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GENERALIZED TOPOLOGIES ON FINITE SETS

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Abstract. The number of topologies on a finite set is a famous open problem. In the present paper we discuss a method of obtaining the number of generalized topologies on finite sets.

1. Introduction

The number $T(n)$ of topologies on a finite set of cardinal n is an open question [6]. There is no known simple formula to compute $T(n)$ for arbitrary n. The online Encyclopedia of Integer Sequences presently lists $T(n)$ for $n \leq 18$. Recall that a subset μ of the power set $expX$ of a set X is a *generalized topology* (briefly GT) in X iff $G_i \in \mu$ $(i \in I)$ implies $\cup_{i \in I} G_i \in \mu$ (in particular, I can be empty so that the defnition implies $\emptyset \in \mu$). The pair (X, μ) is called a *generalized topological space* (briefly GTS). A member of μ is called *open* and a subset F of X is called *closed* if $X \setminus F \in \mu$. Sets which are simultaneousely open and closed are called open-closed sets. The theory of generalized topological spaces, which was introduced by Á. Császár [5], is one of the most important development of general topology in recent years. A GT on X is a join-sublattice $(expX, \subseteq)$ with the minimum element $\emptyset$, denoted by 0. Some important counterexamples in topological spaces or GTS can be found in the finite forms (see for example, [1, 3]).

Let X be an n-element set. Then the number $GT(n)$ of generalized topologies on X is exactly the number of join-sublattices of (exp $X, \subseteq$) with 0. There is no known formula giving $GT(n)$. Let $gt(n, k)$ be the set of all labeled generalized topologies on X having k open sets and $GT(n, k) = |gt(n, k)|$. Thus $GT(n) = \sum_{k=1}^{2^n} GT(n, k)$.

Definition 1.1. [7] *Let μ be a GT on X and μ' a GT on X'. A mapping $f : X \to X'$ is (μ, μ')-continuous iff $M' \in \mu'$ implies*

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$f^{-1}(M^{'}) \in \mu$. If f is bijective and $(\mu, \mu^{'})$-continuous, moreover f^{-1} is $(\mu^{'}, \mu)$-continuous, then we say that f is a $(\mu, \mu^{'})$-homeomorphism and (X, μ) and $(X^{'}, \mu^{'})$ are said to be equivalent.

Example 1.2. *There is a unique GT on the empty set. Likewise there are two distinct generalized topologies on $\{a\}$: $g\tau = \{\emptyset\}$ and $g\tau^{'} = \{\emptyset, \{a\}\}$. Let $X = \{a, b\}$ be a two-element set. There are 7 distinct generalized topologies on X but only 5 inequivalent generalized topologies: $g_1 = \{\emptyset\}$, $g_2 = \{\emptyset, \{a\}\}$, $g_3 = \{\emptyset, X\}$, $g_4 = \{\emptyset, \{a\}, X\}$ and $g_5 = \{\emptyset, \{a\}, \{b\}, X\}$.*

2. A GTS (X, μ), Where $|\mu| \geqq 2^{|X|} - 7$

Recall that a GTS (X, μ) is said to be a μ-T_1 space [9] if for any pair of distinct points x and y of X, there exists a $U \in \mu$ such that $x \notin U$ and $y \in U$. As [9] a GTS (X, μ) is called μ-T_2 if for every distinct points $x, y \in X$, there exist disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in Uy$. It is well known that a finite Hausdorff topological space, i.e., a finite μ-T_2 topological space, is discrete, but in [2] there is a non-discrete μ-T_2 GTS which is finite. Here, we give another example.

Example 2.1. *$\mu = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is a μ-T_2 GT on $X = \{a, b, c\}$, that is not discrete.*

Definition 2.2. *If a GTS (X, μ) which is T_1 has a base consisting of open-and-closed sets, then it is called zero-dimensional. A GTS (X, μ) which is T_1 is called completely regular if for every $x \in X$ and every closed subset $F \subset X$ such that $x \notin F$ there exists a continuous function f from X to $\mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$. If the cardinality of every nonempty member of μ is greater than one, then we say that (X, μ) is crowded.*

We give an example of a crowded completely regular GTS which is finite.

Example 2.3. *Let $X = \{a, b, c, d\}$ and $\mu = expX \backslash \{\{a\}, \{b\}, \{c\}, \{d\}\}$. Then (X, μ) is a zero-dimensional space since*

$$\beta = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

is a base for the GTS such that every member of β is closed. Therefore the generalized topological space is completely regular.

Remark 2.4. *For $n \geq 2$, the calculations of $GT(n,k)$ are complicated. In the following proposition we give some results about them.*

Proposition 2.5. *For $n \geq 3$ we have the following.*

(I) $GT(n,1) = GT(n,2^n) = 1$;
(II) $GT(n,2) = 2^n - 1$;
(III) $GT(n,2^n-1) = n$ *for* $n \geq 2$;
(IV) $GT(n,2^n-2) = \frac{3n(n-1)}{2}$;
(V) $GT(n,2^n-3) = \binom{n}{3} + \binom{n}{2}(2n-3) + \binom{n}{1}\binom{n-1}{2}$;
(VI) $GT(n,2^n-4) = \binom{n}{4} + \binom{n}{3}[3n-6] + \binom{n}{2}(\binom{2n-3}{2} + \frac{n^2(n-1)(n-2)}{6}$;
(VII) $GT(n,2^n-5) = \binom{n}{5} + \binom{n}{4}[4n-10] + \binom{n}{3}(\binom{3n-6}{2} + \binom{n}{2}[\binom{2n-3}{3} + 2\binom{n-1}{2}] + \binom{n}{1}[3\binom{n-1}{3} + \binom{n-1}{4}]$;
(VIII) $GT(n,2^n-6) = \binom{n}{6} + (5n-15)\binom{n}{5} + \binom{n}{4}\binom{4n-10}{2} + \binom{n}{3}[\binom{3n-6}{3} + \binom{3n-6}{2}] + \binom{n}{2}[1+\binom{2n-3}{4}) + 3\binom{n-1}{3}] + n[\binom{n-1}{5} + \binom{n-1}{4}\binom{4}{2} + \binom{n-1}{3}\binom{3}{2}]$;
(IX) $GT(n,2^n-7) = \binom{n}{7} + (6n-21)\binom{n}{6} + \binom{n}{5}\binom{5n-15}{2} + \binom{n}{4}[\binom{4n-10}{3} + 4\binom{n-1}{2}] + \binom{n}{3}[\binom{3n-6}{4} + 1 + 3\binom{n-1}{3}\binom{3}{2}] + \binom{n}{2}[\binom{2n-3}{5} + 2\binom{n-1}{4}\binom{4}{2} + 3\binom{n-1}{3}\binom{3}{2}] + n[\binom{n-1}{6} + \binom{n-1}{5}\binom{5}{2} + \binom{n-1}{4}\binom{4}{2} + \binom{n-1}{3}\binom{3}{3}]$.

Proof. Let μ be a GT in X. (I) and (II). The proofs of (I) and (II) are clear. (III). If $|\mu| = 2^n - 1$, then $\mu = expX \setminus \{x\}$ for some $x \in X$. (IV). If $|\mu| = 2^n - 2$, then there exist two elements x, y of X such that $\mu = expX \setminus \{\{x\},\{y\}\}$ or $\mu = expX \setminus \{\{x\},\{x,y\}\}$. (V). Let $|\mu| = 2^n - 3$. Then there exist three elements $a, b, x \in X$ such that $\mu = expX \setminus \{\{a\},\{b\},\{x\}\}$, $\mu = expX \setminus \{\{a\},\{b\},\{a,x\}\}$, or $\mu = expX \setminus \{\{a\},\{a,x\}\{a,y\}\}$. (VI). Let $|\mu| = 2^n - 4$. Then μ has one of the following forms:

Case (1): Let $\mu = expX \setminus A$; where $A = \{\{a\},\{b\},\{c\},\{d\}\} \subset expX$. The set $A = \{\{a\},\{b\},\{c\},\{d\}\}$ is chosen in $\binom{n}{4}$ ways.

Case (2): Let $\mu = expX \setminus \{\{a\},\{b\},\{c\},D\}$; where $a, b, c \in X$ and $D \subset X$. Then D is a two-element set and $D \cap \{a,b,c\} \neq \emptyset$, D is chosen in $3n-6$ different ways and the subsets $\{a\}, \{b\}$, and $\{c\}$ in $\binom{n}{3}$ different Ways.

Case (3): Let $\mu = expX \setminus \{\{a\},\{b\},C,D\}$; where $a, b \in X$ and $C, D \subset X$. Then D and C intersect $\{a,b,c\}$ and $|C| = |D| = 2$.

Case (4): Let $\mu = expX \setminus \{\{a\},B,C,D\}$; where $a \in X$ and $B, C, D \subset X$. Then $a \in B \cap C \cap D$ and so $|B| = |C| = |D| = 2$ or $|B| = |C| = 2$ and $D = B \cup C$. $\{a\}$ is chosen in n different ways and B, C, D are chosen in $\frac{n(n-1)(n-2)}{6}$ different ways. Similarly, (VII), (VIII) and (IX) hold. $\square$

Example 2.6. *Let* $X = \{a, b, c\}$*. Then by the above proposition* $GT(3,7) = 3,\ GT(3,6) = 9,\ GT(3,5) = 13,\ GT(3,4) = 15,\ GT(3,3) = 12$ *and* $GT(3,2) = 7\ GT(3,1) = 1 = GT(3,8)$*. Thus, the total number of generalized topologies on X is*

$$GT(3) = \sum_{i=1}^{8} GT(3,i) = 61.$$

Example 2.7. *Let* $X = \{a, b, c, d\}$*. Then by the above proposition* $GT(4,15) = 4,\ GT(4,14) = 18,\ GT(4,13) = 46,\ GT(4,12) = 51,\ GT(4,11) = 174,\ GT(4,10) = 221,\ GT(4,9) = 196,\ GT(4,2) = 15$ *and* $GT(4,16) = 1 = GT(4,1)$*.*

3. A GT with less than seven open sets

Recall that a chain topology on a finite set X, is a topology whose open sets are totally ordered by inclusion. For generalized topological spaces, we have the following definition.

Definition 3.1. *A GT-chain on X, is a generalized topology whose open sets are totally ordered by inclusion.*

Proposition 3.2. [4, 8] *Let* $C(n, k)$ *be the number of chain topologies on X having* k *open sets. Then,*

$$C(n,k) = \sum_{i=1}^{n-1} \binom{n}{i} C(i, k-1) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-1-i)^n.$$

Corollary 3.3. *Let* $CGT(n)$ *be the totall number of GT-chains on X, where* $|X| \geq 3$*. Then,*

$$CGT(n) = 1 + \sum_{i=1}^{n} \sum_{k=2}^{i+1} \sum_{j=0}^{k-1} (-1)^j \binom{n}{i} \binom{k-1}{j} (k-1-j)^i.$$

Proof. If μ is a GT-chain on X, then μ is a chain topology on $A = \cup\mu$. It is clear that any chain topology on a subset B of X is a GT-chain on X. Thus, there is a bijective correspondence between GT-chains on X and chain topologies on subsets of X. Therefore,

$$CGT(n) = 1 + \sum_{i=1}^{n} \binom{n}{i} \sum_{k=2}^{i+1} C(i,k),$$

and so by the above proposition we are done. □

Corollary 3.4. *Let $CGT(n,m)$ be the totall number of GT-chains on X with m open sets, where $|X| = n$ and $m \leq n+1$. Then,*

$$CGT(n,m) = \binom{m-1}{0} m^n - \binom{m-1}{1}(m-1)^n + \cdots + (-1)^{m-1}\binom{m-1}{m-1}1^n.$$

Proof. Let μ be a GT-chain on X with m open sets. If $X \notin \mu$, then $\mu \cup \{X\}$ is a chain topology on X with $m+1$ open sets. Otherwise, μ is a chain topology on X with m open sets. Thus by Proposition 3.2,

$$\begin{aligned} CGT(n,m) &= C(n,m+1) + C(n,m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i}(m-i)^n \\ &+ \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i}(m-1-i)^n = \binom{m}{0} m^n \\ &- [\binom{m}{1} - \binom{m-1}{0}](m-1)^n + \cdots + (-1)^{m-1}[\binom{m}{m-1} \\ &- \binom{m-1}{m-2}]1^n = \binom{m-1}{0} m^n - \binom{m-1}{1}(m-1)^n \\ &+ \cdots + (-1)^{m-1}\binom{m-1}{m-1}1^n. \end{aligned}$$

□

Example 3.5. *Let X be a set and $|X| = n$ where $n \in \mathbb{N}$. Then by the above corollary, the total number of GT-chains on X with three open sets is $CGT(n,3) = 3^n - 2^{n+1} + 1$. Similarly $CGT(n,4) = 4^n - 3^{n+1} + 3.2^n - 1$, $CGT(n,5) = 5^n - 4^{n+1} + 6.3^n - 4.2^n + 1$, $CGT(n,6) = 6^n - 5^{n+1} + 10.4^n - 10.3^n + 5.2^n - 1$, and $CGT(n,7) = 7^n - 6^{n+1} + 15.5^n - 20.4^n + 15.3^n - 6.2^n + 1$.*

Proposition 3.6. *For every $2 \leq n \in \mathbb{N}$ we have*

(a) $GT(n,3) = 3^n - 2^{n+1} + 1$;

(b) $GT(n,4) = 4^n - 3^{n+1} + 3.2^n - 1 + \frac{1}{2}\sum_{m=2}^{n}\sum_{i=0}^{m-1}\binom{n}{m}\binom{m}{i}(2^i - 1)$.

Proof. . (a) For every $\mu \in gt(n,3)$ there are a subset B of X and a subset A of B such that $\emptyset \neq A \subsetneq B \subset X$ and $\mu = \{\emptyset, A, B\}$. Thus by Example 3.5, $GT(n,3) = 3^n - 2^{n+1} + 1$.

(b) If $\mu = \{\emptyset, A, B, C\} \in gt(n,4)$, then either $A \subset B \subset C$ or $C = A \cup B$. The two cases are disjoint.

Case (1) : This is the number of GT-chains on X having 4 open sets; so by Example 3.5, the total number of generalized topologies in this case is $4^n - 3^{n+1} + 3.2^n - 1$.

Case (2) : Let C be a subset of X such that

$$2 \leq m = |C| \leq n$$

and A be proper and non-empty subset of C. Then, $1 \leq |A| = i < m$ and so $B = (C \setminus A) \cup B_1$, where $B_1 \subsetneq A$. Therefore, the total number of generalized topologies in this case is

$$\frac{1}{2}\sum_{m=2}^{n}\sum_{i=1}^{m-1}\binom{n}{m}\binom{m}{i}(2^i - 1).$$

□

Proposition 3.7. *For every $n = |X| \geq 3$ we have $GT(n, 5) =$*

$$5^n - 4^{n+1} + 6.3^n - 4.2^n + 1 + \frac{1}{2}\sum_{m=2}^{n-1}\sum_{i=1}^{m-1}\binom{n}{m}\binom{m}{i}(2^i - 1)(2^{n-m} - 1)$$
$$+\frac{5}{3}\sum_{m=3}^{n}\sum_{i=2}^{m-1}\sum_{j=1}^{i-1}\binom{n}{m}\binom{m}{i}\binom{i}{j}(2^j - 1).$$

Proof. Let $\mu \in gt(n, 5)$. If μ is a GT-chain, then by Example 3.5, the total number of GT-chains on X is $5^n - 4^{n+1} + 6.3^n - 4.2^n + 1$. If μ is not a GT-chain, then there are $A, B \in \mu$ such that $A \nsubseteq B$ and $B \nsubseteq A$, and so $C = A \cup B \in \mu$. Thus, the non-empty members of μ has one of the following forms:

(1) $A \cup B = C \subsetneq D$.
(2) $\emptyset \neq D \subsetneq C = A \cup B$.
The two cases are disjoint.

Case (1) : D can be chosen in $2^{n-m} - 1$ ways, where

$$2 \leq m = |C| \leq n - 1.$$

Thus by the proof of the above proposition, the total number of generalized topologies in this case is

$$\frac{1}{2}\sum_{m=2}^{n-1}\sum_{i=1}^{m-1}\binom{n}{m}\binom{m}{i}(2^i - 1)(2^{n-m} - 1).$$

Case (2) : Since in this case $3 \leq m = |C| \leq n$, we have $2 \leq i = |A| \leq m-1$. In this case, $D \cap (A \cap B) \neq \emptyset$ and so

$$1 \leq |A \cap B| = j \leq i - 1.$$

Thus the total number of generalized topologies in this case is $n_1 + n_2 + n_3$, where n_1, n_2 and n_3 can be computed as follows:

(2I) If D is a nonempty subset of $A \cap B$. Then, by the proof of the above proposition

$$n_1 = \frac{1}{2} \sum_{m=3}^{n} \sum_{i=2}^{m-1} \sum_{j=1}^{i-1} \binom{n}{m} \binom{m}{i} \binom{i}{j} (2^j - 1).$$

(2II) If $D = (A \setminus B) \cup (B \setminus A) \cup D_1$, where D_1 is a proper subset of $A \cap B$. Then, $n_2 = \frac{1}{3} n_1$.

(2II) If $D = (A \setminus B) \cup D_1$, where D_1 is a proper subset of $A \cap B$. Then $n_3 = 2n_1$. The proof is complete.

□

Notation 3.8. *Let Y be a subset of $A \cup B$. Then Y can be written as $Y = Y' \cup Y_1 \cup Y''$, where $Y' \subset A \setminus B$, $Y_1 \subset A \cap B$ and $Y'' \subset B \setminus A$.*

Theorem 3.9. *Let $\mu = \{\emptyset, A, B, A \cup B, D, E\}$ be in $gt(n, 6)$, such that $A \not\subseteq B$ and $B \not\subseteq A$ and $D, E \subsetneq A \cup B$. If two of the sets D', D'' and D_1 are empty and one of them is non-empty, then μ has one of the following forms:*

(1) $\mu = \{\emptyset, A, B, A \cup B, D_1, ((A \cap B) \setminus I) \cup E_2\}$, *where I is a non-empty subset of D_1 and $E_2 = A \setminus B$ or $E_2 = (A \setminus B) \cup (B \setminus A)$.*
(2) $\mu = \{\emptyset, A, B, A \cup B, D_1, E\}$, *where $\emptyset \neq D1 \subset E_1 = E \subset A \cap B$, $E = E_1 \cup (B \setminus A) \cup (A \setminus B)$ or $E = E_1 \cup (B \setminus A)$, where $\emptyset \neq D_1 \subset E_1 \subsetneq A \cap B$.*
(3) $\mu = \{\emptyset, A, B, A \cup B, D_1, A \cup E''\}$, *where $\emptyset \subsetneq E'' \subsetneq B \setminus A$.*
(4) $\mu = \{\emptyset, A, B, A \cup B, D', D' \cup B\}$, *where $\emptyset \neq D' \subsetneq A \setminus B$.*
(5) $\mu = \{\emptyset, A, B, A \cup B, A \setminus B, B \cup E'\}$, *where $\emptyset \neq E' \subsetneq A \setminus B$.*
(6) $\mu = \{\emptyset, A, B, A \cup B, A \setminus B, E_1 \cup E_2\}$, *where $E_2 = (A \setminus B) \cup (B \setminus A)$ or $E_2 = A \setminus B$.*

Proof. Let $C = A \cup B$. If $D = D_1$, then $D \cup E = E' \cup (D_1 \cup E_1) \cup E''$ and so $|\mu| = 6$ implies that there are the following cases which are disjoint:

(1) If $D \cup E = A$, then $E'' = \emptyset$, and so $A = E' \cup (D_1 \cup E_1)$, i.e., $D_1 \cup E_1 = A \cap B$ and $E' = A \setminus B$. Thus $E_1 = (A \cap B) \setminus I$, where I is a non-empty subsets of D_1. We note that if $D \cup E = B$, then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $D \cup E = A$.
(2) If $D \cup E = C$, then $(D_1 \cup E_1) \cup E' \cup E'' = C$. Thus, $A \setminus B = E', B \setminus A = E''$ and $D_1 \cup E_1 = A \cap B$, i.e., $E = (A \setminus B) \cup (B \setminus A) \cup E_1$ and $E_1 = (A \cap B) \setminus I$, where I is a non-empty subset of D_1.

(3) If $E = D \cup E$, then $E = (D_1 \cup E_1) \cup E' \cup E''$, and so $D_1 \subset E_1$. We note that $E \cup A = E'' \cup A \in \mu$ can not be equal to B or $D = D_1$ so there are the following cases which are disjoint.

(3I) If $E \cup A = E'' \cup A = A$, then $E'' = \emptyset$ and $E \subset A$. But $B \cup E = B \cup E' \in \mu$ implies that $E' = \emptyset$ or $E' = A \setminus B$. Thus $D = D_1 \subsetneq E = E_1 \subset A \cap B$ or $E = E_1 \cup (A \setminus B)$, where $D_1 \subset E_1 \subsetneq A \cap B$.

(3II) If $E \cup A = E'' \cup A = C$, then $E'' = B \setminus A$ and $B \cup E' = B \cup E \in \mu$ implies that $E' = \emptyset$ or $E' = A \setminus B$. Thus $E = E_1, E = E_1 \cup (A \setminus B) \cup (B \setminus A)$ or $E = E_1 \cup (B \setminus A)$. We note that if $E = E_1 \cup (B \setminus A)$, then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $E = E_1 \cup (A \setminus B)$.

(3III) If $E \cup A = E'' \cup A = E$, then $A \subset E$. Thus $E' = A \setminus B$, $E_1 = A \cap B$ and E'' is a non-empty and proper subset of $B \setminus A$.

Let $D_1 = D'' = \emptyset$ and $\emptyset \neq D'$. Then $D \cup A = A$ and $D' \cup B = D \cup B$ is equal to E or C. Thus there are the following cases which are disjoint.

(4) If $D' \cup B = E$, then D' is a non-empty and proper subset of $A \setminus B$.

(5) If $D' \cup B = C$, then $D' = D = A \setminus B$. Since $E \cup D = E_1 \cup (A \setminus B) \cup E'' \in \mu$, we have the following cases which are disjoint:

(5I) If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = A$. Then $E'' = \emptyset$ and $E_1 = A \cap B$ and so $E = E' \cup (A \cap B)$. Thus $E \cup B = E' \cup B$ implies that $E' = \emptyset$ and so $E = A \cap B$; which is not a new GT.

(5II) If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = C$, then $E = B \cup E'$. Since $E \neq$B and $E \neq C$, we have $\emptyset \neq E' \subsetneq A \setminus B$.

If $E_1 \cup (A \setminus B) \cup E'' = E \cup D = E$, then $E' = A \setminus B = D$ and so $E = (A \setminus B) \cup E_1 \cup E''$. Since $A \cup E = A \cup E''$ there are two cases which are disjoint:

(5III) If $A \cup E'' = C$, then $E'' = B \setminus A$ and so $E = (A \setminus B) \cup E_1 \cup (B \setminus A)$.

(5IV) If $A \cup E'' = A$, then $E'' = \emptyset$ and so $E = (A \setminus B) \cup E_1$.

We note that if $D_1 = \emptyset = D'$ and $D'' \neq \emptyset$, then the set of gener alized topologies in this case is coincided with the set of generalized topologies in the case $D_1 = \emptyset = D''$ and $D' \neq \emptyset$.

□

Theorem 3.10. *Let $\mu = \{\emptyset, A, B, A \cup B, D, E\}$ be in $gt(n, 6)$, such that $A \nsubseteq B$ and $B \nsubseteq A$ and $B \nsubseteq A$ and $D, E \subsetneq A \cup B$. If one of the sets D', D'' and D_1 is empty and two of them are non-empty, then μ has one of the following forms:*

(1) *$\mu = \{\emptyset, A, B, A \cup B, D_1 \cup D', D' \cup B\}$, where D_1 is non-empty and $\emptyset \neq D' \subsetneq A \setminus B$.*

(2) *$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \setminus B) \cup ((A \cap B) \setminus I)\}$, where I is a non-empty subset of $A \cap B$ or $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \cap B) \setminus I\}$, where $I \subset A \cap B$.*

(3) *$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup E_1\}$, where $E_2 = A \setminus B$ or $E_2 = (A \setminus B) \cup (B \setminus A)$ and $D1 \subsetneq E_1 \subsetneq A \cap B$.*

(4) *$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup ((A \cap B) \setminus I)\}$, where E_2 is $B \setminus A$ or $(B \setminus A) \cup (A \setminus B)$ and I is a non-empty subset of D_1.*

(5) *$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D'', A \cup D''\}$, where $\emptyset \neq D'' \subsetneq B \setminus A$.*

Proof. Let $C = A \cup B$. If D_1 and D' are non-empty and $D'' = \emptyset$. Then $D = D_1 \cup D', D \cup A = A$ and $D \cup B = D' \cup B$ is equal to E or C. Thus we have the following cases which are disjoint:

(1) If $D' \cup B = D \cup B = E$. Then $\emptyset \neq D'$ is a proper subset of $A \setminus B$ since $E \neq C$. Thus the general form of μ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, D_1 \cup D', D' \cup B\}.$$

(2) If $D \cup B = C$, then $D' = A \setminus B$ and so $D = (A \setminus B) \cup D_1$. Thus $\emptyset \neq D_1$ is a proper subset of $A \cap B$. Since $D \cup E = (D_1 \cup E_1) \cup (A \setminus B) \cup E'' \in \mu$, we have the following cases which are disjoint:

(2I) If $D \cup E = A$, then$(D_1 \cup E_1) \cup (A \setminus B) \cup E'' = A$ and so $E'' = \emptyset$, $D_1 \cup E_1 = A \cap B$ and $E = E_1 \cup E'$. If $E' \subsetneq A \setminus B$, then $E' = \emptyset$. Thus μ has the following form:

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \cap B) \setminus I)\},$$

where I is a subset of D_1. If $E' = A \setminus B$, then $E_1 = (A \cap B) \setminus I$, where I is a non-empty subset of D_1. Thus μ has the following form:

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, (A \setminus B) \cup ((A \cap B) \setminus I)\}.$$

(3) If $D_1 \cup (A \setminus B) \cup E = D \cup E = E$, then $E' = A \setminus B$. $A \cup E = A \cup E''$ implies that E'' is equal to $\emptyset$ or $B \setminus A$ and so $E = (A \setminus B) \cup E_1$ or $E = (A \setminus B) \cup (B \setminus A) \cup E_1$. Thus the general form of μ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_1 \cup E_2\},$$

where $E_2 = (A \setminus B)$ or $E_2 = (A \setminus B) \cup (B \setminus A)$ and $D_1 \subsetneq E1 \subsetneq A \cap B$.

(4) If $D_1 \cup (A \setminus B) \cup E = D \cup E = C$, then $D_1 \cup E_1 = A \cap B$ and $E'' = B \setminus A$. Thus $E \cup B = E' \cup$B implies that $E' = \emptyset$ or $E' = A \setminus B$

and $E_1 = (A \cap B) \setminus I$, where I is a non-empty subset of D_1 and so the general form of μ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1, E_2 \cup ((A \cap B) \setminus I)\},$$

where E_2 is $B \setminus A$ or $(B \setminus A) \cup (A \setminus B)$, and I is a non-empty subset of D_1. We note that if D_1 and D'' are nonempty and $D' = \emptyset$. Then the set of generalized topologies in this case is coincided with the set of generalized topologies in the case $D_1 \neq \emptyset \neq D'$ and $D'' = \emptyset$. Let D' and D'' be non-empty and $D_1 = \emptyset$. Then $D = D' \cup D''$ and so $B \cup D = B \cup D' \in \mu$ and $A \cup D = A \cup D'' \in \mu$. Since $A \cup D''$ is equal to E or C, the following cases are disjoint:

(5) If $A \cup D'' = E$, then $B \cup D' = C$ and so $D' = A \setminus B$. Thus the general form of μ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D'', A \cup D''\},$$

where $\emptyset \neq D'' \subsetneq B \setminus A$. We note that if $A \cup D'' = C$, then $D'' = B \setminus A$. Thus $D = D' \cup (B \setminus A)$ and $D \cup B = D' \cup B$, and so the set of generalized topologies in this case is coincided with the set of generalized topologies in the case$A \cup D'' = E$.

□

Theorem 3.11. *Let $\mu = \{\emptyset, A, B, A \cup B, D, E\}$ be in $gt(n, 6)$, such that $A \nsubseteq B$ and $B \nsubseteq A$, and each of the sets D_1, D' and D'' are non-empty. Then $\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1 \cup D'', A \cup D''\}$, where $\emptyset \neq D_1 \subsetneq A \cap B$; and $\emptyset \neq D'' \subsetneq B \setminus A$.*

Proof. Let D_1, D' and D'' be non-empty, then $A \cup D = A \cup D'' \in \mu$ implies that $A \cup D$ is equal to C or E. If $A \cup D = E$, then from $B \cup D = B \cup D'$ we conclude that $D' = A \setminus B$. Thus the general form of μ in this case is

$$\mu = \{\emptyset, A, B, A \cup B, (A \setminus B) \cup D_1 \cup D'', A \cup D''\},$$

where $\emptyset \neq D_1 \subsetneq A \cap B$; and $\emptyset \neq D'' \subsetneq B \setminus A$.
We note that if $A \cup D = E$, then the set of all generalized topologies obtained in this case is equal to the set of all generalized topologies obtained in the case $A \cup D = E$. □

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