

ON SOME SERIES IDENTITIES

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Abstract. In this paper, using a transformation formula for a certain series which comes from the generalized non-analytic Eisenstein series, we obtained some infinite series identities which contain Ramanujan’s formula and author’s previous results.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta function and $B_n(x)$ denote the n -th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi).$$

The n -th Bernoulli number B_n , $n \geq 0$, is defined by $B_n = B_n(0)$. For a real number x , $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. One of Ramanujan’s remarkable theorems is the arithmetic series relation with $\zeta(2n + 1)$, which is given by the following identity.

Ramanujan’s Formula ([8]); Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Then, for any positive integer n ,

$$\begin{aligned} & \alpha^{-n} \left(\frac{1}{2} \zeta(2n + 1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\alpha k} - 1} \right) \\ &= (-\beta)^{-n} \left(\frac{1}{2} \zeta(2n + 1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\beta k} - 1} \right) \\ (1.1) \quad & - 2^{2n} \sum_{k=0}^{n+1} \frac{B_{2k} B_{2n+2-2k}}{(2k)! (2n+2-2k)!} \alpha^{n+1-k} (-\beta)^k. \end{aligned}$$

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An interesting history of (1.1) and many authors who have given various proofs of (1.1) are written very well in [2]. Using modular transformation formulas given by B. C. Berndt [3], the author [5] derived a more generalized series relation than (1.1).

Theorem 1.1. ([5]). *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Let c denote a positive integer. Then, for any integer n ,*

$$\alpha^{-n} \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2k(\alpha-\pi i)/c} - 1} = (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2k(\beta+\pi i)/c} - 1} - 2^{2n} \sum_{j=1}^c \sum_{k=0}^{2n+2} \frac{B_k(j/c) \bar{B}_{2n+2-k}(j/c)}{k!(2n+2-k)!} \alpha^{n-k+1} (-\pi i)^k + I_0(n),$$

where

$$I_0(n) := \begin{cases} \frac{1}{2} ((-\beta)^{-n} - \alpha^{-n}) \zeta(1+2n), & \text{if } n \neq 0, \\ -\frac{1}{4}(\log \beta - \log \alpha) + \frac{1}{4}\pi i, & \text{if } n = 0. \end{cases}$$

Note that Theorem 1.1 implies (1.1) when $c = 1$ and $n > 0$. Replacing c by $2c$ in Theorem 1.1, we also obtain the following corollary.

Corollary 1.2. *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Let c denote a positive integer. Then, for any integer n ,*

$$\alpha^{-n} \sum_{k=0}^{\infty} \frac{(2k+1)^{-2n-1}}{e^{(2k+1)(\alpha-\pi i)/c} - 1} = (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{(2k+1)^{-2n-1}}{e^{(2k+1)(\beta+\pi i)/c} - 1} - 2^{2n} \sum_{j=1}^{2c} \sum_{k=0}^{2n+2} \frac{B_k(j/(2c)) \bar{B}_{2n+2-k}(j/(2c))}{k!(2n+2-k)!} \alpha^{n-k+1} (-\pi i)^k + 2^{-1} \sum_{j=1}^c \sum_{k=0}^{2n+2} \frac{B_k(j/c) \bar{B}_{2n+2-k}(j/c)}{k!(2n+2-k)!} \alpha^{n-k+1} (-\pi i)^k + (1 - 2^{-2n-1})I_0(n).$$

Furthermore, we see the following corollary putting $c = 1$ in Corollary 1.2.

Corollary 1.3. *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Then, for any integer n ,*

$$\alpha^{-n} \sum_{k=0}^{\infty} \frac{(2k+1)^{-2n-1}}{e^{(2k+1)\alpha} + 1} = (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{(2k+1)^{-2n-1}}{e^{(2k+1)\beta} + 1} + I_1(n) + \sum_{k=0}^{n+1} (2^{2n+1} - 2^{2k-1})(1 - 2^{-2k}) \frac{B_{2k} B_{2n+2-2k}}{(2k)!(2n+2-2k)!} \alpha^{n-k+1} (-\beta)^k,$$

where

$$I_1(n) := \begin{cases} (2^{-2n-2} - 2^{-1})((-\beta)^{-n} - \alpha^{-n})\zeta(2n + 1), & \text{if } n \neq 0, \\ \frac{1}{8}(\log \beta - \log \alpha), & \text{if } n = 0. \end{cases}$$

Corollary 1.3 was first established by Malurkar [7] and later by Bruce C. Berndt [3].

Recently, Komori, Matsumoto and Tsumura [4] derived (1.1) from functional equations for Barnes zeta-functions. They also obtained generalized formulas of certain series involving hyperbolic functions and the Bernoulli polynomials. In fact, we see (1.1), Theorem 1.1 and Corollary 1.3 as series relations involving hyperbolic functions and the Bernoulli polynomials. For example, by using the equality

$$\frac{e^x}{e^x + 1} = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{x}{2}\right),$$

Corollary 1.3 can be changed to the following theorem.

Corollary 1.4. ([3]). *Let α and β be positive real numbers with $\alpha\beta = \pi^2$. Then, for any positive integer n ,*

$$\begin{aligned} \alpha^{-n} \sum_{k=0}^{\infty} \frac{\tanh((2k + 1)\alpha/2)}{(2k + 1)^{2n+1}} &= (-\beta)^{-n} \sum_{k=0}^{\infty} \frac{\tanh((2k + 1)\beta/2)}{(2k + 1)^{2n+1}} \\ &+ \sum_{k=0}^{n+1} (2^{2n+1} - 2^{2k-1})(2^{-2k+1} - 2) \frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n + 2 - 2k)!} \alpha^{n-k+1} (-\beta)^k. \end{aligned}$$

In [6], the author computed transformation formulas for certain series with confluent hypergeometric functions of the second kind, which come from a transformation formula for generalized non-analytic Eisenstein series. In this paper, using those transformation formulas, we obtain some identities of series with hyperbolic functions and Eulerian numbers, which contain the equations in Theorem 1.1 and Corollary 1.2. It is interesting to note that some moderately good symmetric(for α, β) identities are derived in a class of series.

2. Notation

In this paper, the branch of the argument for a complex w is defined by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi iw}$ and λ denote the characteristic function of the integers. For real x, α and $\text{Re}(t) > 1$, let

$$\psi(x, \alpha, t) = \sum_{n+\alpha>0} \frac{e(nx)}{(n + \alpha)^t}.$$

Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n,$$

where $(x)_n$ denotes the rising factorial defined by

$$(x)_n = x(x+1)\cdots(x+n-1) \text{ for } n > 0, (x)_0 = 1.$$

Let $\Gamma(s)$ denote the Gamma function. We see that ${}_2F_1(\alpha, \beta; \gamma; z)/\Gamma(\gamma)$ can be analytically continued to all $\alpha, \beta, \gamma \in \mathbb{C}$ and all $z \in \mathbb{C} \setminus [1, \infty)$, p. 99 in [10]. The confluent hypergeometric function of the second kind $U(\alpha, \beta, z)$ is defined to be

$$U(\alpha; \beta; z) = \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta} {}_1F_1(1+\alpha-\beta; 2-\beta; z) + \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)} {}_1F_1(\alpha; \beta; z),$$

where

$${}_1F_1(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n.$$

The function $U(\alpha; \beta; z)$ can be analytically continued to all values of α, β and z real or complex, even when β is zero or a negative integer [9].

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, the upper half-plane. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors. For $\tau \in \mathbb{H}$ and arbitrary $s_1, s_2 \in \mathbb{C}$, define

$$\mathcal{A}(\tau, s_1, s_2; r, h) = \sum_{m+r_1>0} \sum_{n-h_2>0} \frac{e(mh_1 + ((m+r_1)\tau + r_2)(n-h_2))}{(n-h_2)^{1-s}} \times U(s_2; s; 4\pi(m+r_1)(n-h_2)\text{Im}(\tau))$$

and

$$\bar{\mathcal{A}}(\tau, s_1, s_2; r, h) = \sum_{m+r_1>0} \sum_{n+h_2>0} \frac{e(mh_1 - ((m+r_1)\bar{\tau} + r_2)(n+h_2))}{(n+h_2)^{1-s}} \times U(s_1; s; 4\pi(m+r_1)(n+h_2)\text{Im}(\tau)),$$

where $s = s_1 + s_2$. The functions $\mathcal{A}(\tau, s_1, s_2; r, h)$ and $\bar{\mathcal{A}}(\tau, s_1, s_2; r, h)$ are well-defined for all $s_1, s_2 \in \mathbb{C}$. Let

$$\mathcal{H}(\tau, s_1, s_2; r, h) = \mathcal{A}(\tau, s_1, s_2; r, h) + e^{\pi is} \mathcal{A}(\tau, s_1, s_2; -r, -h)$$

and

$$\bar{\mathcal{H}}(\tau, s_1, s_2; r, h) = \bar{\mathcal{A}}(\tau, s_1, s_2; r, h) + e^{\pi is} \bar{\mathcal{A}}(\tau, s_1, s_2; -r, -h).$$

Let

$$\mathbf{H}(\tau, \bar{\tau}, s_1, s_2; r, h) = \frac{1}{\Gamma(s_1)} \mathcal{H}(\tau, s_1, s_2; r, h) + \frac{1}{\Gamma(s_2)} \bar{\mathcal{H}}(\tau, s_1, s_2; r, h).$$

For real x, α and $t \in \mathbb{C}$, let

$$\begin{aligned} \Psi(x, \alpha, t) &= \psi(x, \alpha, t) + e^{\pi i t} \psi(-x, -\alpha, t), \\ \Psi_{-1}(x, \alpha, t) &:= \psi(x, \alpha, t - 1) + e^{\pi i t} \psi(-x, -\alpha, t - 1). \end{aligned}$$

In the sequel, $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ always denotes a modular transformation with $c > 0$ for every complex τ . Let R and H be real vectors defined by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

We now state the principal theorem which shall be applied to obtain series relations.

Theorem 2.1. ([6]). *Let $Q = \{\tau \in \mathbb{H} \mid \operatorname{Re} \tau > -d/c\}$ and $\varrho = c\{R_2\} - d\{R_1\}$. Then for $\tau \in Q$ and $s_1, s_2 \in \mathbb{C}$ with $s := s_1 + s_2$, not integers ≤ 1 ,*

$$\begin{aligned} z^{-s_1} \bar{z}^{-s_2} \mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) &= \mathbf{H}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &+ \lambda(R_1)(2\pi i)^{-s} e^{-\pi i(2R_1 H_1 + s_2)} \Psi(-H_2, -R_2, s) \\ &- \lambda(r_1)(2\pi i)^{-s} e^{-\pi i(2r_1 h_1 - s_1)} z^{-s_1} \bar{z}^{-s_2} \Psi(h_2, r_2, s) \\ &+ \lambda(H_2)(4\pi \operatorname{Im}(\tau))^{1-s} \frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)} \Psi_{-1}(H_1, R_1, s) \\ &- \lambda(h_2)(4\pi \operatorname{Im}(\tau))^{1-s} \frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)} z^{s_2-1} \bar{z}^{s_1-1} \Psi_{-1}(h_1, r_1, s) \\ &+ \frac{(2\pi i)^{-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H), \end{aligned}$$

where $z = c\tau + d$ and

$$\begin{aligned} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) &:= e(-H_1([R_1] - c) - H_2([R_2] + 1 + -d)) \\ &\times \sum_{j=1}^c e(-H_1 j - H_2[(jd + \varrho)/c]) \int_0^1 v^{s_1-1} (1-v)^{s_2-1} \\ &\times \int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)}} \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e^{-2\pi i H_2}} dudv, \end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)})(e^u - e^{-2\pi i H_2})$$

lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Remark 2.2. If s_1 is an integer, then Theorem 2.1 is true for all $s \in \mathbb{C}$. In Theorem 2.1, if s is an integer, then integrals may be computed using the residue theorem.

Using the analytic continuation of ${}_2F_1$ -hypergeometric function, we observe that, for $\tau \in \mathbb{Q}$,

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2, ; R, H)$$

can be defined for every integer s . Note that after evaluation of the transformation formula in Theorem 2.1 for integers s_1 and s_2 with $s_1 > 0$ and $s_2 \leq 0$, or $s \geq 3$, the resultant formula will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

3. Infinite series identities

In this section, we obtain some infinite series identities from Theorem 2.1. We use the Eulerian numbers $E(n, j)$, the number of permutations of numbers from 1 to n such that exactly j numbers are greater than the previous elements. For any integer n , the polylogarithm function $\text{Li}_n(z)$ is defined by

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

where $z \in \mathbb{C}$ and $|z| < 1$. Specially, for any positive integer n , we see that

$$\text{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} E(n, j) z^{n-j}.$$

We need the following lemma for values of $U(\alpha; \beta; z)$ when α and β are integers.

Lemma 3.1. *Let u, v and t be integers with $u + v = t$. Let*

$$U(v; t; z) = \lim_{s \rightarrow t} U(s - u; s; z).$$

Then, for $u \geq 1, v \geq 1$ and $t \geq 2$, we have that

$$U(v; t; z) = \frac{1}{(v-1)!} \sum_{k=0}^{u-1} \binom{u-1}{k} (t-k-2)! z^{k-t+1}.$$

Proof. Since $1 - u \leq 0$,

$$\frac{1}{\Gamma(1 - u)} = 0$$

and

$$(1 - u)_k = \begin{cases} 0, & k > u - 1 \\ (-1)^k \frac{(u-1)!}{(u-1-k)!}, & k \leq u - 1. \end{cases}$$

Then

$$\begin{aligned} & \lim_{s \rightarrow t} U(s - u; s; z) \\ &= \lim_{s \rightarrow t} \left(\frac{\Gamma(1 - s)}{\Gamma(1 - u)} \sum_{k=0}^{\infty} \frac{(s - u)_k}{(s)_k} \frac{z^k}{k!} + \frac{\Gamma(s - 1)}{\Gamma(s - u)} z^{1-s} \sum_{k=0}^{\infty} \frac{(1 - u)_k}{(2 - s)_k} \frac{z^k}{k!} \right) \\ &= \lim_{s \rightarrow t} \frac{\Gamma(s - 1)}{\Gamma(s - u)} z^{1-s} \sum_{k=0}^{u-1} (-1)^k \binom{u-1}{k} \frac{z^k}{(2 - s)_k} \\ &= \frac{\Gamma(t - 1)}{\Gamma(v)} z^{1-t} \sum_{k=0}^{u-1} \binom{u-1}{k} \frac{(t - 2 - k)!}{(t - 2)!} z^k \\ &= \frac{1}{(v - 1)!} \sum_{k=0}^{u-1} \binom{u-1}{k} (t - k - 2)! z^{k-t+1}. \end{aligned}$$

□

Let

$$\hat{k} = \left\lfloor \frac{k - 1}{2} \right\rfloor \text{ and } \tilde{k} = \left\lceil \frac{k - 1}{2} \right\rceil$$

for any integer k .

Theorem 3.2. Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $A \geq 0, B \geq 0$ and $N \geq 1$ with $A + B = 2N$,

$$\begin{aligned} & (-1)^B \alpha^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N - k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})(\alpha - i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\alpha - i\pi)/c)} \\ &+ (-1)^B \alpha^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N - k)!}{(2\alpha/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})(\alpha + i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\alpha + i\pi)/c)} \\ &+ (-1)^B (2N)! \alpha^{-N} \left(\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\alpha - i\pi)/c} - 1} + \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\alpha + i\pi)/c} - 1} + \zeta(2N + 1) \right) \\ &+ (-1)^{B+1} 2^{2N} A! B! c^{-2N-1} \alpha^{-N-1} \zeta(2N + 2) \\ &= (-\beta)^{-N} \sum_{k=1}^A \binom{A}{k} \frac{(2N - k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})(\beta + i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\beta + i\pi)/c)} \end{aligned}$$

$$\begin{aligned}
 &+(-\beta)^{-N} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\beta/c)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j+\hat{k})(\beta-i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\beta-i\pi)/c)} \\
 &+(2N)!(-\beta)^{-N} \left(\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\beta-i\pi)/c}-1} + \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\beta+i\pi)/c}-1} + \zeta(2N+1) \right) \\
 &+2^{2N} A!B!c^{-2N-1}(-\beta)^{-N-1}\zeta(2N+2),
 \end{aligned}$$

where the dash ' means that if k is odd, then the value of the term $j = 0$ is multiplied by $1/2$.

Proof. Let $s_1 = A + 1$ for an integer $A \geq 0$. Let $r = (0, 0)$, $h = (0, 0)$ and

$$V\tau = \frac{\tau - 1}{c\tau - c + 1} \text{ and } c\tau - c + 1 = \frac{\pi i}{\alpha}.$$

Then $R = H = (0, 0)$ and

$$V\tau = \frac{1}{c} \left(1 + \frac{\alpha i}{\pi} \right).$$

Thus we have

$$\mathcal{A}(V\tau, s_1, s_2; r, h) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-2mn(\alpha-\pi i)/c}}{n^{1-s}} U \left(s-1-A; s; \frac{4mn\alpha}{c} \right).$$

Put $s = 2N + 2$ for an integer $N \geq 1$. Applying Lemma 3.1,

$$U \left(s-1-A; s; \frac{4mn\alpha}{c} \right) = \frac{1}{B!} \sum_{k=0}^A \binom{A}{k} (2N-k)! \left(\frac{4mn\alpha}{c} \right)^{k-2N-1}.$$

Hence we obtain that

$$\mathcal{A}(V\tau, s_1, s_2; r, h) = \frac{1}{B!} \sum_{k=0}^A \binom{A}{k} \frac{(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{m=1}^{\infty} \frac{\text{Li}_{-k}(e^{-2m(\alpha-\pi i)/c})}{m^{2N-k+1}}.$$

For brevity's sake, let $w = (\alpha - \pi i)/c$. Then for $k > 0$,

$$\begin{aligned}
 \text{Li}_{-k}(e^{-2m(\alpha-\pi i)/c}) &= \frac{1}{(1-e^{-2mw})^{k+1}} \sum_{j=0}^{k-1} E(k, j) e^{-2mw(k-j)} \\
 &= \left(\frac{e^{mw}}{e^{mw}-e^{-mw}} \right)^{k+1} \sum_{j=0}^{k-1} E(k, j) e^{-2mw(k-j)} \\
 &= \frac{2^{-k-1}}{\sinh^{k+1}(mw)} \sum_{j=0}^{k-1} E(k, j) e^{2mw(j-(k-1)/2)}.
 \end{aligned}$$

For k even, applying $E(k, j) = E(k, k - j - 1)$, we see that

$$\begin{aligned}
 & \sum_{j=0}^{k-1} E(k, j) e^{2mw(j-(k-1)/2)} \\
 &= \sum_{j=0}^{\hat{k}} E(k, j) \left(e^{-mw(k-2j-1)} + e^{mw(k-2j-1)} \right) \\
 &= 2 \sum_{j=0}^{\hat{k}} E(k, j) \cosh(mw(k-2j-1)) \\
 &= 2 \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \cosh(2mw(j+\tilde{k})).
 \end{aligned}$$

For k odd, applying $E(k, j) = E(k, k - j - 1)$,

$$\begin{aligned}
 & \sum_{j=0}^{k-1} E(k, j) e^{2mw(j-(k-1)/2)} \\
 &= \sum_{j=0}^{\hat{k}-1} E(k, j) \left(e^{-mw(k-2j-1)} + e^{mw(k-2j-1)} \right) + E(k, \hat{k}) \\
 &= 2 \sum_{j=0}^{\hat{k}-1} E(k, j) \cosh(mw(k-2j-1)) + E(k, \hat{k}) \\
 &= 2 \sum_{j=1}^{\hat{k}} E(k, \hat{k}-j) \cosh(2mw(j+\tilde{k})) + E(k, \hat{k}) \\
 &= 2 \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \cosh(2mw(j+\tilde{k})).
 \end{aligned}$$

Thus, for $k > 0$, we have that

$$\text{Li}_{-k}(e^{-2mw}) = \frac{2^{-k}}{\sinh^{k+1}(mw)} \sum_{j=0}^{\hat{k}'} E(k, \hat{k}-j) \cosh(2mw(j+\tilde{k}))$$

and

$$\text{Li}_0(e^{-2mw}) = \sum_{\ell=1}^{\infty} e^{-2m\ell w} = \frac{1}{e^{2mw} - 1}.$$

Therefore we obtain that

$$\begin{aligned} \mathcal{A}(V\tau, s_1, s_2; r, h) &= \frac{1}{B!} \frac{(2N)!}{(4\alpha/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2mw} - 1} \\ &+ \frac{1}{B!} \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{(2\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2mw(j + \tilde{k}))}{m^{2N-k+1} \sinh^{k+1}(mw)}. \end{aligned}$$

Similarly we see that

$$\begin{aligned} \bar{\mathcal{A}}(V\tau, s_1, s_2; r, h) &= \frac{1}{A!} \frac{(2N)!}{(4\alpha/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m\bar{w}} - 1} \\ &+ \frac{1}{A!} \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{(2\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m\bar{w}(j + \tilde{k}))}{m^{2N-k+1} \sinh^{k+1}(m\bar{w})}. \end{aligned}$$

Furthermore, by the same way, we can compute $\bar{\mathcal{A}}(V\tau, s_1, s_2; r, h)$, $\mathcal{A}(\tau, s_1, s_2; R, H)$ and $\bar{\mathcal{A}}(\bar{\tau}, s_1, s_2; R, H)$. It is easy to see that

$$\Psi(0, 0, 2N + 2) = 2\zeta(2N + 2), \quad \Psi_{-1}(0, 0, 2N + 2) = 2\zeta(2N + 1).$$

Finally we compute $\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H)$. By the residue theorem, using the same manner in [1], we find that

$$\begin{aligned} &\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))ju/c}}{e^{-(zv+\bar{z}(1-v))u} - 1} \frac{e^{\{j/c\}u}}{e^u - 1} dudv \\ (3.1) \quad &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k(j/c)\bar{B}_{-s+2-k}(j/c)}{k!(N+2-k)!} (-(z-\bar{z})v - \bar{z})^{k-1}. \end{aligned}$$

Since $s = 2N + 2 \geq 4$, the summation in (3.1) is 0. Thus we have

$$\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) = 0.$$

If we apply the above results to Theorem 2.1, then the proof is done. \square

Note that Theorem 3.2 contains Theorem 1.1 for $n > 3$. If B is even, then, applying Theorem 1.1, we may remove the part in parentheses on both sides of the identity in Theorem 3.2 and instead, put sums of the Bernoulli polynomials into the identity.

Corollary 3.3. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned} &\alpha^{-N} \sum_{k=1}^{2N} \frac{(2\alpha/c)^k}{k!} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})(\alpha - i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\alpha - i\pi)/c)} \\ &= (-\beta)^{-N} \sum_{k=1}^{2N} \frac{(2\beta/c)^k}{k!} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})(\beta + i\pi)/c)}{m^{2N-k+1} \sinh^{k+1}(m(\beta + i\pi)/c)} \end{aligned}$$

$$\begin{aligned}
 &+2^{2N+1} \sum_{j=1}^c \sum_{k=0}^{N+1} \frac{B_{2k}(j/c) \bar{B}_{2N+2-2k}(j/c)}{(2k)!(2N+2-2k)!} \alpha^{N-k+1} (-\beta)^k \\
 &+2^{2N} c^{-2N-1} (\alpha^{-N-1} + (-\beta)^{-N-1}) \zeta(2N+2).
 \end{aligned}$$

Proof. Let $B = 0$ in Theorem 3.2. Applying Theorem 1.1, we have that

$$\begin{aligned}
 &\alpha^{-N} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\alpha-\pi i)/c} - 1} - (-\beta)^{-N} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\beta+\pi i)/c} - 1} \\
 &+ \alpha^{-N} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\alpha+\pi i)/c} - 1} - (-\beta)^{-N} \sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m(\beta-\pi i)/c} - 1} \\
 &+ (\alpha^{-N} - (-\beta)^{-N}) \zeta(1+2N) \\
 &= -2^{2N} \sum_{j=1}^c \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \alpha^{N-k+1} (-\pi i)^k \\
 (3.2) \quad &+ (2i)^{2N} \sum_{j=1}^c \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \beta^{N-k+1} (-\pi i)^k.
 \end{aligned}$$

Revercing the order of summation over k , namely, replacing k by $2N+2-k$ and using $\alpha\beta = \pi^2$, we see that, for $j < c$,

$$\begin{aligned}
 &(-1)^N \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \beta^{N-k+1} (-\pi i)^k \\
 &= (-1)^N \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \beta^{-N+k-1} (-\pi i)^{2N+2-k} \\
 &= - \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \alpha^{N-k+1} (\pi i)^k.
 \end{aligned}$$

In case $j = c$, employing $B_n(1) = (-1)^n B_n$ and $B_n(0) = B_n$,

$$\begin{aligned}
 &(-1)^N \sum_{k=0}^{2N+2} \frac{B_k(c/c) \bar{B}_{2N+2-k}(c/c)}{k!(2N+2-k)!} \beta^{N-k+1} (-\pi i)^k \\
 &= (-1)^N \sum_{k=0}^{2N+2} \frac{(-1)^k B_k B_{2N+2-k}}{k!(2N+2-k)!} \beta^{-N+k-1} (-\pi i)^{2N+2-k} \\
 &= - \sum_{k=0}^{2N+2} \frac{(-1)^k B_k B_{2N+2-k}}{k!(2N+2-k)!} \alpha^{N-k+1} (\pi i)^k.
 \end{aligned}$$

We now obtain that

$$\begin{aligned}
 & (-1)^N \sum_{j=1}^c \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \beta^{N-k+1} (-\pi i)^k \\
 (3.3) \quad & = - \sum_{j=1}^c \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \alpha^{N-k+1} (\pi i)^k.
 \end{aligned}$$

Thus, it follows from (3.3) that

$$\begin{aligned}
 \text{RHS in (3.2)} & = - \sum_{j=1}^c \sum_{k=0}^{2N+2} \frac{B_k(j/c) \bar{B}_{2N+2-k}(j/c)}{k!(2N+2-k)!} \alpha^{N-k+1} ((-\pi i)^k + (\pi i)^k) \\
 & = -2^{2N+1} \sum_{j=1}^c \sum_{k=0}^{N+1} \frac{B_{2k}(j/c) \bar{B}_{2N+2-2k}(j/c)}{(2k)!(2N+2-2k)!} \alpha^{N-k+1} (-\beta)^k,
 \end{aligned}$$

where ‘RHS’ denote the right hand side. □

Corollary 3.4. *For any integer $N \geq 1$,*

$$\begin{aligned}
 & \sum_{k=1}^{4N-2} \frac{(2\pi)^k}{k!} \sum_{j=0}^{\hat{k}} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})\pi)}{m^{4N-k-1} \sinh^{k+1}(m\pi)} \\
 & = 2^{-1} (2\pi)^{4N-1} \sum_{k=0}^{2N} \frac{(-1)^k B_{2k} B_{4N-2k}}{(2k)!(4N-2k)!} + 2^{4N-2} \pi^{-1} \zeta(4N).
 \end{aligned}$$

Proof. Let $\alpha = \beta = \pi$ and $c = 1$ in Corollary 3.3. Replace N by $2N - 1$ and apply

$$\begin{aligned}
 \sinh^{k+1}(m(\pi \pm \pi i)) & = (-1)^{m(k+1)} \sinh(m\pi) \\
 & = \begin{cases} (-1)^m \sinh(m\pi), & k \text{ even} \\ \sinh(m\pi), & k \text{ odd,} \end{cases} \\
 \cosh(2m(j + \tilde{k})(\pi \pm \pi i)) & = e(\tilde{k}m) \cosh(2m\pi(j + \tilde{k})) \\
 & = \begin{cases} (-1)^m \cosh(2m\pi(j + \tilde{k})), & k \text{ even} \\ \cosh(2m\pi(j + \tilde{k})), & k \text{ odd.} \end{cases}
 \end{aligned}$$

□

If we put $N = 1$ in Corollary 3.4, then

$$\sum_{m=1}^{\infty} \frac{\operatorname{csch}^2(m\pi)}{m^2} + \pi \sum_{m=1}^{\infty} \frac{\operatorname{csch}^2(m\pi) \operatorname{coth}(m\pi)}{m} = \frac{1}{45} \pi^2.$$

Corollary 3.5. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned} & \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\alpha)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})\alpha)}{m^{2N-k+1} \sinh^{k+1}(m\alpha)} \\ & + (2N)! \alpha^{-N} \left(\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m\alpha} - 1} + \frac{1}{2} \zeta(2N + 1) \right) - 2^{2N-1} (N!)^2 \alpha^{-N-1} \zeta(2N + 2) \\ & = \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{(2\beta)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})\beta)}{m^{2N-k+1} \sinh^{k+1}(m\beta)} \\ & + (2N)! \beta^{-N} \left(\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m\beta} - 1} + \frac{1}{2} \zeta(2N + 1) \right) - 2^{2N-1} (N!)^2 \beta^{-N-1} \zeta(2N + 2). \end{aligned}$$

Proof. Let $c = 1$ and $A = B$ in Theorem 3.2. □

We see that Ramanujan’s formula (1.1) is included in Corollary 3.5. Thus, in case of N even in Corollary 3.5, we may apply (1.1) to cancel out the sum $\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m\alpha} - 1}$ and $\zeta(2N + 1)$.

For j and m integral and x real, let

$$f_k(j, m, x) := \begin{cases} (-1)^{\hat{k}+j+1} \sinh((2m + 1)(j + 1/2)x), & k \text{ even,} \\ (-1)^{\hat{k}+j+1} \cosh((2m + 1)jx), & k \text{ odd.} \end{cases}$$

Corollary 3.6. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned} & \alpha^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{\alpha^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=0}^{\infty} \frac{(2m + 1)^{k-2N-1} f_k(j, m, \alpha)}{\cosh^{k+1}((2m + 1)\alpha/2)} \\ & - (2N)! \alpha^{-N} \left(\sum_{m=0}^{\infty} \frac{(2m + 1)^{-2N-1}}{e^{(2m+1)\alpha} + 1} + (2^{-2N-2} - 2^{-1}) \zeta(2N + 1) \right) \\ & = \beta^{-N} \sum_{k=1}^N \binom{N}{k} \frac{(2N-k)!}{\beta^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=0}^{\infty} \frac{(2m + 1)^{k-2N-1} f_k(j, m, \beta)}{\cosh^{k+1}((2m + 1)\beta/2)} \\ & - (2N)! \beta^{-N} \left(\sum_{m=0}^{\infty} \frac{(2m + 1)^{-2N-1}}{e^{(2m+1)\beta} + 1} + (2^{-2N-2} - 2^{-1}) \zeta(2N + 1) \right). \end{aligned}$$

Proof. Let $c = 2$ and $A = B$ in Theorem 3.2. Apply Corollary 3.5 and use the followings;

$$\begin{aligned} \cosh((2m + 1)(j + 1/2)(\alpha + \pi i)) &= (-1)^{m+j} i \sinh((2m + 1)(j + 1/2)\alpha), \\ \cosh((2m + 1)(j + 1/2)(\alpha - \pi i)) &= (-1)^{m+j+1} i \sinh((2m + 1)(j + 1/2)\alpha), \\ \sinh^{k+1}((m + 1/2)(\alpha + \pi i)) &= (-1)^{m(k+1)} i^{k+1} \cosh((2m + 1)\alpha/2), \end{aligned}$$

$$\sinh^{k+1}((m + 1/2)(\alpha - \pi i)) = (-1)^{(m+1)(k+1)} i^{k+1} \cosh((2m + 1)\alpha/2).$$

□

Corollary 3.6 contains Corollary 1.3. Therefore, in case of N even in Corollary 3.6, Corollary 1.3 can be applied to replace the sums of the Bernoulli numbers for the sum $\sum_{m=0}^{\infty} \frac{(2m+1)^{-2N-1}}{e^{(2m+1)\alpha+1}}$ and the terms with $\zeta(2N + 1)$.

Corollary 3.5 and Corollary 3.6 show elegant symmetric series relations for α and β . In case of $N = 1$ in Corollary 3.5 and Corollary 3.6, we obtain

$$\begin{aligned} & \alpha^{-1} \sum_{m=1}^{\infty} \frac{\operatorname{csch}(m\alpha)}{m^3 e^{m\alpha}} + \sum_{m=1}^{\infty} \frac{\operatorname{csch}^2(m\alpha)}{m^2} + \alpha^{-1} \zeta(3) + \frac{1}{45} \alpha^2 \\ (3.4) \quad & = \beta^{-1} \sum_{m=1}^{\infty} \frac{\operatorname{csch}(m\beta)}{m^3 e^{m\beta}} + \sum_{m=1}^{\infty} \frac{\operatorname{csch}^2(m\beta)}{m^2} + \beta^{-1} \zeta(3) + \frac{1}{45} \beta^2 \end{aligned}$$

and

$$\begin{aligned} & \alpha^{-1} \sum_{m=0}^{\infty} \frac{\operatorname{sech}((2m + 1)\alpha/2)}{(2m + 1)^3 e^{(2m+1)\alpha/2}} + \sum_{m=1}^{\infty} \frac{\operatorname{sech}^2((2m + 1)\alpha/2)}{2(2m + 1)^2} - \frac{7}{8} \alpha^{-1} \zeta(3) \\ & = \beta^{-1} \sum_{m=0}^{\infty} \frac{\operatorname{sech}((2m + 1)\beta/2)}{(2m + 1)^3 e^{(2m+1)\beta/2}} + \sum_{m=1}^{\infty} \frac{\operatorname{sech}^2((2m + 1)\beta/2)}{2(2m + 1)^2} - \frac{7}{8} \beta^{-1} \zeta(3). \end{aligned}$$

The identity (3.4) also comes from Theorem 2.17 in [2].

Corollary 3.7. Assume that $A \not\equiv B \pmod{4}$. Then

$$\begin{aligned} & \sum_{k=1}^A \binom{A}{k} \frac{(2N - k)!}{(2\pi)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})\pi)}{m^{2N-k+1} \sinh^{k+1}(m\pi)} \\ & = - \sum_{k=1}^B \binom{B}{k} \frac{(2N - k)!}{(2\pi)^{-k}} \sum_{j=0}^{\hat{k}'} E(k, \hat{k} - j) \sum_{m=1}^{\infty} \frac{\cosh(2m(j + \tilde{k})\pi)}{m^{2N-k+1} \sinh^{k+1}(m\pi)} \\ & \quad - (2N)! \left(\sum_{m=1}^{\infty} \frac{2m^{-2N-1}}{e^{2m\pi} - 1} + \zeta(2N + 1) \right) + 2^{2N} \pi^{-1} A! B! \zeta(2N + 2). \end{aligned}$$

Proof. Let $c = 1$ and let $\alpha = \beta = \pi$ in Theorem 3.2. Use the followings;

$$\cosh(2m(j + \tilde{k})(\pi - \pi i)) = \begin{cases} \cosh(2m(j + \tilde{k})\pi), & k \text{ odd} \\ (-1)^m \cosh(2m(j + \tilde{k})\pi), & k \text{ even} \end{cases}$$

and

$$\sinh^{k+1}(m(\pi - \pi i)) = (-1)^{m(k+1)} \sinh^{k+1}(m\pi).$$

□

If N is odd in Corollary 3.7, then we may apply (1.1) to replace the sum $\sum_{m=1}^{\infty} \frac{m^{-2N-1}}{e^{2m\pi}-1}$ by the sums of the Bernoulli numbers, i.e.,

$$\sum_{m=1}^{\infty} \frac{2m^{-2N-1}}{e^{2m\pi}-1} + \zeta(2N+1) = -2^{2N} \pi^{2N+1} \sum_{k=0}^{N+1} \frac{(-1)^k B_{2k} B_{2N+2-2k}}{(2N)!(2N+2-2k)!}.$$

Corollary 3.8. Assume that $A \not\equiv B \pmod{4}$. Then

$$\begin{aligned} & \sum_{k=1}^A \binom{A}{k} \frac{(2N-k)!}{\pi^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \sum_{m=0}^{\infty} \frac{(2m+1)^{k-2N-1} f_k(j, m, \pi)}{\cosh^{k+1}((2m+1)\pi/2)} \\ &= - \sum_{k=1}^B \binom{B}{k} \frac{(2N-k)!}{\pi^{-k}} \sum_{j=0}^{\hat{k}} E(k, \hat{k}-j) \sum_{m=0}^{\infty} \frac{(2m+1)^{k-2N-1} f_k(j, m, \pi)}{\cosh^{k+1}((2m+1)\pi/2)} \\ & \quad + 2(2N)! \sum_{m=0}^{\infty} \frac{(2m+1)^{-2N-1}}{e^{(2m+1)\pi} + 1} + (2N)!(2^{-2N-1} - 1)\zeta(2N+1). \end{aligned}$$

Proof. Let $c = 2$ and let $\alpha = \beta = \pi$ in Theorem 3.2. Apply Corollary 3.7 and the equations in the proof of Corollary 3.6. □

If N is odd in Corollary 3.8, then we may apply Theorem 1.3 to replace the sum $\sum_{m=0}^{\infty} \frac{(2m+1)^{-2N-1}}{e^{(2m+1)\pi} + 1}$ by the sums of the Bernoulli numbers, i.e.,

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{2(2m+1)^{-2N-1}}{e^{(2m+1)\pi} + 1} + (2^{-2N-1} - 1)\zeta(2N+1) \\ &= \pi^{2N+1} \sum_{k=0}^{N+1} (2^{2N+1} - 2^{2k-1})(1 - 2^{-2k}) \frac{(-1)^k B_{2k} B_{2N+2-2k}}{(2N)!(2N+2-2k)!}. \end{aligned}$$

Other types of infinite series identities may be obtained from Theorem 2.1, which are comparable to those in Ramanujan’s notebook([8]).

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