(q^{12})+\frac{1232\cdot 4}{31\cdot 3}C(q^3)L(q^{12})
\end{aligned}
$$

then ahead we should calculate some terms $C(q^3)L(q^3)$, $D(q^3)L(q^3)$, $C(q^6)L(q^3)+32D(q^6)L(q^3)$, and $C(q^6)L(q^{12})+32D(q^6)L(q^{12})$ in (45). So by (1), (2), (6), Proposition 3.2 (j), and (38) we have

$$
\begin{aligned}
&L(q)C(q)\\
&= \left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\sum_{m=1}^{\infty}c(m)q^m\\
&= \sum_{N=1}^{\infty}\left\{c(N) - 24\sum_{m=1}^{N-1}c(m)\sigma_1(N-m)\right\}q^N\\
&= \frac{7}{1796600}M^3(q) - \frac{7}{1796600}M^3(q^2) + \frac{5}{2263716}N^2(q)\\
&\quad - \frac{5}{2263716}N^2(q^2) - \frac{693}{3455}\Delta(q) - \frac{26256}{3455}\Delta(q^2)\\
&\quad - \frac{16384}{5}\Delta(q^4) - \frac{131072}{5}F(q) + \frac{6}{5}\sum_{n=1}^{\infty}nc(n)q^n
\end{aligned}
\tag{46}
$$

also from (1), (6), Proposition 3.2 (k), and (38) we obtain

$$
\begin{aligned}
&L(q)D(q)\\
&= \left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\sum_{m=1}^{\infty}d(m)q^m\\
&= \sum_{N=1}^{\infty}\left\{d(N) - 24\sum_{m=1}^{N-1}d(m)\sigma_1(N-m)\right\}q^N\\
&= -\frac{7}{57491200}M^3(q) + \frac{7}{57491200}M^3(q^2) - \frac{5}{72438912}N^2(q)\\
&\quad + \frac{5}{72438912}N^2(q^2) + \frac{1}{55280}\Delta(q) - \frac{1883}{1382}\Delta(q^2) + \frac{512}{5}\Delta(q^4)\\
&\quad + \frac{4096}{5}F(q) + \frac{6}{5}\sum_{n=1}^{\infty}nd(n)q^n
\end{aligned}
\tag{47}
$$

similarly by (1), (2), (6), Proposition 3.2 (l), and (38) we note that

(48)

$$
\begin{aligned}
&L(q)C(q^2)+32L(q)D(q^2)\\
&=\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\sum_{m=1}^{\infty}c(m)q^{2m}\\
&\quad+32\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\sum_{m=1}^{\infty}d(m)q^{2m}\\
&=\sum_{N=1}^{\infty}\left\{c(\frac{N}{2})+32d(\frac{N}{2})-24\sum_{m<\frac{N}{2}}c(m)\sigma_1(N-2m)\right.\\
&\qquad\qquad\left.-32\cdot 24\sum_{m<\frac{N}{2}}d(m)\sigma_1(N-2m)\right\}q^N\\
&=\sum_{N=1}^{\infty}\left[c(\frac{N}{2})+32d(\frac{N}{2})-24\sum_{m<\frac{N}{2}}\{d(2m)-32d(m)\}\sigma_1(N-2m)\right.\\
&\qquad\qquad\left.-32\cdot 24\sum_{m<\frac{N}{2}}d(m)\sigma_1(N-2m)\right]q^N\\
&=\sum_{N=1}^{\infty}\left\{c(\frac{N}{2})+32d(\frac{N}{2})-24\sum_{m<\frac{N}{2}}d(2m)\sigma_1(N-2m)\right\}q^N\\
&=-\frac{21}{22996480}M^3(q)+\frac{21}{22996480}M^3(q^2)-\frac{25}{48292608}N^2(q)\\
&\quad+\frac{25}{48292608}N^2(q^2)+\frac{3}{22112}\Delta(q)-\frac{15463}{13820}\Delta(q^2)\\
&\quad+\frac{2048}{5}\Delta(q^4)+6144F(q)+\frac{384}{5}\sum_{n=1}^{\infty}nd(n)q^{2n}+\frac{12}{5}\sum_{n=1}^{\infty}nc(n)q^{2n}.
\end{aligned}
$$

Furthermore owing to Proposition 2.1 (k) and Lemma 3.3 (c) we observe that

$$
\begin{aligned}
&L(q)L(q^2)M^2(q^2) = L(q)M^2(q^2)\cdot L(q^2)\\
&= \left(2L(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q) - \frac{342}{341}M(q^2)N(q^2) - \frac{720}{31}C(q)\right.\\
&\qquad \left. - \frac{23040}{31}D(q)\right) L(q^2)\\
&= 2L^2(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q)L(q^2) - \frac{342}{341}M(q^2)N(q^2)L(q^2)\\
&\qquad - \frac{720}{31}\left\{C(q)L(q^2) + 32D(q)L(q^2)\right\}
\end{aligned}
$$

and so

$$
\begin{aligned}
&C(q)L(q^2) + 32D(q)L(q^2)\\
&= \frac{31}{720}\left(2L^2(q^2)M^2(q^2) + \frac{1}{341}M(q)N(q)L(q^2)\right.\\
&\qquad \left. - \frac{342}{341}M(q^2)N(q^2)L(q^2) - L(q)L(q^2)M^2(q^2)\right)\\
&= \frac{3481}{11940480}M^3(q) + \frac{16067}{1492560}M^3(q^2) - \frac{3481}{11940480}N^2(q)\\
&\qquad - \frac{16067}{1492560}N^2(q^2) - \frac{717}{6910}\Delta(q) - \frac{42156}{3455}\Delta(q^2)\\
&\qquad + \frac{96}{5}\sum_{n=1}^{\infty} nd(n)q^n + \frac{3}{5}\sum_{n=1}^{\infty} nc(n)q^n,
\end{aligned}
\tag{49}
$$

where we use Proposition 2.1 (g) for the last line. Therefore applying (46), (47), (48), and (49) into (45) we obtain

$$
\begin{aligned}
&4928C(q^3)L(q^{12}) - 1024D(q^3)L(q^{12})\\
&= 31\cdot 3\Bigg[(1+2p)^6 k^{12} - \frac{1}{1023\cdot 3}M(q^3)N(q^3)L(q^3)\\
&\quad + \frac{4}{1023\cdot 3}M(q^3)N(q^3)L(q^{12}) + \frac{1024}{1023\cdot 3}M(q^{12})N(q^{12})L(q^3)\\
&\quad - \frac{1024\cdot 4}{1023\cdot 3}M(q^{12})N(q^{12})L(q^{12}) - \frac{256}{31\cdot 3}D(q^3)L(q^3)\\
&\quad + \frac{1232}{31\cdot 3}C(q^3)L(q^3) + \frac{39424}{31\cdot 3}\left\{C(q^6)L(q^3) + 32D(q^6)L(q^3)\right\}\\
&\quad - \frac{39424\cdot 4}{31\cdot 3}\left\{C(q^6)L(q^{12}) + 32D(q^6)L(q^{12})\right\}\Bigg]\\
&= -\frac{17045}{35932}M^3(q^3) - \frac{75861296}{1212705}M^3(q^6) - \frac{327700352}{93285}M^3(q^{12})\\
&\quad + \frac{963275}{2263716}N^2(q^3) + \frac{531443672}{8488935}N^2(q^6) + \frac{327700352}{93285}N^2(q^{12})\\
&\quad + \frac{14580368}{3455}\Delta(q^3) + \frac{582466816}{3455}\Delta(q^6) + \frac{111543574528}{3455}\Delta(q^{12})\\
&\quad + 209715200F(q^3) - \frac{1536}{5}\sum_{n=1}^{\infty} nd(n)q^{3n} + \frac{7392}{5}\sum_{n=1}^{\infty} nc(n)q^{3n}
\end{aligned}
$$

then to deduce $4928C(q)L(q^4) - 1024D(q)L(q^4)$ we reduce $q^{3n}$ to $q^n$, that is, we replace $n$ with $3n$ :

$$
\begin{aligned}
&4928C(q)L(q^4) - 1024D(q)L(q^4)\\
&= -\frac{17045}{35932}M^3(q) - \frac{75861296}{1212705}M^3(q^2) - \frac{327700352}{93285}M^3(q^4)\\
&\quad + \frac{963275}{2263716}N^2(q) + \frac{531443672}{8488935}N^2(q^2) + \frac{327700352}{93285}N^2(q^4)\\
&\quad + \frac{14580368}{3455}\Delta(q) + \frac{582466816}{3455}\Delta(q^2) + \frac{111543574528}{3455}\Delta(q^4)\\
&\quad + 209715200F(q) - \frac{1536}{5}\sum_{n=1}^{\infty} nd(n)q^{n} + \frac{7392}{5}\sum_{n=1}^{\infty} nc(n)q^{n}\\
&:= \beta(q).
\end{aligned} \tag{50}
$$

Finally solving two equations (44) and (50) we conclude that

$$\begin{aligned}
&L(q^4)C(q)\\
&= \frac{1}{4960}\left(32\alpha(q)+\beta(q)\right)\\
&= -\frac{17189}{185077440}M^3(q) - \frac{11524411}{925387200}M^3(q^2) - \frac{3283972}{4819725}M^3(q^4)\\
&\quad + \frac{67169}{809713800}N^2(q) + \frac{4036757}{323885520}N^2(q^2) + \frac{3283972}{4819725}N^2(q^4)\\
&\quad + \frac{1821829}{2142100}\Delta(q) + \frac{725740}{21421}\Delta(q^2) + \frac{3482086144}{535525}\Delta(q^4)\\
&\quad + \frac{212992}{5}F(q) + \frac{3}{10}\sum_{n=1}^{\infty} nc(n)q^n.
\end{aligned} \tag{51}$$

(b) From (44) and (51) we can know that

$$\begin{aligned}
&L(q^4)D(q)\\
&= \frac{1}{32}\big(\alpha(q) - L(q^4)C(q)\big)\\
&= \frac{2508569}{153984430080}M^3(q) + \frac{222565469}{192480537600}M^3(q^2) + \frac{17525879}{115673400}M^3(q^4)\\
&\quad - \frac{88056967}{5389455052800}N^2(q) - \frac{77894701}{67368188160}N^2(q^2)\\
&\quad - \frac{17525879}{115673400}N^2(q^4) - \frac{483241}{17136800}\Delta(q) - \frac{170154}{107105}\Delta(q^2)\\
&\quad - \frac{126497592}{535525}\Delta(q^4) + \frac{1024}{5}F(q) + \frac{3}{10}\sum_{n=1}^{\infty} nd(n)q^n.
\end{aligned}$$

□

**Corollary 3.4.** *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$\sum_{m<\frac{n}{4}} \sigma_1(m)c(n-4m)$$
$$= \begin{cases} \dfrac{1}{240}\left\{64\tau(\dfrac{n}{2}) + 1024\tau(\dfrac{n}{4}) - (3n-10)\,c(n)\right\}, & \textit{for even } n, \\ \dfrac{1}{331680}\left\{13\sigma_{11}(n) - 9687\tau(n) - 588709888f(n)\right. & \\ \qquad \left. -1382\,(3n-10)\,c(n)\right\}, & \textit{for odd } n, \end{cases}$$

(b)

$$\sum_{m<\frac{n}{4}} \sigma_1(m)d(n-4m)$$
$$= \begin{cases} -\dfrac{1}{240}\left\{4\tau(\dfrac{n}{2}) + 64\tau(\dfrac{n}{4}) + (3n-10)\,d(n)\right\}, & \textit{for even } n, \\ \dfrac{1}{5306880}\left\{\sigma_{11}(n) - \tau(n) - 45285376f(n)\right. & \\ \qquad \left. -22112\,(3n-10)\,d(n)\right\}, & \textit{for odd } n. \end{cases}$$

*Proof.* (a) By (1) and (6), we have

$$\begin{aligned} & 24\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{4}} \sigma_1(m)c(N-4m)\right)q^N \\ &= 24\left(\sum_{n=1}^{\infty} c(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right) \\ &= C(q)\left(1-L(q^4)\right) \\ &= C(q) - C(q)L(q^4). \end{aligned}$$

So we use Theorem 1.3 (a) and obtain

$$\sum_{m<\frac{n}{4}} \sigma_1(m)c(n-4m)$$

(52)
$$= \frac{1}{331680}\Big\{13\sigma_{11}(n) - 13\sigma_{11}(\frac{n}{2}) - 9687\tau(n) - 170664\tau(\frac{n}{2}) -73588736\tau(\frac{n}{4}) - 588709888f(n) - 1382\,(3n-10)\,c(n)\Big\}.$$

If $n$ is odd then it is obvious that

$$\sum_{m<\frac{n}{4}} \sigma_1(m)c(n-4m) = \frac{1}{331680}\{13\sigma_{11}(n) - 9687\tau(n) - 588709888f(n) -1382\,(3n-10)\,c(n)\}$$

but if $n$ is even then by Proposition 1.1 (b) and (c), Eq. (52) becomes

$$\sum_{m<\frac{n}{4}} \sigma_1(m)c(n-4m) = \frac{1}{240}\Big\{64\tau(\frac{n}{2}) + 1024\tau(\frac{n}{4}) - (3n-10)\,c(n)\Big\}.$$

(b) Proof is similar to Corollary 3.4 (a) except we use Theorem 1.3 (b) and have

$$\sum_{m<\frac{n}{4}} \sigma_1(m)d(n-4m) = \frac{1}{5306880}\Big\{\sigma_{11}(n) - \sigma_{11}(\frac{n}{2}) - \tau(n) - 90520\tau(\frac{n}{2}) - 5660672\tau(\frac{n}{4}) - 45285376f(n) - 22112\,(3n-10)\,d(n)\Big\}.$$

□

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## 4. Appendix

The first twenty values of $\tau(n)$ are given in the Table 1,

| $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ | $n$ | $\tau(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | −6048 | 11 | 534612 | 16 | 987136 |
| 2 | −24 | 7 | −16744 | 12 | −370944 | 17 | −6905934 |
| 3 | 252 | 8 | 84480 | 13 | −577738 | 18 | 2727432 |
| 4 | −1472 | 9 | −113643 | 14 | 401856 | 19 | 10661420 |
| 5 | 4830 | 10 | −115920 | 15 | 1217160 | 20 | −7109760 |

TABLE 1. $\tau(n)$ for $n$ $(1 \leq n \leq 20)$

similarly the first twenty values of $a(n)$, $b(n)$, $c(n)$, $d(n)$, $f(n)$, $c_{1,6}(n)$, and $c_{1,12}(n)$ are listed in the following tables.

| $n$ | $a(n)$ | $n$ | $a(n)$ | $n$ | $a(n)$ | $n$ | $a(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | 0 | 11 | 540 | 16 | 0 |
| 2 | 0 | 7 | −88 | 12 | 0 | 17 | 594 |
| 3 | −12 | 8 | 0 | 13 | −418 | 18 | 0 |
| 4 | 0 | 9 | −99 | 14 | 0 | 19 | 836 |
| 5 | 54 | 10 | 0 | 15 | −648 | 20 | 0 |

TABLE 2. $a(n)$ for $n$ $(1 \leq n \leq 20)$

| $n$ | $b(n)$ | $n$ | $b(n)$ | $n$ | $b(n)$ | $n$ | $b(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | −96 | 11 | 1092 | 16 | 4096 |
| 2 | −8 | 7 | 1016 | 12 | 768 | 17 | 14706 |
| 3 | 12 | 8 | −512 | 13 | 1382 | 18 | 16344 |
| 4 | 64 | 9 | −2043 | 14 | −8128 | 19 | −39940 |
| 5 | −210 | 10 | 1680 | 15 | −2520 | 20 | −13440 |

TABLE 3. $b(n)$ for $n$ $(1 \leq n \leq 20)$

| $n$ | $c(n)$ | $n$ | $c(n)$ | $n$ | $c(n)$ | $n$ | $c(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | 2496 | 11 | −38996 | 16 | −65536 |
| 2 | −16 | 7 | −4536 | 12 | 39936 | 17 | 311442 |
| 3 | 100 | 8 | −4096 | 13 | 37806 | 18 | −74448 |
| 4 | −256 | 9 | 23085 | 14 | 15232 | 19 | 128244 |
| 5 | −154 | 10 | −13920 | 15 | −146472 | 20 | −222720 |

TABLE 4. $c(n)$ for $n$ $(1 \leq n \leq 20)$

| $n$ | $d(n)$ | $n$ | $d(n)$ | $n$ | $d(n)$ | $n$ | $d(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 6 | $-156$ | 11 | $-536$ | 16 | 4096 |
| 2 | 1 | 7 | 112 | 12 | $-2496$ | 17 | $-17472$ |
| 3 | $-8$ | 8 | 256 | 13 | 4384 | 18 | 4653 |
| 4 | 16 | 9 | $-576$ | 14 | $-952$ | 19 | 5848 |
| 5 | 32 | 10 | 870 | 15 | 336 | 20 | 13920 |

TABLE 5. $d(n)$ for $n$ $(1 \le n \le 20)$

| $n$ | $f(n)$ | $n$ | $f(n)$ | $n$ | $f(n)$ | $n$ | $f(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 6 | 8 | 11 | 6296 | 16 | 388608 |
| 2 | 0 | 7 | 44 | 12 | 16384 | 17 | 756822 |
| 3 | 0 | 8 | 192 | 13 | 39569 | 18 | 1419200 |
| 4 | 0 | 9 | 694 | 14 | 89424 | 19 | 2572328 |
| 5 | 1 | 10 | 2208 | 15 | 191028 | 20 | 4521984 |

TABLE 6. $f(n)$ for $n$ $(1 \le n \le 20)$

| $n$ | $c_{1,6}(n)$ | $n$ | $c_{1,6}(n)$ | $n$ | $c_{1,6}(n)$ | $n$ | $c_{1,6}(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | 6 | 11 | 12 | 16 | 16 |
| 2 | $-2$ | 7 | $-16$ | 12 | $-12$ | 17 | $-126$ |
| 3 | $-3$ | 8 | $-8$ | 13 | 38 | 18 | $-18$ |
| 4 | 4 | 9 | 9 | 14 | 32 | 19 | 20 |
| 5 | 6 | 10 | $-12$ | 15 | $-18$ | 20 | 24 |

TABLE 7. $c_{1,6}(n)$ for $n$ $(1 \le n \le 20)$

| $n$ | $c_{1,12}(n)$ | $n$ | $c_{1,12}(n)$ | $n$ | $c_{1,12}(n)$ | $n$ | $c_{1,12}(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | $-36/11$ | 11 | $252/11$ | 16 | $-96/11$ |
| 2 | $12/11$ | 7 | $-56/11$ | 12 | $72/11$ | 17 | $-666/11$ |
| 3 | $-3/11$ | 8 | $48/11$ | 13 | $178/11$ | 18 | $108/11$ |
| 4 | $-24/11$ | 9 | 9 | 14 | $-192/11$ | 19 | $-380/11$ |
| 5 | $-54/11$ | 10 | $72/11$ | 15 | $-378/11$ | 20 | $-144/11$ |

TABLE 8. $c_{1,12}(n)$ for $n$ $(1 \le n \le 20)$