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EFFECTS OF PHASE-LAGS AND VARIABLE THERMAL CONDUCTIVITY IN A THERMOVISCOELASTIC SOLID WITH A CYLINDRICAL CAVITY

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Abstract. This paper investigates the effect of dual-phase-lags on a thermoviscoelastic orthotropic solid with a cylindrical cavity. The cylindrical cavity is subjected to a thermal shock varying heat and its material is taken to be of Kelvin-Voigt type. The phase-lag thermoelastic model, Lord and Shulman's model and the coupled thermoelasticity model are employed to study the thermomechanical coupling, thermal and mechanical relaxation (viscous) effects. Numerical solutions for temperature, displacement and thermal stresses are obtained by using the method of Laplace transforms. Numerical results are plotted to illustrate the effect phase-lags, viscoelasticity, and the variability thermal conductivity parameter on the studied fields. The variations of all field quantities in the context of dual-phase-lags and coupled thermoelasticity models follow similar trends while the Lord and Shulman's model may be different. The influence of viscosity parameter and variability of thermal conductivity is very pronounced on temperature and thermal stresses of the thermoviscoelastic solids.

1. Introduction

Viscoelasticity is of interest in various engineering applications due to a variety of microphysical processes. Most solids exhibit viscous effects when subjected to dynamic loading and for this reason the linear viscoelasticity has been remained as an important area of research. The constitutive relations for many structures can be approximated by the

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Nomenclature

c_{ij}	isothermal elastic constants
C_E	specific heat at constant strain
k_0	thermal conductivity at reference temperature T_0
K_0, K_1, K_2	modified Bessel's functions of first kind of orders 0, 1, 2
$k_1^*(\equiv k_1/k_0)$	slope of the thermal conductivity-temperature curve
K_r	thermal material coefficient
(r, θ, z)	cylindrical coordinates
T	absolute temperature
t_0	viscous damping parameter
T_0	reference temperature
t_Θ, t	finite times required for thermal equilibrium
$\vec{q}$	heat flows vector
u_r	radial displacement
u_z	axial displacement
$\varepsilon_r, \varepsilon_\theta$	radial and hoop strains
u_θ	hoop displacement
β_{ij}	thermal elastic coupling components
ρ	material density
$\sigma_r, \sigma_\theta, \sigma_z$	radial, hoop and axial stresses
$\Theta = T - T_0$	temperature increment ($\lvert\Theta/T_0\rvert \ll 1$)

linear viscoelasticity theory. This theory may be extended to the corresponding one of thermos-viscoelasticity theory at finite strains. Different investigations are dealt with generalized or coupled thermoviscoelastic problems for many applications [1]-[9]. Kovalenko and Karnaukhov [10] have presented a generalized linearized theory of thermoviscoelasticity that included effect of heat formation. Equations of motion are given of state together for the energy with the linearized boundary conditions for large initial deformations. Drozdov [11] has derived a constitutive model for the viscoelastic behavior of polymers at finite strains which is rather simple to be employed in engineering applications. Kundu and Mukhopadhyay [12] have considered the distribution of displacements, temperature, and stresses in a homogeneous isotropic viscoelastic medium with a spherical cavity. They have taken into account the relaxation effect and solve this problem in the context of generalized thermoelasticity and used the Laplace transform. Baksi et al. [13] have obtained the basic equations of the problems of generalized thermoelasticity in an infinite rotating magneto-thermo-viscoelastic media including heat sources with one relaxation parameter. The eigenvalue approach has been used to solve these equations to determine the field quantities. Kar and Kanoria [14] have studied the thermoviscoelastic stresses, in the context of generalized theories of thermoelasticity, in a homogeneous viscoelastic isotropic spherical shell. Kanoria and Mallik [15] have obtained the thermoviscoelastic field quantities in a homogeneous, infinite Kelvin-Voigt-type viscoelastic, thermally conducting medium under periodically varying heat sources. Ezzat et al. [16] have applied the governing coupled fractional relaxation equations in the frame of the thermo-viscoelasticity with fractional order heat transfer to the one-dimensional problem with heat sources. Deswal and Kalkal [17] have presented a two-temperature model for a half-space problem in the context of fractional order micro-polar thermoviscoelasticity. Deswal and Kalkal [18] have discussed the effects of phase-lags on wave propagation in a 3D thermoelastic medium in the domain of three-phase-lag theory with viscosity and two-temperature parameter. This article deals with the thermo-viscoelastic interaction of a conducting orthotropic solid of variable thermal conductivity including a cylinder cavity. The boundaries of the cylinder are subjected to a time-dependent thermal shock and its surface is traction free. The thermoelastic interactions in this solid in the context of a generalized thermoelasticity with dual-phase-lags (DPLs) [19]-[23] has been investigated. The present DPLs model developed by Tzou [24],[25] is an extension to the well-known generalized thermoelasticity theory [26]-[28]. The numerical estimates of the radial displacement, temperature and thermal stresses are obtained for thermoviscoelastic material. A comparison of the results for different theories (DPL model, LS model and CTE model) is presented and the effects of viscosity and variability of thermal conductivity parameters are also shown. Neglecting the viscosity coefficient and variability of thermal conductivity to illustrated some special cases of the problem.

2. Basic equations

Let us consider a viscoelastic orthotropic solid with a cylindrical cavity at a reference temperature T_0. The surface of solid is traction-free and the solid itself is under a time-dependent thermal shock. Kelvin-Voigt model of linear viscoelasticity may be employed to describe the viscoelastic nature of the material of the solid. The present problem is considered as axially symmetric one and accordingly the displacement field of the body is reduced to

$$u_r = u(r,t), \qquad u_\theta(r,t) = u_z(r,t) = 0, \tag{1}$$

and their non-vanished strain components are given by

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r}. \tag{2}$$

The constitutive stress-strain relations for a Kelvin-Voigt-type solid take the form [29]

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = \left(1 + t_0 \frac{\partial}{\partial t}\right) \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \\ c_{13} & c_{23} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} \end{Bmatrix} - \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \end{Bmatrix} \Theta. \tag{3}$$

The viscous damping parameter t_0 represents the mechanical relaxation time due to viscosity. Neglecting the body forces to get the equation of motion of the cylindrical cavity in the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4}$$

Equation (3) can be used in the above equation of motion to get

$$\begin{aligned} &\left(1 + t_0 \frac{\partial}{\partial t}\right) \left[c_{11} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) - c_{22} \frac{u}{r^2}\right] \\ &= \rho \frac{\partial^2 u}{\partial t^2} + \beta_{11} \frac{\partial \Theta}{\partial r} + (\beta_{11} - \beta_{22}) \frac{\Theta}{r}. \end{aligned} \tag{5}$$

The modified Fourier's law is given by

$$\left(1 + t_q \frac{\partial}{\partial t}\right) \vec{q} = -K_r \left(1 + t_\Theta \frac{\partial}{\partial t}\right) \nabla \Theta. \tag{6}$$

The delay time t_Θ is called the PL of temperature gradient while the other time t_q is said to be the PL of heat flux. The target of the PL of heat flux t_q is to ensure that the heat conduction equation will predict finite speeds of heat propagation. So, the equation of energy conservation may be written as

$$-\nabla . \vec{q} = \rho C_E \frac{\partial \Theta}{\partial t} + T_0 \frac{\partial}{\partial t} \left(\beta_{11} \frac{\partial u}{\partial r} + \beta_{22} \frac{u}{r}\right). \tag{7}$$

Eliminating $\vec{q}$ by using equations (6) and (7), the heat conduction equation with DPLs and ignoring the heat sources takes the form

$$\left(1 + t_\Theta \frac{\partial}{\partial t}\right) (K_r \Theta_{,r})_{,r} = \left(1 + t_q \frac{\partial}{\partial t}\right) \left[\rho C_E \frac{\partial \Theta}{\partial t} + T_0 \frac{\partial}{\partial t} \left(\beta_{11} \frac{\partial u}{\partial r} + \beta_{22} \frac{u}{r}\right)\right]. \tag{8}$$

The governing field equations in the context of linear generalized thermoelasticity with one relaxation time, i.e., the Lord and Shulman's theory (LS model) can be written from equations (1)-(8) by setting mechanical PLs parameters $t_\Theta = 0$ and $t_q = \tau_0$ (τ_0 is the thermal relaxation time). Upon neglecting the thermal PLs, i.e., $t_\Theta = t_q = 0$, we obtain the governing field equations for coupled theory of thermoelasticity (CTE model). Also, it is clear that by setting the thermal PLs $t_\Theta = t_q = 0$, and the thermomechanical coupling parameters $\beta_{11} = \beta_{22} = 0$, one gets the governing field equations for uncoupled thermoelasticity.

3. Variable thermal conductivity

Thermal properties of thermosensitivity solid should be vary with temperature and leads to a nonlinear heat conduction problem. One of the ways that enable us to solve such problem is by assuming simply nonlinear properties of the material. This means that the thermal material coefficient K_r and the specific heat C_E should be taken to be linearly depending on the temperature [30], but the thermal diffusivity k $(= \frac{K_r}{\rho C_E})$ may be assumed constant. That is

$$K_r = K_r(\Theta) = k_0 + k_1^*\Theta. \tag{9}$$

Now, let us consider a new function ψ to express the heat conduction in Kirchhoff transformation in the form [30]

$$\psi = \frac{1}{k_0}\int_0^\Theta K_r(\Theta)\mathrm{d}\Theta. \tag{10}$$

The above equations with the aid of equation (9) gives

$$\psi = \Theta\left(1 + \frac{1}{2}k_1\Theta\right). \tag{11}$$

From equation (11), it follows that

$$\nabla\psi = \frac{K_r(\Theta)}{k_0}\nabla\Theta, \qquad \frac{\partial\psi}{\partial t} = \frac{K_r(\Theta)}{k_0}\frac{\partial\Theta}{\partial t}. \tag{12}$$

Finally, the general heat equation with variable thermal conductivity is given, after substituting equation (12) into equation (8), in the form

$$\begin{aligned}\left(1 + t_\Theta\frac{\partial}{\partial t}\right)&\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\\ &= \left(1 + t_q\frac{\partial}{\partial t}\right)\left[\rho C_E\frac{\partial\psi}{\partial t} + \frac{T_0}{k_0}\frac{\partial}{\partial t}\left(\beta_{11}\frac{\partial u}{\partial r} + \beta_{22}\frac{u}{r}\right)\right].\end{aligned} \tag{13}$$

From equations (11), the equation of motion will be

$$\left(1+t_0\frac{\partial}{\partial t}\right)\left[c_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_{22}\frac{u}{r^2}\right] = \rho\frac{\partial^2 u}{\partial t^2}+\frac{\beta_{11}}{1+2k_1\Theta}\frac{\partial \psi}{\partial r}+\frac{\beta_{11}-\beta_{22}}{k_1 r}\left[\sqrt{1+2k_1\psi}-1\right], \tag{14}$$

or in an expanding form

$$\begin{aligned}\left(1+t_0\frac{\partial}{\partial t}\right)&\left[c_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_{22}\frac{u}{r^2}\right]\\ &= \rho\frac{\partial^2 u}{\partial t^2}+\beta_{11}\frac{\partial \psi}{\partial r}\left[1-(2k_1\Theta)+(2k_1\Theta)^2-\cdots\right]\\ &+\frac{\beta_{11}-\beta_{22}}{k_1 r}\left[1+\frac{1}{2}(2k_1\psi)-\frac{1}{8}(2k_1\psi)^2+\cdots-1\right].\end{aligned} \tag{15}$$

For linearity, since $\Theta = T - T_0$ such that $|\Theta/T_0| \ll 1$, then the above equation will be reduced to

$$\left(1+t_0\frac{\partial}{\partial t}\right)\left[c_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-c_{22}\frac{u}{r^2}\right] = \rho\frac{\partial^2 u}{\partial t^2}+\beta_{11}\frac{\partial \psi}{\partial r}+(\beta_{11}-\beta_{22})\frac{\psi}{r}, \tag{16}$$

which is the same as equation (5) just Θ is replaced with ψ. Consequently, the thermal stresses may be obtained by using equation (3) with replacing Θ by ψ. In what follows we will consider the following non-dimensional variables

$$\begin{aligned}&\{r', u', R'\} = \frac{c_0}{k}\{r, u, R\}, \qquad \{t', t_0', t_q', t_\Theta'\} = \frac{c_0^2}{k}\{t, t_0, t_q, t_\Theta\},\\ &\psi' = \frac{\psi}{T_0}, \quad \sigma_j' = \frac{\sigma_j}{c_{11}}, \quad k_1' = T_0 k_1, \quad c_0^2 = \frac{c_{11}}{\rho}, \quad (j = r, \theta, z).\end{aligned} \tag{17}$$

Using the above dimensionless quantities in the governing equations and suppressing dashes, we obtain

$$\left(1+t_0\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-C_2\frac{u}{r^2}\right) = \frac{\partial^2 u}{\partial t^2}+\beta_1\frac{\partial \psi}{\partial r}+\beta_6\frac{\psi}{r}, \tag{18}$$

$$\left(1+t_\Theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = \left(1+t_q\frac{\partial}{\partial t}\right)\left[\frac{\partial \psi}{\partial t}+\frac{\partial}{\partial t}\left(\beta_4\frac{\partial u}{\partial r}+\beta_5\frac{u}{r}\right)\right]. \tag{19}$$

However, the dimensionless stresses are given by

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\end{Bmatrix} = \left(1 + t_0\frac{\partial}{\partial t}\right)\begin{bmatrix}1 & c_1\\ c_1 & c_2\\ c_3 & c_4\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial r}\\ \frac{u}{r}\end{Bmatrix} - \begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix}\psi, \tag{20}$$

where

(21)

$$\{\beta_1, \beta_2, \beta_3, \beta_6\} = \frac{T_0}{c_{11}}\{\beta_{11}, \beta_{22}, \beta_{33}, (\beta_{11} - \beta_{22})\},$$

$$\{\beta_4, \beta_5\} = \frac{1}{\rho C_E}\{\beta_{11}, \beta_{22}\}, \quad \{c_1, c_2, c_3, c_4\} = \frac{1}{c_{11}}\{c_{12}, c_{22}, c_{13}, c_{23}\}.$$

4. Solution of the problem

To solve the present problem we will firstly considered the initial and regularity conditions. These conditions may be expressed as

$$u(r,0) = \left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0} = 0, \quad \Theta(r,0) = \left.\frac{\partial \Theta(r,t)}{\partial t}\right|_{t=0} = 0, \quad \psi(r,0) = \left.\frac{\partial \psi(r,t)}{\partial t}\right|_{t=0} = 0, \tag{22}$$

$$u(r,t) = \Theta(r,t) = \psi(r,t) = 0 \quad \text{when} \quad r \to \infty. \tag{23}$$

In addition, it is assumed that the disturbances are small and confined to neighborhood of the interface $r = R$ and hence vanish as r tends to infinity. Equations (18) and (19) can be solved by considering that the medium described above is quiescent and the surface of the cylinder is subjected to a time dependent thermal shock and traction free. So, the corresponding boundary conditions may be written as

$$\Theta(R,t) = \Theta_0 H(t), \quad t > 0, \tag{24}$$

$$\sigma_r(R,t) = 0, \tag{25}$$

where Θ_0 is constant. Using equation (11), then one gets

$$\psi(R,t) = \Theta_0 H(t) + \frac{1}{2}k_1[\Theta_0 H(t)]^2. \tag{26}$$

The Laplace transform is applied to equations (18)-(20) taking into consideration the initial conditions given in equation (22) and assuming that $\beta_{11} = \beta_{22}$ (i.e., $\beta_4 = \beta_5 = \beta$) and $c_{11} = c_{22}$ to obtain the following equations:

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} - \frac{s^2}{1+t_0 s}\bar{u} = \frac{\beta_1}{1+t_0 s}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}r}, \tag{27}$$

$$\frac{\mathrm{d}^2\bar{\psi}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}r} = \frac{s(1+t_q s)}{1+t_\Theta s}\left[\bar{\psi} + \beta\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} + \frac{\bar{u}}{r}\right)\right], \tag{28}$$

$$\begin{Bmatrix} \bar{\sigma}_r \\ \bar{\sigma}_\theta \\ \bar{\sigma}_z \end{Bmatrix} = (1+t_0 s)\begin{bmatrix} 1 & c_1 \\ c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{Bmatrix} \frac{\partial\bar{u}}{\partial r} \\ \frac{\bar{u}}{r} \end{Bmatrix} - \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix}\bar{\psi}, \tag{29}$$

where $\bar{u}$, $\bar{\psi}$, $\bar{\sigma_r}$, $\bar{\sigma_\theta}$ and $\bar{\sigma_z}$ are the Laplace transforms of quantities u, ψ, σ_r, σ_θ and σ_z, respectively, and s is the Laplace parameter. Equations (28) and (29) may be simplified as

$$(DD_1 - \tau_1)\bar{u} = \tau_2 D\bar{\psi}, \tag{30}$$

$$\beta\tau_3 D_1\bar{u} = (DD_1 - \tau_3)\bar{\psi}, \tag{31}$$

where

$$D = \frac{\mathrm{d}}{\mathrm{d}r}, D_1 = \frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}, \tau_1 = \frac{s^2}{1+t_0 s}, \tau_2 = \frac{\beta_1}{1+t_0 s}, \tau_3 = \frac{s(1+t_q s)}{1+t_\Theta s}. \tag{32}$$

Now, let the radial displacement u is appeared as a first derivative of a new thermoelastic potential function ϕ in the form

$$u = \frac{\mathrm{d}\phi}{\mathrm{d}r}, \tag{33}$$

then, the Laplace form of the above relation may be introduced into equations (30) and (31) to obtain

$$(DD_1 - \tau_1)\bar{\phi} = \tau_2\bar{\psi}, \tag{34}$$

$$\beta\tau_3 D_1 D\bar{\phi} = (DD_1 - \tau_3)\bar{\psi}. \tag{35}$$

Eliminating $\bar{\psi}$ from the above equations, one gets

$$\{\nabla^4 - [\tau_1 + \tau_3(1+\beta\tau_2)]\nabla^2 + \tau_1\tau_3\}\bar{\phi} = 0, \tag{36}$$

which tends to the following characteristic equation:

$$(\nabla^2 - m_1^2)(\nabla^2 - m_2^2)\bar{\phi} = 0, \tag{37}$$

where m_1^2 and m_2^2 are the roots of the equation

$$m^4 - [\tau_1 + \tau_3(1+\beta\tau_2)]m^2 + \tau_1\tau_3 = 0. \tag{38}$$

The roots of equation (37) are obtained as

$$m_1^2 = \frac{1}{2}(2A + \sqrt{A^2 - 4B}), \quad m_2^2 = \frac{1}{2}(2A - \sqrt{A^2 - 4B}), \tag{39}$$

where

$$A = \tau_1 + \tau_3(1 + \beta\tau_2), \quad B = \tau_1\tau_3. \tag{40}$$

Equation (37) leads to the modified Bessel's equation for $\bar{\phi}$ of zero order

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - m_1^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - m_2^2\right)\bar{\phi} = 0. \tag{41}$$

The solutions of the above equation under the regularity conditions that u, Θ, $\psi \to 0$ as $r \to \infty$ can be written in the form

$$\bar{\phi} = \sum_{i=1}^{2} A_i K_0(m_i r), \tag{42}$$

where A_i, $i = 1, 2$ are two parameters may be given in terms of s. The substitution of equation (42) into equation (34) gives

$$\bar{\psi} = \frac{1}{\tau_2}\sum_{i=1}^{2}(m_i^2 - s^2)A_i K_0(m_i r). \tag{43}$$

Also, the radial displacement according to equations (33) and (42) will be

$$\bar{u} = -\sum_{i=1}^{2} A_i K_1(m_i r). \tag{44}$$

The following well-known expression of Bessel's function

$$xK_n'(x) = -xK_{n\pm1} \pm nK_n(x), \tag{45}$$

is used to derive the stresses in terms of the displacement $\bar{u}$ and the function $\bar{\psi}$. So, one obtains

$$\bar{\sigma}_r = -\sum_{i=1}^{2}\left[s^2 K_0(m_i r) + \frac{m_i(1 - c_1)}{r}K_1(m_i r)\right]A_i, \tag{46}$$

(47)

$$\bar{\sigma}_\theta = -\sum_{i=1}^{2}\left\{[s^2 - m_i^2(1 - c_1)]K_0(m_i r) - \frac{m_i(1 - c_1)}{r}K_1(m_i r)\right\}A_i,$$

$$\bar{\sigma}_z = -\sum_{i=1}^{2}\left\{\left[\frac{m_i^2 c_3}{2} - \frac{\beta 3}{\beta_1}(m_i^2 - s^2)\right] K_0(m_i r) - \frac{m_i c_4}{r} K_1(m_i r) + \frac{m_i^2 c_3}{2} K_2(m_i r)\right\} A_i. \tag{48}$$

The boundary conditions appeared in equations (26) and (25), after using Laplace transform, take the form

$$\bar{\psi}(R,s) = \Theta_0\left(\frac{1}{2} + \frac{k_1}{2s}\right) = \bar{G}(s), \tag{49}$$

$$\bar{\sigma}_r(R,s) = 0. \tag{50}$$

The substitution of equations (43) and (46) into the above conditions gives two equations in the unknown parameters A_i as

$$\sum_{i=1}^{2}(m_i^2 - s^2) A_i K_0(m_i R) = \tau_2 \bar{G}(s), \tag{51}$$

$$\sum_{i=1}^{2}\left[s^2 K_0(m_i R) + \frac{m_i(1-c_1)}{R} K_1(m_i R)\right] A_i = 0. \tag{52}$$

After getting A_i, the solution of the problem may be completed in the Laplace transform domain. Furthermore, the temperature $\bar{\Theta}$ can be obtained by solving equation (11) after application of the Laplace transform as

$$\bar{\Theta}(r,s) = \frac{-1 + \sqrt{1 + 2k_1\bar{\psi}}}{k_1}. \tag{53}$$

5. Discussions of numerical results

The distributions of the field quantities should be obtained inside the medium in their inverted forms. To invert the Laplace transform in equations (44), (46)-(48) and (53), a numerical inversion method based on a Fourier series expansion [24], [25] should be adopted. Any expression in Laplace domain can be inverted in this method to the time domain as

$$f(t) = \frac{e^{ct}}{t}\left\{\frac{1}{2}\bar{f}(c) + \mathrm{Re}\left[\sum_{n=1}^{N}(-1)^n \bar{f}\left(c + \frac{in\pi}{t}\right)\right]\right\}, \tag{54}$$

where the value of c should be satisfy the relation $ct \approx 4.7$ as mentioned in numerous numerical experiments [31]. For numerical purpose, one can use the properties of *Cobalt* material in SI units [32].

The reference temperature $T_0 = 298$ K is used during the numerical results of all quantities. The dimensionless temperature Θ, radial displacement u, thermal stresses σ_r, σ_θ and σ_z are plotted for thermoviscoelastic (TVE) solid ($t_0 \neq 0$) and thermoelastic (TE) solid ($t_0 = 0$) at different values of $r \geq 1$. The results have been illustrated in Figures 1-5 for three cases. The first one is devoted to investigated the effect of the DPLs t_q and t_Θ on the field quantities when the variability thermal conductivity parameter k_1 and the mechanical relaxation time due to the viscosity t_0 remain constants ($k_1 = -0.5$ and $t_0 = 0.1$). The second case is devoted to discuss the effects of the variability thermal conductivity parameter k_1 on the field quantities of TVE solid ($t_0 = 0.1$) when t_q and t_Θ remain constants ($t_q = 0.2$ and $t_\Theta = 0.1$). In this case, three different values of k_1 are considered. The values $k_1 = -1$ and -0.5 are taken for variable thermal conductivity and $k_1 = 0$ for temperature-independent thermal conductivity. Finally, the effects of mechanical relaxation time due to the viscosity parameter t_0 on the field quantities is presented in the third case ($t_q = 0.2$, $t_\Theta = 0.1$ and $k_1 = -0.5$). The comparisons of the dimensionless physical quantities are made for the TVE solid when $t_0 = 0.2$ and 0.1 and the TE solid when $t_0 = 0$. Figure 1 shows three plots of the distribution of temperature Θ along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0. Similar plots for the distributions of radial displacement u and thermal stresses σ_r, σ_θ and σ_z are shown in Figures 2-5, respectively. Figure 1 shows that the temperature Θ decreases along the radial direction. Figure 1(a) shows that the variation of Θ in the context of DPL and CTE models follows similar trends while the LS model may be different. Figure 1(b, c) shows that Θ is increasing as k_1 and t_0 increase. It can be observed from Figure 1(c) that the viscosity parameter t_0 acts to increase the magnitude of the temperature distribution. The temperature distribution in the TVE solid may be larger than the corresponding one in the TE solid. Figure 2 shows that the distribution of radial displacement u starts with negative values in all cases, and it is monotonically increasing to get its maximum values at different positions. Figure 2(a) shows that u_{max} occurs at different positions according to the used model. The variation of u has the same behavior in the context of DPL and CTE model of thermoviscoelasticity while the behavior of u due to the LS model may be different. Figure 2(b) shows that u_{max} occurs at the same position ($r \approx 1.12$) and then u gradually diminishes to zero. In fact, the radial displacement u vanishes twice, the first at $r \approx 1.09$ and the second at $r = 2$. As k_1 decreases u increases in the interval $1 \leq r \leq 1.09$ and decreases in the interval $1.09 \leq r \leq 2$. From Figure 2(c) it is observed that, when the value of the viscosity parameter t_0 increases, the absolute values of the radial displacement u decreases, and the peek takes place when $r = 1.18$. In Figure 3, the distribution of thermal radial stress σ_r starts with a zero value at $r = 1$ for all cases which agrees with the boundary condition. Figure 3(a) shows that the variation of σ_r in the context of DPL and CTE models of thermoelasticity follows similar trends while the LS model may be different. Figure 3(b) shows that σ_r is continuously increasing to attain its highest values at $r \approx 1.07$ then it decreases to attain its lowest values at $r \approx 1.22$. It is to be noted that the increase of k_1 acts to increase the magnitude of the wave of thermal stress σ_r. In Figure 3(c), when the value of the viscosity parameter t_0 increases, the absolute value of the thermal radial stress σ_r increases along the radial direction. Figure 4 shows that the thermal hoop stress σ_θ starts with negative values and continuously vibrates along the radial direction. Figure 4(a) shows that the variation of σ_θ in the context of DPL and CTE models follows similar trends while the LS model may be different. Figure 4(b) shows that σ_θ increases as the parameter k_1 decreases. The difference in the values of σ_θ at a particular point for three different values of viscosity parameter t_0 can easily be observed in Figure 4(c). It is obvious that the thermal hoop stress σ_θ is always compressive along the radial direction. Figure 5 shows that the distribution of thermal axial stress σ_z starts with value above zero at $r = 1$ for all cases. As usual, Figure 5(a) shows that the variation of σ_z in the context of DPL and CTE models follows similar trends while the LS model may be different. Figure 5(b) shows that σ_z is continuously increasing to attain its highest values at $r \approx 1.07$ then it decreases to attain its lowest values at $r \approx 1.22$. It is clear that σ_z vanishes twice, the first at $r \approx 1.1$ and the second at $r = 2$. As k_1 increases the thermal axial stress σ_z increases in the interval $1 \leq r \leq 1.1$ and decreases in the interval $1.1 \leq r \leq 2$. In Figure 3(c), when the value of the viscosity parameter t_0 increases, the thermal axial stress σ_z decreases in the interval $1.19 \leq r \leq 2$. However, in the interval $1 \leq r \leq 1.9$ it is observed that σ_z for TVE solid ($t_0 = 0.2$ and $t_0 = 0.1$) is larger than the corresponding one for TE solid ($t_0 = 0$).

6. Conclusions

In this work, the equations of generalized thermoviscoelasticity are obtained for a homogeneous orthotropic infinite unbounded solid containing a cylindrical cavity with a variable thermal conductivity based on the DPL model. The Lord and Shulman's model and the coupled thermoelasticity model are also employed to study the thermomechanical coupling, thermal and mechanical relaxation effects. The outer surface of the cylindrical cavity is taken to be traction-free and subjected to a time-dependent thermal shock. Numerical results for the field quantities are illustrated in many plots. Comparisons between thermoelasticity models are made and the effects of different parameters are discussed. It is seen that the viscous effect plays an important role and its variation is more pronounced in the thermoviscoelastic solid. The speed of the wave propagation of all field quantities is very sensitive to the variation of variability thermal conductivity parameter. The results presented in this article should prove useful for investigators in the development of mechanics of solids.

References

[1] M. A. Ezzat, M. I. Othman and A. S. El-Karamany, *State space approach to generalized thermo-viscoelasticity with two relaxation times*, Int. J. Eng. Sci. **40(3)** (2002), 283–302.

[2] A. M. Abd-Alla, H.A.H. Hammad and S. M. Abo-Dahab, *Magneto-thermo-viscoelastic interactions in an unbounded body with a spherical cavity subjected to a periodic loading*, Appl. Math. Comput. **155(1)** (2004), 235–248.

[3] M. Aouadi and A.S. El-Karamany, *The relaxation effects of volume properties in two-dimensional generalized thermoviscoelastic problem*, Appl. Math. Comput. **151(3)** (2004), 689–711.

[4] M. I. A. Othman, *Effect of rotation and relaxation time on a thermal shock problem for a half-space in generalized thermo-viscoelasticity*, Acta Mech, **174(3-4)** (2005), 129–143.

[5] X. Tian and Y. Shen, *Study on generalized magneto-thermoelastic problems by FEM in time domain*, Acta Mech. Sinica **21(4)** (2005), 380–387.

[6] M. Rakshit and B. Mukhopadhyay, *A two dimensional thermoviscoelastic problem due to instantaneous point heat source*, Math. Comput. Model. **46(11-12)** (2007), 1388–1397.

[7] N. Sarkar and A. Lahiri, *The effect of fractional parameter on a perfect conducting elastic half-space in generalized magneto-thermoelasticity*, Mecc, **48(1)** (2013), 231–245.

[8] M. A. Ezzat, A. S. El-Karamany and A. A. El-Bary, *Generalized thermo-viscoelasticity with memory-dependent derivatives*, Int. J. Mech. Sci. **89** (2014), 470–475.

[9] A. S. El-Karamany and M. A. Ezzat, *Two-temperature GreenNaghdi theory of type III in linear thermoviscoelastic anisotropic solid*, Appl. Math. Model. **39(8)** (2015), 2155–2171.

[10] A. D. Kovalenko and V. G. Karnaukhov, *A linearized theory of thermoviscoelasticity*, Polymer Mech. **8(2)** (1972), 194–199.

[11] A. D. Drozdov, *A constitutive model in finite thermoviscoelasticity based on the concept of transient networks*, Acta Mech. **133(1)** (1999), 13–37.

[12] M. Rakshit Kundu and B. Mukhopadhyay, *A thermoviscoelastic problem of an infinite medium with a spherical cavity using generalized theory of thermoelasticity*, Math. Comput. Model. **41(1)** (2005), 25–32.

[13] A. Baksi, B. K. Roy and R. K. Bera, *Eigenvalue approach to study the effect of rotation and relaxation time in generalized magneto-thermo-viscoelastic medium in one dimension*, Math. Comput. Model. **44(11-12)** (2006), 1069–1079.

[14] A. Kar and M. Kanoria, *Generalized thermo-visco-elastic problem of a spherical shell with three-phase-lag effect*, Appl. Math. Model. **33(8)** (2009), 3287–3298.

[15] M. Kanoria and S. H. Mallik, *Generalized thermoviscoelastic interaction due to periodically varying heat source with three-phase-lag effect*, Europ. J. Mech. A/Solids **29(4)** (2010), 695–703.

[16] M. A. Ezzat, A. S. El-Karamany, A. A. El-Bary and, M. A. Fayik, *Fractional calculus in one-dimensional isotropic thermo-viscoelasticity*, Compt. Rend. Mec. **341(7)** (2013), 553–566.

[17] S. Deswal and K. K. Kalkal, *Fractional order heat conduction law in micropolar thermo-viscoelasticity with two temperatures*, Int. J. Heat Mass Transfer **66** (2013), 451–460.

[18] S. Deswal and K. K. Kalkal, *Three-dimensional half-space problem within the framework of two-temperature thermo-viscoelasticity with three-phase-lag effects*, Appl. Math. Model. **39(23-24)** (2015), 7093–7112.

[19] A. E. Abouelregal, *Generalized thermoelasticity for an isotropic solid sphere in dual-phase-lag of heat transfer with surface heat flux*, Int. J. Comput. Meth. Eng. Sci. Mech. **12(2)** (2011), 96–105.

[20] A. M. Zenkour, D. S. Mashat and A. E. Abouelregal, *The effect of dual-phase-lag model on reflection of thermoelastic waves in a solid half space with variable material properties*, Acta Mech. Solida Sinica **26(6)** (2013), 659–670.

[21] I. A. Abbas and A. M. Zenkour, *Dual-phase-lag model on thermoelastic interactions in a semi-infinite medium subjected to a ramp-type heating*, J. Comput. Theor. Nanosci. **11(3)** (2014), 642–645.

[22] A. E. Abouelregal and A. M. Zenkour, *Effect of phase lags on thermoelastic functionally graded microbeams subjected to ramp-type heating*, IJST, Trans. Mech. Eng. **38(M2)** (2014), 321–335.

[23] A. M. Zenkour, Two-dimensional coupled solution for thermoelastic beams via generalized dual-phase-lags model, Math. Model. Analys **21(3)** (2016), 319–335.

[24] D. Y. Tzou, *A unified approach for heat conduction from macro- to micro-scales*, J. Heat Transfer **117(1)** (1995), 8–16.

[25] D. Y. Tzou, *Macro to Micro-scale Heat Transfer: The Lagging Behavior*, Taylor and Francis, Washington DC, 1996.

[26] H. W. Lord and Y. Shulman, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids **15(5)** (1967), 299–309.

[27] E. Green and K. A. Lindsay, *Thermoelasticity*, J. Elast. **2(1)** (1972), 1–7.
[28] A. E. Green and P. M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elast. **31(3)** (1993), 189–209.
[29] A. C. Eringen, *Mechanic of Continua*, John Wiley, Sons Inc., New York, 1967.
[30] N. Noda, *Thermal Stresses in Materials with Temperature-dependent Properties*, Thermal Stresses I, R.B. Hetnarski (Editor), North-Holland, Amsterdam, 1986.
[31] G. Honig and U. Hirdes, *A method for the numerical inversion of Laplace transform*, J. Comp. Appl. Math. **10(1)** (1984), 113–132.
[32] J. C. Misra, N. C. Chattopadhyay and S. C. Samanta, *Study of the thermoelastic interactions in an elastic half space subjected to a ramp-type heatinga statespace approach*, Int. J. Eng. Sci. **34(5)** (1996), 579–596.

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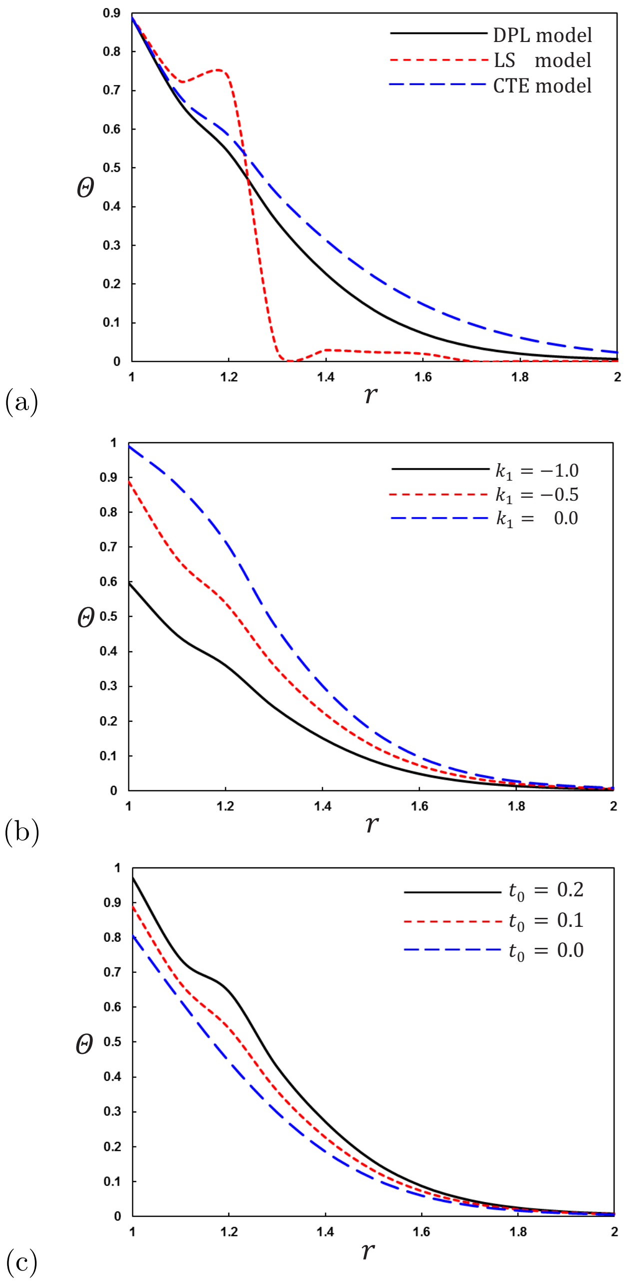


FIGURE 1. The distribution of temperature Θ along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0.

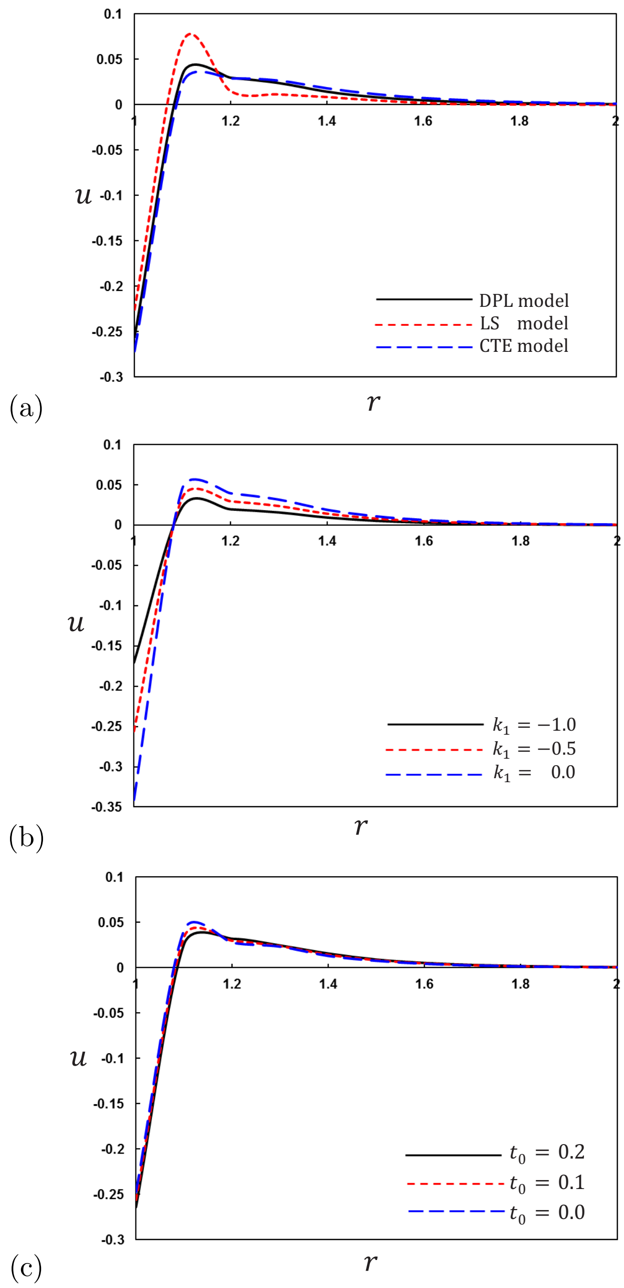


FIGURE 2. The distribution of displacement u along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0.

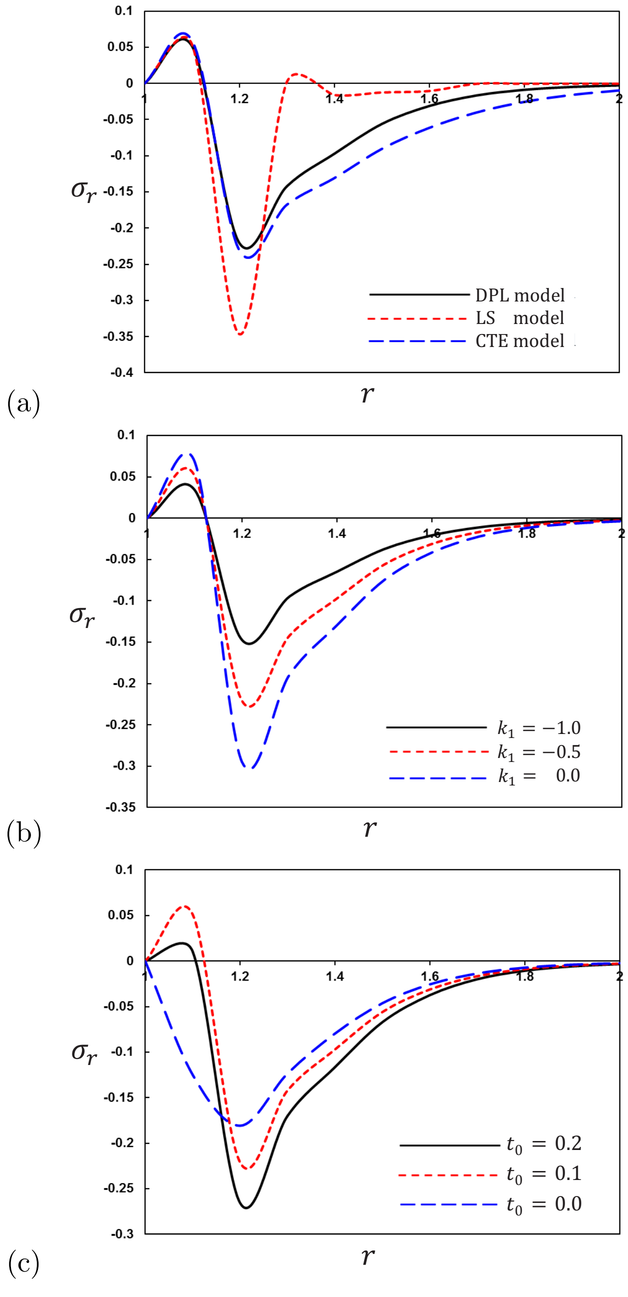
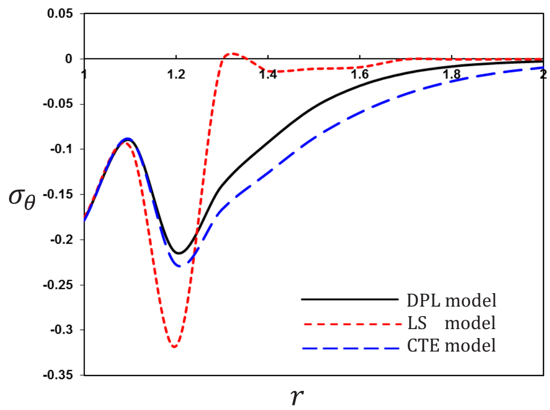
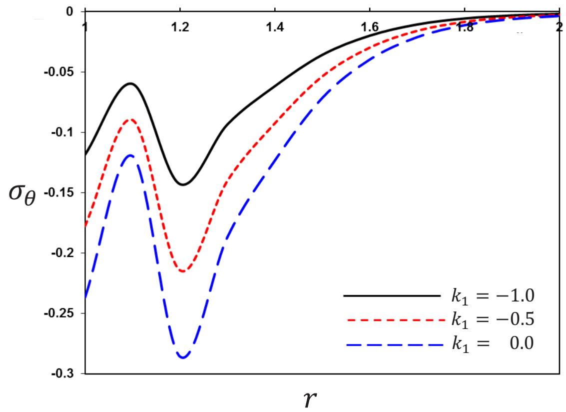


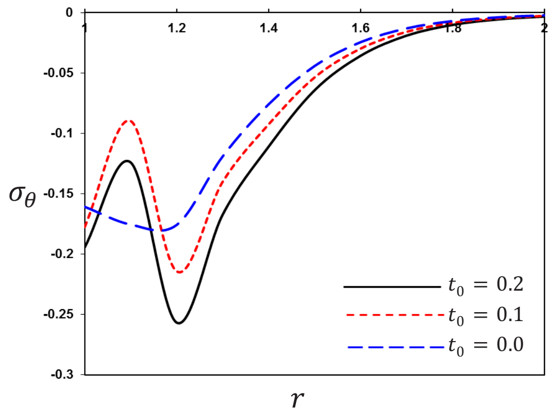
FIGURE 3. The distribution of radial stress σ_r along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0.



(a)



(b)



(c)

FIGURE 4. The distribution of hoop stress σ_θ along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0.

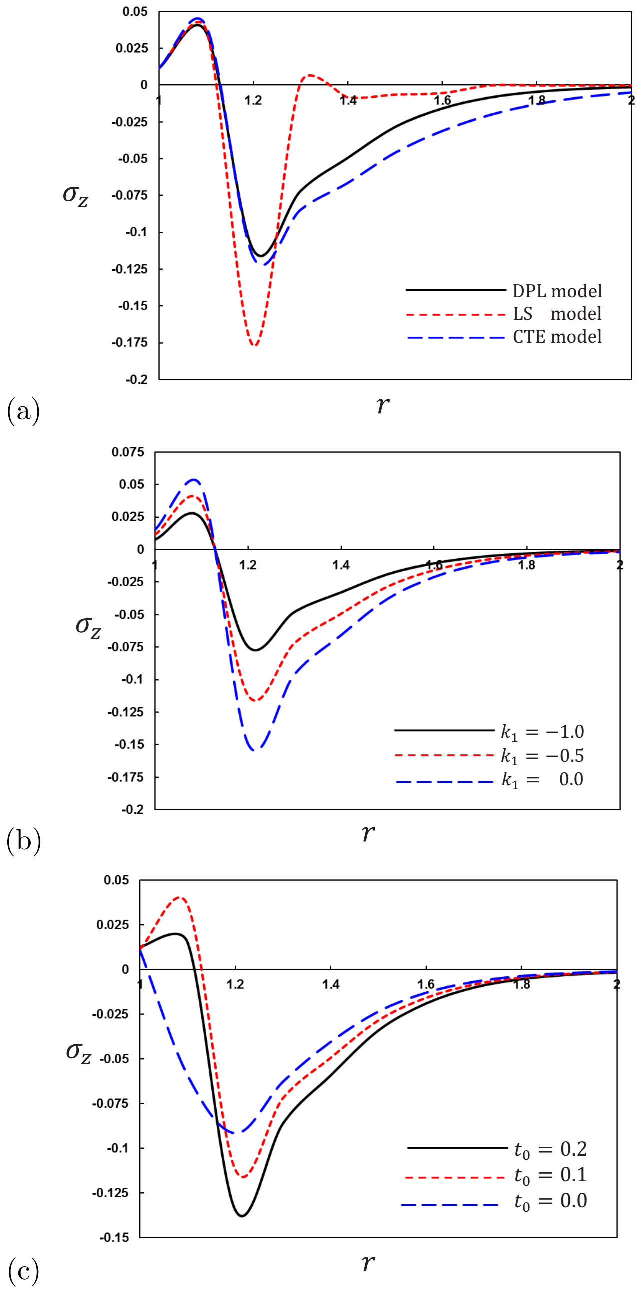


FIGURE 5. The distribution of axial stress σ_z along the radial direction for different: (a) theories of thermoelasticity, (b) thermal conductivity parameter k_1, and (c) viscosity parameter t_0.