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FRACTIONAL CALCULUS OPERATORS AND THEIR IMAGE FORMULAS

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Abstract. During the past four decades or so, due mainly to a wide range of applications from natural sciences to social sciences, the so-called fractional calculus has attracted an enormous attention of a large number of researchers. Many fractional calculus operators, especially, involving various special functions, have been extensively investigated and widely applied. Here, in this paper, in a systematic manner, we aim to establish certain image formulas of various fractional integral operators involving diverse types of generalized hypergeometric functions, which are mainly expressed in terms of Hadamard product. Some interesting special cases of our main results are also considered and relevant connections of some results presented here with those earlier ones are also pointed out.

1. Introduction

We begin by giving a brief outline of fractional calculus and its development. Fractional calculus is a branch of mathematics, which has a long history and has recently gone through a period of rapid development. Many earlier works on the subject of fractional calculus contain interesting accounts of the theories of fractional calculus operators and their applications in diverse research areas (see, *e.g.*, Caputo [11], Oldham and Spanier [36], Ross [43], McBride and Roach [31], Nishimoto [35], Miller and Ross [32]), Podlubny [40], Samko *et al.* [48], Hilfer [18], Kilbas *et al.* [21], and the five volume works written by Nishimoto [33]). The fractional calculus operators have been extensively used in describing and solving various integral equations, ordinary differential equations and partial differential equations in applied sciences such as fluid mechanics, rheology, diffusive transport, electrical networks, electromagnetic

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theory, probability, turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, non-linear control theory, image processing, non-linear biological systems and astrophysics.

In recent years, fractional integral and differential operators involving the various special functions have been investigated by many authors, for example, Kalla and Saxena [19], Kilbas and Saigo [20], Saigo [44], Kiryakova [22, 23, 24, 25], Saigo and Kilbas [46], in particular, Srivastava and Saxena [59] presented a survey-cum-expository paper which gives a remarkably clear, insightful, and systematic exposition of the investigations carried out by various authors in the field of fractional calculus and its applications and contains a fairly comprehensive bibliography of as many as 190 *further* references on the subject (see also [16]).

Due mainly to their various applications, image formulas of fractional calculus operators have attracted not only mathematicians and statisticians with diverse research interests but also electrical engineers, biologists, economists, psychologists, and sociologists. Here, in this paper, in a systematic manner, we establish certain image formulas of fractional integral operators involving some new generalized Gauss hypergeometric type functions. Also importance of the image formulas of the fractional calculus operators is highlighted and shared with the interested readers.

2. Generalized special functions

Many important functions in applied sciences (which are popularly known as special functions) are defined via improper integrals or infinite series (or infinite products). During last four decades or so, several interesting and useful extensions of many of the familiar special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been considered by many authors (see, *e.g.*, in a chronological way, [12], [13], [15], [37], [38], [53], [55], [56]). Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, integers, and positive integers, respectively, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For our present investigation, we recall some required special functions. The generalized hypergeometric series ${}_pF_q$ (p, $q \in \mathbb{N}_0$) is defined by (see [42, p. 73] and [57, pp. 71–75]):

$$ {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \tag{2.1} $$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [57, p. 2 and p. 5]):

$$ (\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \tag{2.2} $$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

and $\Gamma(\lambda)$ is the familiar Gamma function.

The generalized Beta function $B_p^{(\alpha,\beta;\kappa,\mu)}(x,y)$ is defined by (see [55])

$$B_p^{(\alpha,\beta;\kappa,\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa(1-t)^\mu}\right)dt \tag{2.3}$$

$$\big(\Re(p) \geqq 0;\ \min\{\Re(x),\ \Re(y),\ \Re(\alpha),\ \Re(\beta)\} > 0;\ \min\{\Re(\kappa),\ \Re(\mu)\} > 0\big).$$

When $\kappa = \mu$, (2.3) reduces to the generalized extended beta function

$$B_p^{(\alpha,\beta;\mu)}(x,y)$$

defined by (see [39, p. 37])

$$B_p^{(\alpha,\beta;\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t^\mu(1-t)^\mu}\right)dt \tag{2.4}$$

$$\big(\Re(p) \geqq 0;\ \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\} > 0;\ \Re(\mu) > 0\big).$$

The special case of (2.4) when $\mu = 1$ reduces immediately to the generalized Beta type function as follows (see [38, p. 4602]):

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y) &= B_p^{(\alpha,\beta;1)}(x,y) \\ &:= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt \end{aligned} \tag{2.5}$$

$$\big(\Re(p) \geqq 0;\ \min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\} > 0\big).$$

The further special case of (2.5) when $\alpha = \beta$ reduces obviously to the extended Beta type function $B_p(x,y)$ due to Chaudhry *et al.* [12]:

$$\begin{aligned} B_p(x,y) &= B_p^{(\alpha,\alpha)}(x,y) \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)dt \qquad \big(\Re(p) \geqq 0\big). \end{aligned} \tag{2.6}$$

The classical beta function $B(x,y)$ is defined by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad \big(\Re(x) > 0;\ \Re(y) > 0\big). \tag{2.7}$$

It is clear to see the following relationship between the classical Beta function $B(x,y)$ and its extensions:

$$B(x,y) = B_0(x,y) = B_0^{(\alpha,\beta)}(x,y) = B_0^{(\alpha,\beta;1)}(x,y) = B_0^{(\alpha,\beta;1,1)}(x,y).$$

Chaudhry *et al.* [13, p. 591, Eqs. (2.1) and (2.2)] made use of the extended Beta function $B_p(x,y)$ in (2.6) to extend the Gauss hypergeometric function

${}_2F_1$ as follows: The extended Gauss hypergeometric function $F_p(a,b;c;z)$ is defined by

$$F_p(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{2.8}$$

$$\big(|z|<1;\ \Re(c)>\Re(b)>0;\ \Re(p) \geqq 0\big).$$

Similarly, by appealing to the generalized Beta function $B_p^{(\alpha,\alpha)}(x,y)$ in (2.5), Özergin [12] and Özergin *et al.* [13] introduced and investigated a further potentially useful extension of the generalized Gauss hypergeometric functions as follows: The extended generalized Gauss hypergeometric functions $F_p^{(\alpha,\beta)}(\cdot)$ is defined by

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{2.9}$$

$$\big(|z|<1;\ \min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(c)>\Re(b)>0;\ \Re(p) \geqq 0\big).$$

Based upon the generalized Beta function in (2.4), Parmar [39] introduced and studied a family of the generalized Gauss hypergeometric functions

$$F_p^{(\alpha,\beta;\mu)}(\cdot)$$

defined by

$$F_p^{(\alpha,\beta;\mu)}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\mu)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{2.10}$$

$$\big(|z|<1;\ \min\{\Re(\alpha),\Re(\beta),\Re(\mu))>0;\ \Re(c)>\Re(b)>0;\ \Re(p) \geqq 0\big).$$

Recently, Srivastava *et al.* [53] used the generalized Beta function in (2.3) to introduce a family of some extended generalized Gauss hypergeometric functions defined by

$$F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{2.11}$$

$$\big(|z|<1;\ \min\{\Re(\alpha),\Re(\beta),\Re(\kappa),\Re(\mu))>0;\ \Re(c)>\Re(b)>0;\ \Re(p) \geqq 0\big).$$

It is easy to see the following relationships:

$$F_p^{(\alpha,\beta;1,1)}(a,b;c;z) = F_p^{(\alpha,\beta)}(a,b;c;z);$$

$$F_p^{(\alpha,\beta;1)}(a,b;c;z) = F_p^{(\alpha,\beta)}(a,b;c;z);$$

$$F_p^{(\alpha,\alpha;1)}(a,b;c;z) = F_p(a,b;c;z);$$

and

$$F_0^{(\alpha,\alpha;1)}(a,b;c;z) = {}_2F_1(a,b;c;z).$$

Very recently, Luo *et al.* [28] investigated various properties of these extended functions and established some connections with the Laguerre polynomial and Fox's H-function.

In recent years the incomplete Gamma type functions like $\gamma(s,x)$ and $\Gamma(s,x)$ have been investigated by a number of researchers. It is noted that both $\gamma(s,x)$ and $\Gamma(s,x)$ which are given in (2.12) and (2.13), respectively, are certain generalizations of the classical Gamma function $\Gamma(z)$ and have proved to be important for physicists and engineers as well as mathematicians. For more details, one may refer to the following literature: [1], [14], [15], [27], [50], [51], [52], [53], [54], [60] and [62].

The incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ are defined by

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt \quad (\Re(s) > 0;\ x \geq 0) \tag{2.12}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt(x \geq 0;\ \Re(s) > 0 \text{ when } x = 0). \tag{2.13}$$

The (2.12) and (2.13) satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \quad (\Re(s) > 0). \tag{2.14}$$

The theory of the incomplete Gamma functions, as a part of the theory of confluent hypergeometric functions, has received its first systematic exposition by Tricomi [61]. Al-Musallam and Kalla (see [8] and [9]) considered a more general incomplete gamma function involving the Gauss hypergeometric function and established a number of analytic properties including recurrence relations, asymptotic expansions and computation for special values of the parameters. Very recently, Srivastava *et al.* [56] introduced and studied some fundamental properties and characteristics of a family of two potentially useful and generalized incomplete hypergeometric functions defined as follows:

$${}_p\gamma_q\left[\begin{matrix}(a_1,x), a_2, \ldots, a_p; \\ b_1, \ldots, b_q\,;\end{matrix}\ z\right] := \sum_{n=0}^{\infty} \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!} \tag{2.15}$$

and

$${}_p\Gamma_q\left[\begin{matrix}(a_1,x), a_2, \ldots, a_p; \\ b_1, \ldots, b_q\,;\end{matrix}\ z\right] := \sum_{n=0}^{\infty} \frac{[a_1;x]_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!}, \tag{2.16}$$

where $(a_1;x)_n$ and $[a_1;x]_n$ which are interesting generalizations of the Pochhammer symbol $(\lambda)_n$ are defined in terms of the incomplete gamma type functions $\gamma(s,x)$ and $\Gamma(s,x)$ as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\ \nu \in \mathbb{C};\ x \geq 0) \tag{2.17}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\ \nu \in \mathbb{C};\ x \geq 0). \tag{2.18}$$

These incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \quad (\lambda,\nu\in\mathbb{C};\ x\geq 0). \tag{2.19}$$

Remark 2.1. For the convergence of the infinite series (2.15) and (2.16), one may refer to Srivastava *et al.* [56, Remark 7].

3. Operators of fractional integration

A number of fractional integral operators have been developed and investigated extensively, due mainly to the importance and usefulness in both theoretical and applicable senses. For our present investigation, we recall some well-known fractional integral operators.

Appell's hypergeometric function F_3 in two variables (see, *e.g.*, [10, p. 14] and [58, p. 23]) is defined by

$$\begin{aligned} &F_3(\alpha,\alpha',\beta,\beta';\eta;x;y)\\ &\quad=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\quad(\max\{|x|,|y|\}<1). \end{aligned} \tag{3.1}$$

Let α, α', β, β', $\eta\in\mathbb{C}$. Then the fractional integral operators $I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}$ and $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}$ of a function $f(x)$ are defined, for $\Re(\eta)>0$, as follows (see Saigo and Maeda [47]; see also Choi and Kumar [17]):

$$\begin{aligned} &\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)\\ &\quad=\frac{x^{-\alpha}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}\,t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-t/x,1-x/t\right)f(t)\,dt \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)\\ &\quad=\frac{x^{-\alpha'}}{\Gamma(\eta)}\int_x^\infty (t-x)^{\eta-1}\,t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-x/t,1-t/x\right)f(t)\,dt, \end{aligned} \tag{3.3}$$

where the function $f(t)$ is so constrained that the defining integrals in (3.2) and (3.3) exist.

The above fractional integral operators can be written as follows:

$$\begin{aligned} \left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)&=\left(\frac{d}{dx}\right)^k\left(I_{0,x}^{\alpha,\alpha',\beta+k,\beta',\eta+k}f\right)(x)\\ &(\Re(\eta)>0;k=[\Re(\eta)]+1) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)&=\left(-\frac{d}{dx}\right)^k\left(I_{0,x}^{\alpha,\alpha',\beta,\beta'+k,\eta+k}f\right)(x)\\ &(\Re(\eta)>0;k=[\Re(\eta)]+1). \end{aligned} \tag{3.5}$$

The operators or integral transforms in (3.2) and (3.3) were introduced by Marichev [29] as Mellin type convolution operators with the Appell function F_3 in their kernel. These operators were rediscovered and studied by Saigo [45] as generalizations of the so-called Saigo fractional integral operators (see also Kiryakova [25]). Such further properties as (for example) their relations with the Mellin transform and with the hypergeometric operators (or the Saigo fractional integral operators), together with their decompositional, operational and other properties in the McBride space Fp (see [31]) were studied by Saigo and Maeda [47] (see also some recent investigations on the subject of fractional calculus in Agarwal [2, 3, 4], Agarwal and Jain [6], Agarwal *et al.* [7], Agarwal *et al.* [5] and [34]).

The Appell function F_3 in (3.2) and (3.3) satisfies a system of two linear partial differential equations of the second order and reduces to the Gauss hypergeometric function ${}_2F_1$ as follows (see [10, p. 25, Eq. (35)] and [58, p. 301, Eq. 9.4 (87)]):

$$F_3(\alpha, \eta-\alpha, \beta, \eta-\beta; \eta; x; y) = {}_2F_1(\alpha, \beta; \eta; x+y-xy). \tag{3.6}$$

Further it is easy to see that

$$F_3(\alpha, 0, \beta, \beta', \eta; x, y) = {}_2F_1(\alpha, \beta; \eta; x) \tag{3.7}$$

and

$$F_3(0, \alpha', \beta, \beta', \eta; x, y) = {}_2F_1(\alpha', \beta'; \eta; y). \tag{3.8}$$

In view of the obvious reduction formula (3.7), the general operators reduce to the aforementioned Saigo operators $I_{0,x}^{\alpha,\beta,\eta}$ and $I_{x,\infty}^{\alpha,\beta,\eta}$ (see, for details, [44] and the references cited therein) defined as follows:

$$\left(I_{0,x}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \qquad (\Re(\alpha) > 0) \tag{3.9}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt \qquad (\Re(\alpha) > 0), \tag{3.10}$$

where the function $f(t)$ is so constrained that the defining integrals in (3.9) and (3.10) exist.

The Saigo fractional integral operators (3.9) and (3.10) can also be written in the following form:

$$\left(I_{0,x}^{\alpha,\beta,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\alpha+k,\beta-k,\eta-k} f\right)(x) \qquad (\Re(\alpha) \leq 0;\ k = [\Re(-\alpha)] + 1) \tag{3.11}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}f\right)(x)=\left(-\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{\alpha-k,\beta-k,\eta}f\right)(x) \tag{3.12}$$
$$(\Re(\alpha)\leq 0; k=[\Re(-\alpha)]+1).$$

The Erdélyi-Kober type fractional integral operators are defined as follows (see Kober [26]):

$$\left(\mathcal{E}_{0,x}^{\alpha,\eta}f\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,t^{\eta}\,f(t)dt \quad (\Re(\alpha)>0) \tag{3.13}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\alpha,\eta}f\right)(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\eta}\,f(t)dt \quad (\Re(\alpha)>0)\,, \tag{3.14}$$

where the function $f(t)$ is so constrained that the defining integrals in (3.13) and (3.14) converge.

The Riemann–Liouville fractional integral operator and the Weyl fractional integral operator are defined as follows (see, *e.g.*, [36]):

$$\left(\mathcal{R}_{0,x}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,f(t)dt \quad (\Re(\alpha)>0) \tag{3.15}$$

and

$$\left(\mathcal{W}_{x,\infty}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt \quad (\Re(\alpha)>0)\,, \tag{3.16}$$

provided both integrals converge.

4. Relations among the operators

We recall some known relationships between the fractional integral operators provided in the previous section. In view of the reduction formula (3.7), Saxena and Saigo [49, p. 93, Eqs. (2.15) and (2.16)] found the following relationship between the Marichev-Saigo-Maeda and the Saigo fractional integral operators:

$$\left(I_{0,x}^{\alpha,0,\beta,\beta',\eta}f\right)(x)=\left(I_{0,x}^{\eta,\alpha-\eta,-\beta}f\right)(x) \quad (\eta\in\mathbb{C}) \tag{4.1}$$

and

$$\left(I_{x,\infty}^{\alpha,0,\beta,\beta',\eta}f\right)(x)=\left(I_{x,\infty}^{\eta,\alpha-\eta,-\beta}f\right)(x) \quad (\eta\in\mathbb{C}). \tag{4.2}$$

The operator $I_{0,x}^{\alpha,\beta,\eta}(\cdot)$ contains both the Riemann-Liouville and Erdélyi-Kober fractional integral operators by means of the following relationships (see Kilbas [21]):

$$\left(\mathcal{R}_{0,x}^{\alpha}f\right)(x)=\left(I_{0,x}^{\alpha,-\alpha,\eta}f\right)(x) \tag{4.3}$$

and

$$\left(\mathcal{E}_{0,x}^{\alpha,\eta}f\right)(x)=\left(I_{0,x}^{\alpha,0,\eta}f\right)(x), \tag{4.4}$$

while the operator $I_{x,\infty}^{\alpha,\beta,\eta}(\cdot)$ unifies the Weyl and Erdélyi-Kober fractional integral operators as follows:

$$\left(\mathcal{W}_{x,\infty}^{\alpha} f\right)(x) = \left(I_{x,\infty}^{\alpha,-\alpha,\eta} f\right)(x) \tag{4.5}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\alpha,\eta} f\right)(x) = \left(I_{x,\infty}^{\alpha,0,\eta} f\right)(x). \tag{4.6}$$

5. Power function formulas

Some required power function formulas of the familiar fractional integral operators are recalled as in the following Lemma 5.1 (see [47] and [49]) and Lemma 5.2 (see [44]).

Lemma 5.1. *Let α, α', β, β' and $\eta \in \mathbb{C}$ with $\Re(\eta) > 0$. Then the following formulas hold true*:

$$\begin{aligned} &\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta} x^{\rho-1}\right)(x) \\ &= \frac{\Gamma(\rho)\Gamma(\rho+\eta-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\eta-\alpha-\alpha')\Gamma(\rho+\eta-\alpha'-\beta)} x^{\rho+\eta-\alpha-\alpha'-1} \\ &\left(\Re(\rho) > \max\left\{0, \Re(\alpha+\alpha'+\beta-\eta), \Re(\alpha'-\beta')\right\}\right) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta} x^{\rho-1}\right)(x) \\ &= \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\eta+\alpha+\alpha')\Gamma(1-\rho+\alpha+\beta'-\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\eta)\Gamma(1-\rho+\alpha-\beta)} x^{\rho+\eta-\alpha-\alpha'-1} \\ &\left(\Re(\rho) < 1 + \min\left\{\Re(-\beta), \Re(\alpha+\alpha'-\eta), \Re(\alpha+\beta'-\eta)\right\}\right). \end{aligned} \tag{5.2}$$

Lemma 5.2. *Let α, β, η, $\rho \in \mathbb{C}$ with $\Re(\alpha) > 0$. Then the following formulas hold true*:

$$\left(I_{0,x}^{\alpha,\beta,\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\eta+\alpha)} x^{\rho-\beta-1} \quad \left(\Re(\rho) > \max\left\{0, \Re(\beta-\eta)\right\}\right) \tag{5.3}$$

and

$$\left(I_{x,\infty}^{\alpha,\beta,\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\beta)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\beta+\eta)} x^{\rho-\beta-1} \quad \left(\Re(\rho) < 1 + \min\left\{\Re(\beta), \Re(\eta)\right\}\right). \tag{5.4}$$

The special case of (5.3) and (5.4) when $\beta = -\alpha$ yields, respectively, two power function formulas involving the Riemann-Liouville and the Weyl type fractional integral operators as in the following lemma (see [30]).

Lemma 5.3. *Let α, $\rho \in \mathbb{C}$. Then the following formulas hold true*:

$$\left(\mathcal{R}_{0,x}^{\alpha} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)} x^{\rho+\alpha-1} \quad \left(\Re(\alpha)>0,\ \Re(\rho)>0\right) \tag{5.5}$$

and

$$\left(\mathcal{W}_{x,\infty}^{\alpha} x^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\alpha)}{\Gamma(1-\rho)} x^{\alpha+\rho-1} \quad \left(\Re(\rho)>\Re(\alpha)>-1\right). \tag{5.6}$$

Setting $\beta = 0$ in (5.3) and (5.4) gives, respectively, two power function formulas involving the Erdélyi-Kober type fractional integral operators as in the following lemma (see [30]).

Lemma 5.4. *Let α, ρ, $\eta \in \mathbb{C}$. Then the following formulas hold true*:

$$\left(\mathcal{E}_{0,x}^{\alpha,\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\alpha+\eta)} x^{\rho-1} \quad \left(\Re(\rho+\eta)>0\right) \tag{5.7}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\alpha,\eta} x^{\rho-1}\right)(x) = \frac{\Gamma(1+\eta-\rho)}{\Gamma(1+\alpha+\eta-\rho)} x^{\rho-1} \quad \left(\Re(\eta)>\Re(\rho)>-1\right). \tag{5.8}$$

6. Fractional integral operators and their image formulas

We present certain fractional integral formulas involving the generalized special functions by using certain general pair of fractional integral operators. To establish our image formulas we require the following concept of the Hadamard products (see [41]).

Definition. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ be two power series whose radii of convergence are given by R_f and R_g, respectively. Then their Hadamard product is the power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n,$$

whose radius of convergence R satisfies $R_f \cdot R_g \leq R$.

If, in particular, one of the power series defines an entire function and the radius of convergence of the other one is greater than 0, then the Hadamard product series defines an entire function, too. We can use the Hadamard product to decompose a newly-emerged function into two known functions. For example, the function ${}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}[z;b]$ can be decomposed as follows:

$$\begin{aligned} &{}_pF_{p+r}^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_1,\ldots,y_{p+r}\end{matrix};z;b\right] \\ &= {}_1F_r\left[\begin{matrix} 1 \\ y_1,\ldots,y_r\end{matrix};z\right] * {}_pF_p^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\ldots,x_p \\ y_{1+r},\ldots,y_{p+r}\end{matrix};z;b\right] \quad (|z|<\infty). \end{aligned}$$

We establish image formulas for the generalized Gauss hypergeometric function involving Saigo-Meada fractional integral operators (3.2) and (3.3), which are expressed in terms of the new generalized Gauss hypergeometric type function $F_p^{(\alpha,\beta;\kappa,\mu)}$, given in Theorems 6.1 and 6.2 below.

Theorem 6.1. *Let $x>0$, $\Re(c)>\Re(b)>0$, $\Re(p)\geq 0$ and the parameters σ, σ', ν, ν', η, ρ, $\gamma\in\mathbb{C}$ such that $\Re(\eta)>0$ and*

$$\Re(\rho)>\max\{0,\Re(\sigma+\sigma'+\nu-\eta),\Re(\sigma'-\nu')\}.$$

Then the following fractional integral formula holds:

$$\begin{aligned}
&\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;\gamma\,t)\right]\right)(x)\\
&= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho+\nu'-\sigma')\Gamma(\rho+\eta-\sigma-\nu-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\nu-\sigma')}\\
&\quad\times F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a,\ b;\\ c;\end{matrix}\gamma\,x\right]\\
&\quad *\ {}_3F_3\left[\begin{matrix}\rho,\rho+\nu'-\sigma',\rho+\eta-\sigma-\nu-\sigma';\\ c,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\nu-\sigma';\end{matrix}\gamma\,x\right].
\end{aligned}\tag{6.1}$$

Proof. For convenience, we denote the left-hand side of the result (6.1) by $\Delta(x)$. Then, using (2.11) and changing the order of integration and summation, which is valid under the conditions given in Theorem 6.1, we get

$$\begin{aligned}
\Delta(x) &:= \left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;\gamma\,t)\right]\right)(x)\\
&= \left(I_{0,x}^{(\sigma,\sigma',\nu,\nu',\eta)}\left[t^{\rho-1}\sum_{n=0}^{\infty}a_n\frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{(\gamma\,t)^n}{n!}\right]\right)(x)\\
&= \sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}.\frac{\gamma^n}{n!}\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho+n-1}\right]\right)(x).
\end{aligned}\tag{6.2}$$

Now, we can make use of (5.1) with ρ replaced by $\rho+n$ $(n\in\mathbb{N}_0)$ to find from (6.2) that

$$\begin{aligned}
\Delta(x) &= x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\\
&\quad\times\frac{\Gamma(\rho+n)\Gamma(\rho+\nu'-\sigma'+n)\Gamma(\rho+\eta-\sigma-\nu-\sigma'+n)}{\Gamma(\rho+\nu'+n)\Gamma(\rho+\eta-\sigma-\sigma'+n)\Gamma(\rho+\eta-\nu-\sigma'+n)}\frac{(\gamma\,x)^n}{n!}\\
&= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho+\nu'-\sigma')\Gamma(\rho+\eta-\sigma-\nu-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\nu-\sigma')}\\
&\quad\times\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}.\\
&\quad\times\frac{(\rho)_n(\rho+\nu'-\sigma')_n(\rho+\eta-\sigma-\nu-\sigma')_n}{(\rho+\nu')_n(\rho+\eta-\sigma-\sigma')_n(\rho+\eta-\nu-\sigma')_n}\frac{(\gamma\,x)^n}{n!}.
\end{aligned}\tag{6.3}$$

Finally applying the definition of Hadamard product to the last expression of (6.3) with the aid of (2.11) is easily seen to yield the right hand side of (6.1). This completes the proof of Theorem 6.1. □

Theorem 6.2. *Let $x > 0$, $\Re(c) > \Re(b) > 0$, $\Re(p) \geq 0$ and the parameters σ, σ', ν, ν', η, ρ, $\gamma \in \mathbb{C}$ satisfying $\Re(\eta) > 0$ and*

$$0 < \Re(\rho) < 1 + \min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}.$$

Then the following fractional integral formula holds true:

$$\begin{aligned}
&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) = x^{\rho+\eta-\sigma-\sigma'-1}\\
&\quad\times\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')}\\
&\quad\times F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b; & \dfrac{\gamma}{x}\\ c; & \end{matrix}\right]\\
&\quad *\ {}_3F_3\left[\begin{matrix} 1-\rho-\nu, 1-\rho-\eta+\sigma+\sigma', 1-\rho-\eta+\sigma+\nu'; & \dfrac{\gamma}{x}\\ 1-\rho, 1-\rho+\sigma-\nu, 1-\rho-\eta+\sigma+\sigma'+\nu'; & \end{matrix}\right].
\end{aligned}\tag{6.4}$$

Proof. For simplicity, we denote the left-hand side of the result (6.4) by $\Omega(x)$. Then, by making use of (2.11), and changing the order of integration and summation, which is justified under the conditions stated in Theorem 6.2, we get

(6.5)

$$\begin{aligned}
\Omega(x) &= \left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\
&= \left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\sum_{n=0}^{\infty} a_n \frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{(\frac{\gamma}{t})^n}{n!}\right]\right)(x)\\
&= \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}.\frac{\gamma^n}{n!}\left(I_{x,\infty}^{(\sigma,\sigma',\nu,\nu',\eta)}\left[t^{\rho-n-1}\right]\right)(x).
\end{aligned}$$

Now, we can make use of (5.2) with ρ replaced by $\rho - n$ $(n \in \mathbb{N}_0)$ to find from (6.5) that

(6.6)

$$\begin{aligned}
\Omega(x) &= x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(1-\rho-\nu+n)}{\Gamma(1-\rho+n)}\\
&\quad\times\frac{\Gamma(1-\rho-\eta+\sigma+\sigma'+n)\Gamma(1-\rho-\eta+\sigma+\nu'+n)}{\Gamma(1-\rho+\sigma-\nu+n)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu'+n)}.\frac{(\frac{\gamma}{x})^n}{n!}\\
&= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma-\nu)}
\end{aligned}$$

$$\times \frac{\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}$$
$$\times \frac{(1-\rho-\nu)_n(1-\rho-\eta+\sigma+\sigma')_n(1-\rho-\eta+\sigma+\nu')_n}{(1-\rho)_n(1-\rho+\sigma-\nu)_n(1-\rho-\eta+\sigma+\sigma'+\nu')_n} \cdot \frac{(\frac{\gamma}{x})^n}{n!}.$$

Finally interpreting the last member of (6.6) by means of Hadamard product and (2.11) is seen to arrive at the right-hand side of (6.4). This completes the proof of Theorem 6.2. □

Corollary 6.3. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$, $\Re(p) \geq 0$ *and the parameters* σ, σ', ν, ν', η, ρ, $\gamma \in \mathbb{C}$ *such that* $\Re(\eta) > 0$ *and*

$$\Re(\rho) > \max\{0, \Re(\sigma+\sigma'+\nu-\eta), \Re(\sigma'-\nu')\}.$$

Then the following fractional integral formula holds true:

$$\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\mu)}(a,b;c;\gamma t)\right]\right)(x) = x^{\rho+\eta-\sigma-\sigma'-1}$$
$$\times \frac{\Gamma(\rho)\Gamma(\rho+\nu'-\sigma')\Gamma(\rho+\eta-\sigma-\nu-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\nu-\sigma')} F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix} a,b; \\ c; \end{matrix}\gamma x\right] \tag{6.7}$$
$$* \; {}_3F_3\left[\begin{matrix} \rho,\rho+\nu'-\sigma',\rho+\eta-\sigma-\nu-\sigma'; \\ c,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\nu-\sigma'; \end{matrix}\gamma x\right].$$

Proof. Setting $\kappa = \mu$ in Theorem 6.1 and using (2.10) yields (6.7). □

Corollary 6.4. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$, $\Re(p) \geq 0$ *and the parameters* σ, σ', ν, ν', η, ρ, $\gamma \in \mathbb{C}$ *satisfying* $\Re(\eta) > 0$ *and*

$$0 < \Re(\rho) < 1 + \min\{\Re(-\nu), \Re(\sigma+\sigma'-\eta), \Re(\sigma+\nu'-\eta)\}.$$

Then the following fractional integral formula holds true:

$$\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.8}$$
$$= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma-\nu)}$$
$$\times \frac{\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')} F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\frac{\gamma}{x}\right]$$
$$* \; {}_3F_3\left[\begin{matrix} 1-\rho-\nu,1-\rho-\eta+\sigma+\sigma',1-\rho-\eta+\sigma+\nu'; \\ 1-\rho,1-\rho+\sigma-\nu,1-\rho-\eta+\sigma+\sigma'+\nu'; \end{matrix}\frac{\gamma}{x}\right].$$

Proof. Setting $\kappa = \mu$ in Theorem 6.2 and using (2.10) yields (6.8). □

Corollary 6.5. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$, $\Re(p) \geq 0$ *and the parameters* σ, σ', ν, ν', η, ρ, $\gamma \in \mathbb{C}$ *such that* $\Re(\eta) > 0$ *and*

$$\Re(\rho) > \max\{0, \Re(\sigma+\sigma'+\nu-\eta), \Re(\sigma'-\nu')\}.$$

Then the following fractional integral formula holds true:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\,F_p^{(\alpha,\beta)}\left(a,b;c;\gamma\,t\right)\right]\right)(x)=x^{\rho+\eta-\sigma-\sigma'-1}\\ &\times\frac{\Gamma(\rho)\Gamma(\rho+\nu'-\sigma')\Gamma(\rho+\eta-\sigma-\nu-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\nu-\sigma')}\,F_p^{(\alpha,\beta)}\left[\begin{matrix}a,b;\\ c;\end{matrix}\gamma\,x\right]\\ &*\;{}_3F_3\left[\begin{matrix}\rho,\rho+\nu'-\sigma',\rho+\eta-\sigma-\nu-\sigma';\\ c,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\nu-\sigma';\end{matrix}\gamma\,x\right].\end{aligned}\tag{6.9}$$

Proof. Setting $\kappa=\mu=1$ in Theorem 6.1 with the aid of (2.9) proves (6.9). □

Corollary 6.6. *Let $x>0$, $\Re(c)>\Re(b)>0$, $\Re(p)\geq 0$ and the parameters σ, σ', ν, ν', η, ρ, $\gamma\in\mathbb{C}$ satisfying $\Re(\eta)>0$ and*

$$0<\Re\left(\rho\right)<1+\min\left\{\Re(-\nu),\,\Re(\sigma+\sigma'-\eta),\,\Re(\sigma+\nu'-\eta)\right\}.$$

Then the following fractional integral formula holds true:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\,F_p^{(\alpha,\beta)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\ &=x^{\rho+\eta-\sigma-\sigma'-1}\,\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma-\nu)}\\ &\times\frac{\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')}\,F_p^{(\alpha,\beta)}\left[\begin{matrix}a,\;b;\frac{\gamma}{x}\\ c;x\end{matrix}\right]\\ &*\;{}_3F_3\left[\begin{matrix}1-\rho-\nu,1-\rho-\eta+\sigma+\sigma',1-\rho-\eta+\sigma+\nu';\\ 1-\rho,1-\rho+\sigma-\nu,1-\rho-\eta+\sigma+\sigma'+\nu';\end{matrix}\frac{\gamma}{x}\right].\end{aligned}\tag{6.10}$$

Proof. Setting $\kappa=\mu=1$ in Theorem 6.2 with the aid of (2.9) proves (6.10). □

Corollary 6.7. *Let $x>0$, $\Re(c)>\Re(b)>0$, $\Re(p)\geq 0$ and the parameters σ, σ', ν, ν', η, ρ, $\gamma\in\mathbb{C}$ such that $\Re(\eta)>0$ and*

$$\Re\left(\rho\right)>\max\left\{0,\Re(\sigma+\sigma'+\nu-\eta),\,\Re(\sigma'-\nu')\right\}.$$

Then the following fractional integral formula holds true:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\,F_p\left(a,b;c;\gamma\,t\right)\right]\right)(x)\\ &=x^{\rho+\eta-\sigma-\sigma'-1}\,\frac{\Gamma(\rho)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')}\\ &\times\frac{\Gamma(\rho+\eta-\sigma-\nu-\sigma')}{\Gamma(\rho+\eta-\nu-\sigma')}\,F_p\left[\begin{matrix}a,b;\\ c;\end{matrix}\gamma\,x\right]\\ &*\;{}_3F_3\left[\begin{matrix}\rho,\rho+\nu'-\sigma',\rho+\eta-\sigma-\nu-\sigma';\\ c,\rho+\nu',\rho+\eta-\sigma-\sigma',\rho+\eta-\nu-\sigma';\end{matrix}\gamma\,x\right].\end{aligned}\tag{6.11}$$

Proof. Setting $\kappa=\mu=1$ and $\alpha=\beta$ in Theorem 6.1 and using (2.8) yields (6.11). □

Corollary 6.8. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$, $\Re(p) \geq 0$ *and the parameters* σ, σ', ν, ν', η, ρ, $\gamma \in \mathbb{C}$ *satisfying* $\Re(\eta) > 0$ *and*

$$0 < \Re(\rho) < 1 + \min\{\Re(-\nu),\, \Re(\sigma + \sigma' - \eta),\, \Re(\sigma + \nu' - \eta)\}.$$

Then the following fractional integral formula holds true:

$$\begin{aligned}(6.12)\qquad &\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left[t^{\rho-1}\, F_p\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\ &= x^{\rho+\eta-\sigma-\sigma'-1}\,\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma-\nu)}\\ &\quad\times \frac{\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')}\, F_p\left[\begin{matrix} a, b; \dfrac{\gamma}{x}\\ c; \end{matrix}\right]\\ &\quad * {}_3F_3\left[\begin{matrix}1-\rho-\nu, 1-\rho-\eta+\sigma+\sigma', 1-\rho-\eta+\sigma+\nu'; \dfrac{\gamma}{x}\\ 1-\rho, 1-\rho+\sigma-\nu, 1-\rho-\eta+\sigma+\sigma'+\nu';\end{matrix}\right].\end{aligned}$$

Proof. Setting $\kappa = \mu = 1$ and $\alpha = \beta$ in Theorem 6.2 and using (2.8) yields (6.12). □

We establish certain image formulas for the generalized Gauss hypergeometric functions involving Saigo fractional integral operators (3.9) and (3.10) which are given in Theorems (6.9) and (6.10) below.

Theorem 6.9. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$ *and the parameters* σ, ν, η, ρ, $\gamma \in \mathbb{C}$ *such that* $\Re(p) \geq 0$, $\Re(\sigma) > 0$ *and* $\Re(\rho) > \max\{0, \Re(\nu - \eta)\}$. *Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.13)\qquad &\left(I_{0,x}^{\sigma,\nu,\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}(a,b;c;\gamma\, t)\right]\right)(x)\\ &= x^{\rho-\nu-1}\,\frac{\Gamma(\rho)\Gamma(\rho-\nu+\eta)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}\\ &\quad\times F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,\ b; \gamma\, x\\ c; \end{matrix}\right] * {}_2F_2\left[\begin{matrix}\rho, \rho+\eta-\nu; \gamma\, x\\ \rho-\nu, \rho+\eta+\sigma;\end{matrix}\right].\end{aligned}$$

Proof. A similar argument as in the proof of Theorem 6.1 with the Saigo fractional integral operators (3.9) and (3.10) will easily establish (6.13). So details of the proof are omitted. □

Theorem 6.10. *Let* $x > 0$, $\Re(c) > \Re(b) > 0$ *and the parameters* σ, ν, η, ρ, $\gamma \in \mathbb{C}$ *satisfying* $\Re(\sigma) > 0$, $\Re(p) \geq 0$ *and* $\Re(\rho) < 1 + \min\{\Re(\nu),\, \Re(\eta)\}$. *Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.14)\qquad &\left(I_{x,\infty}^{\sigma,\nu,\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\ &= x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\nu+\sigma)}\\ &\quad\times F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,\ b; \dfrac{\gamma}{x}\\ c; \end{matrix}\right] * {}_2F_2\left[\begin{matrix}1-\rho+\nu, 1-\rho+\eta; \dfrac{\gamma}{x}\\ 1-\rho, 1-\rho+\eta+\nu+\sigma;\end{matrix}\right].\end{aligned}$$

Proof. A similar argument as in the proof of Theorem 6.2 with the Saigo fractional integral operators (3.9) and (3.10) will easily establish (6.13). So details of the proof are omitted. □

Some obvious special cases of Theorems 6.9 and 6.10, which are interesting and (potentially) useful, are given in Corollaries 6.11-6.16.

Corollary 6.11. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, and $\Re(p) \geq 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > \max\{0, \Re(\nu-\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.15)\qquad &\left(I_{0,x}^{\sigma,\nu,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\mu)}(a,b;c;\gamma\, t)\right]\right)(x)\\ &= x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho-\nu+\eta)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}\\ &\quad\times F_p^{(\alpha,\beta;\mu)}\begin{bmatrix} a,\ b; & \\ c; & \gamma\, x\end{bmatrix} *\ {}_2F_2\begin{bmatrix}\rho,\rho+\eta-\nu; & \\ \rho-\nu,\rho+\eta+\sigma; & \gamma\, x\end{bmatrix}.\end{aligned}$$

Corollary 6.12. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, and $\Re(\rho) < 1 + \min\{\Re(\nu), \Re(\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.16)\qquad &\left(I_{x,\infty}^{\sigma,\nu,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\ &= x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\nu+\sigma)}\\ &\quad\times F_p^{(\alpha,\beta;\mu)}\begin{bmatrix} a,\ b; & \frac{\gamma}{x} \\ c; & \end{bmatrix} *\ {}_2F_2\begin{bmatrix}1-\rho+\nu,1-\rho+\eta; & \frac{\gamma}{x}\\ 1-\rho,1-\rho+\eta+\nu+\sigma; & \end{bmatrix}.\end{aligned}$$

Corollary 6.13. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > \max\{0, \Re(\nu-\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.17)\qquad &\left(I_{0,x}^{\sigma,\nu,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}(a,b;c;\gamma\, t)\right]\right)(x)\\ &= x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho-\nu+\eta)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}\\ &\quad\times F_p^{(\alpha,\beta)}\begin{bmatrix} a,\ b; & \\ c; & \gamma\, x\end{bmatrix} *\ {}_2F_2\begin{bmatrix}\rho,\rho+\eta-\nu; & \\ \rho-\nu,\rho+\eta+\sigma; & \gamma\, x\end{bmatrix}.\end{aligned}$$

Corollary 6.14. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, and $\Re(\rho) < 1 + \min\{\Re(\nu), \Re(\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned}(6.18)\qquad &\left(I_{x,\infty}^{\sigma,\nu,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x)\\ &= x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\nu+\sigma)}\end{aligned}$$

$$\times\, F_p^{(\alpha,\beta)}\left[\begin{matrix} a,\, b; \frac{\gamma}{x} \\ c; \end{matrix}\right] * {}_2F_2\left[\begin{matrix} 1-\rho+\nu, 1-\rho+\eta; \frac{\gamma}{x} \\ 1-\rho, 1-\rho+\eta+\nu+\sigma; \end{matrix}\right].$$

Corollary 6.15. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$, and, $\Re(\rho) > \max\{0, \Re(\nu-\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned} &\left(I_{0,x}^{\sigma,\nu,\eta}\left[t^{\rho-1}\, F_p\left(a,b;c;\gamma\, t\right)\right]\right)(x) \qquad (6.19)\\ &= x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho-\nu+\eta)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)} \\ &\quad\times F_p\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\gamma\, x\right] * {}_2F_2\left[\begin{matrix} \rho, \rho+\eta-\nu; \\ \rho-\nu, \rho+\eta+\sigma; \end{matrix}\gamma\, x\right]. \end{aligned}$$

Corollary 6.16. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, ν, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, $\Re(\rho) < 1 + \min\{\Re(\nu), \Re(\eta)\}$. Then the following fractional integral formula holds true*:

$$\begin{aligned} &\left(I_{x,\infty}^{\sigma,\nu,\eta}\left[t^{\rho-1}\, F_p\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \qquad (6.20)\\ &= x^{\rho-\nu-1}\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\nu+\sigma)} \\ &\quad\times F_p\left[\begin{matrix} a,\, b; \frac{\gamma}{x} \\ c; \end{matrix}\right] * {}_2F_2\left[\begin{matrix} 1-\rho+\nu, 1-\rho+\eta; \frac{\gamma}{x} \\ 1-\rho, 1-\rho+\eta+\nu+\sigma; \end{matrix}\right]. \end{aligned}$$

Certain image formulas for the generalized Gauss hypergeometric functions involving the Erdélyi-Kober fractional integral operators (3.13) and (3.14) are given in Theorems 6.17 and 6.18 below.

Theorem 6.17. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$ and $\Re(\rho) > \Re(-\eta)$. Then the following fractional integral formula holds true*:

$$\begin{aligned} &\left(\mathcal{E}_{0,x}^{\sigma,\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\gamma\, t\right)\right]\right)(x) \qquad (6.21)\\ &= x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\eta+\sigma)}F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\gamma\, x\right] * {}_1F_1\left[\begin{matrix} \rho+\eta; \\ \rho+\eta+\sigma; \end{matrix}\gamma\, x\right]. \end{aligned}$$

Proof. Setting $\nu = 0$ in Theorem 6.1 with the operators (3.13) and (3.14) will establish (6.21). So its proof details are omitted. □

Theorem 6.18. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, $\Re(\rho) < 1 + \Re(\eta)$. Then the following fractional integral formula holds true*:

$$\begin{aligned} &\left(\mathcal{K}_{x,\infty}^{\sigma,\eta}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \qquad (6.22)\\ &= x^{\rho-\nu-1}\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)} \end{aligned}$$

$$\times F_p^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,\ b; & \dfrac{\gamma}{x} \\ c; & \dfrac{\gamma}{x}\end{matrix}\right] * {}_1F_1\left[\begin{matrix} 1-\rho+\eta; & \dfrac{\gamma}{x} \\ 1-\rho+\eta+\sigma; & \dfrac{\gamma}{x}\end{matrix}\right].$$

Proof. Setting $\nu = 0$ in Theorem 6.2 with the operators (3.13) and (3.14) will establish (6.21). So its proof details are omitted. □

Some obvious and interesting special cases of Theorems 6.17 and 6.18 are given in the following corollaries.

Corollary 6.19. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$ and $\Re(\rho) > \Re(-\eta)$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{E}_{0,x}^{\sigma,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\gamma\, t\right)\right]\right)(x) \tag{6.23}$$
$$= x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\eta+\sigma)}F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix} a,\ b; & \\ c; & \gamma\, x\end{matrix}\right] * {}_1F_1\left[\begin{matrix} \rho+\eta; & \\ \rho+\eta+\sigma; & \gamma\, x\end{matrix}\right].$$

Corollary 6.20. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, and $\Re(\rho) < 1+\Re(\eta)$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{K}_{x,\infty}^{\sigma,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.24}$$
$$= x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)}$$
$$\times F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix} a,b; & \dfrac{\gamma}{x} \\ c; & \dfrac{\gamma}{x}\end{matrix}\right] * {}_1F_1\left[\begin{matrix} 1-\rho+\eta; & \dfrac{\gamma}{x} \\ 1-\rho+\eta+\sigma; & \dfrac{\gamma}{x}\end{matrix}\right].$$

Corollary 6.21. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > \Re(-\eta)$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{E}_{0,x}^{\sigma,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}\left(a,b;c;\gamma\, t\right)\right]\right)(x) \tag{6.25}$$
$$= x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\eta+\sigma)}F_p^{(\alpha,\beta)}\left[\begin{matrix} a,\ b; & \\ c; & \gamma\, x\end{matrix}\right] * {}_1F_1\left[\begin{matrix} \rho+\eta; & \\ \rho+\eta+\sigma; & \gamma\, x\end{matrix}\right].$$

Corollary 6.22. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(p) \geq 0$, and $\Re(\rho) < 1+\Re(\eta)$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{K}_{x,\infty}^{\sigma,\eta}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.26}$$
$$= x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)}F_p^{(\alpha,\beta)}\left[\begin{matrix} a,\ b; & \dfrac{\gamma}{x} \\ c; & \dfrac{\gamma}{x}\end{matrix}\right] * {}_1F_1\left[\begin{matrix} 1-\rho+\eta; & \dfrac{\gamma}{x} \\ 1-\rho+\eta+\sigma; & \dfrac{\gamma}{x}\end{matrix}\right].$$

Corollary 6.23. *Let $x > 0$, $\Re(c) > \Re(b) > 0$ and the parameters σ, η, ρ, $\gamma \in \mathbb{C}$, $\Re(p) \geq 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > \Re(-\eta)$. Then the following fractional* *integral formula holds true*:

$$\left(\mathcal{E}_{0,x}^{\sigma,\eta}\left[t^{\rho-1}\, F_p\left(a,b;c;\gamma\, t\right)\right]\right)(x) = x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\eta+\sigma)}F_p\begin{bmatrix} a,\, b; & \gamma\, x \\ c; & \end{bmatrix} *\, {}_1F_1\begin{bmatrix} \rho+\eta; & \gamma\, x \\ \rho+\eta+\sigma; & \end{bmatrix}. \tag{6.27}$$

Corollary 6.24. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters σ, η, ρ, $\gamma\in\mathbb{C}$, $\Re(\sigma)>0$, $\Re(p)\geq 0$, and $\Re(\rho)<1+\Re(\eta)$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{K}_{x,\infty}^{\sigma,\eta}\left[t^{\rho-1}\, F_p\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) = x^{\rho-\nu-1}\,\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)}F_p\begin{bmatrix} a,\, b; & \dfrac{\gamma}{x} \\ c; & \end{bmatrix} *\, {}_1F_1\begin{bmatrix} 1-\rho+\eta; & \dfrac{\gamma}{x} \\ 1-\rho+\eta+\sigma; & \end{bmatrix}. \tag{6.28}$$

Certain image formulas for the generalized Gauss hypergeometric functions involving the Riemann-Liouville and Weyl type fractional integral operators (3.15) and (3.16) are given in Theorems 6.25 and 6.26 below.

Theorem 6.25. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\,\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{R}_{0,x}^{\sigma}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\gamma\, t\right)\right]\right)(x) = x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}F_p^{(\alpha,\beta;\kappa,\mu)}\begin{bmatrix} a,\, b; & \gamma\, x \\ c; & \end{bmatrix} *\, {}_1F_1\begin{bmatrix} \rho; & \gamma\, x \\ \rho+\sigma; & \end{bmatrix}. \tag{6.29}$$

Proof. Setting $\nu=-\sigma$ in Theorem 6.1 with operators (3.15) and (3.16) will easily establish (6.29). So the detailed account of proof is omitted. □

Theorem 6.26. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\,\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{W}_{x,\infty}^{\sigma}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\kappa,\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) = x^{\rho+\sigma-1}\,\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}F_p^{(\alpha,\beta;\kappa,\mu)}\begin{bmatrix} a,\, b; & \dfrac{\gamma}{x} \\ c; & \end{bmatrix} *\, {}_1F_1\begin{bmatrix} 1-\rho+\sigma; & \dfrac{\gamma}{x} \\ 1-\rho; & \end{bmatrix}. \tag{6.30}$$

Proof. Setting $\nu=-\sigma$ in Theorem 6.2 with operators (3.15) and (3.16) will easily establish (6.30) So the detailed account of proof is omitted. □

Some obvious and interesting special cases of Theorems 6.25 and 6.26 are given in the following corollaries.

Corollary 6.27. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\,\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{R}_{0,x}^{\sigma}\left[t^{\rho-1}\, F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\gamma\, t\right)\right]\right)(x) \tag{6.31}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix}a,\ b; \\ c;\end{matrix}\gamma x\right] * {}_1F_1\left[\begin{matrix}\rho; \\ \rho+\sigma;\end{matrix}\gamma x\right].$$

Corollary 6.28. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{W}_{x,\infty}^{\sigma}\left[t^{\rho-1}\ F_p^{(\alpha,\beta;\mu)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.32}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}F_p^{(\alpha,\beta;\mu)}\left[\begin{matrix}a,\ b;\frac{\gamma}{x} \\ c;x\end{matrix}\right] * {}_1F_1\left[\begin{matrix}1-\rho+\sigma;\frac{\gamma}{x} \\ 1-\rho;x\end{matrix}\right].$$

Corollary 6.29. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{R}_{0,x}^{\sigma}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}(a,b;c;\gamma t)\right]\right)(x) \tag{6.33}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}F_p^{(\alpha,\beta)}\left[\begin{matrix}a,\ b; \\ c;\end{matrix}\ \gamma x\right] * {}_1F_1\left[\begin{matrix}\rho; \\ \rho+\sigma;\end{matrix}\ \gamma x\right].$$

Corollary 6.30. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{W}_{x,\infty}^{\sigma}\left[t^{\rho-1}\ F_p^{(\alpha,\beta)}\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.34}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}F_p^{(\alpha,\beta)}\left[\begin{matrix}a,\ b;\ \frac{\gamma}{x} \\ c;\ x\end{matrix}\right] * {}_1F_1\left[\begin{matrix}1-\rho+\sigma;\ \frac{\gamma}{x} \\ 1-\rho;\ x\end{matrix}\right].$$

Corollary 6.31. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{R}_{0,x}^{\sigma}\left[t^{\rho-1}\ F_p(a,b;c;\gamma t)\right]\right)(x) \tag{6.35}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}F_p\left[\begin{matrix}a,\ b; \\ c;\end{matrix}\ \gamma x\right] * {}_1F_1\left[\begin{matrix}\rho; \\ \rho+\sigma;\end{matrix}\ \gamma x\right].$$

Corollary 6.32. *Let $x>0$, $\Re(c)>\Re(b)>0$ and the parameters p, σ, ρ, $\gamma\in\mathbb{C}$ with $\Re(p)\geq 0$ and $\min\{\Re(\sigma),\Re(\rho)\}>0$. Then the following fractional integral formula holds true*:

$$\left(\mathcal{W}_{x,\infty}^{\sigma}\left[t^{\rho-1}\ F_p\left(a,b;c;\left(\frac{\gamma}{t}\right)\right)\right]\right)(x) \tag{6.36}$$

$$= x^{\rho+\sigma-1}\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}F_p\left[\begin{matrix}a,\ b;\ \frac{\gamma}{x} \\ c;\ x\end{matrix}\right] * {}_1F_1\left[\begin{matrix}1-\rho+\sigma;\ \frac{\gamma}{x} \\ 1-\rho;\ x\end{matrix}\right].$$

The Marichev-Saigo-Maeda fractional integrations (3.2) of the product of $t^{\rho-1}$ and generalized incomplete hypergeometric functions (2.15) and (2.16), respectively, are given in the following theorem (see [54]).

Theorem 6.33. *Let* α, α', β, β', γ, $\rho \in \mathbb{C}$, $x > 0$ *such that* $\min\{\Re(\gamma), \Re(\rho)\} > 0$ *and*

$$\Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')].$$

Then the following formulas hold true:

(6.37)

$$\begin{aligned}&\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\,{}_p\gamma_q(at)\right]\right)(x)\\ &= x^{\rho+\gamma-\alpha-\alpha'-1}\frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\gamma-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\beta-\alpha')}\\ &\quad\times{}_{p+3}\gamma_{q+3}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho),(\rho+\beta'-\alpha'),(\rho+\gamma-\alpha-\beta-\alpha');\\ b_1,\ldots,b_q,(\rho+\beta'),(\rho+\gamma-\alpha-\alpha'),(\rho+\gamma-\beta-\alpha');\end{matrix}\ ax\right]\end{aligned}$$

and

(6.38)

$$\begin{aligned}&\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\,{}_p\Gamma_q(at)\right]\right)(x)\\ &= x^{\rho+\gamma-\alpha-\alpha'-1}\frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\gamma-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\beta-\alpha')}\\ &\quad\times{}_{p+3}\Gamma_{q+3}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho),(\rho+\beta'-\alpha'),(\rho+\gamma-\alpha-\beta-\alpha');\\ b_1,\ldots,b_q,(\rho+\beta'),(\rho+\gamma-\alpha-\alpha'),(\rho+\gamma-\beta-\alpha');\end{matrix}\ ax\right].\end{aligned}$$

Proof. Let the left-hand side of (6.37) be denoted by $\mathcal{I}$. Applying (2.15) with (3.2) and changing the order of integration and summation, we find

$$\begin{aligned}\mathcal{I} &= \left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{[a_1;x]_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(at)^n}{n!}\right]\right)(x) \qquad (6.39)\\ &= \sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(a)^n}{n!}\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left\{t^{\rho+n-1}\right\}\right)(x).\end{aligned}$$

Using the stated conditions here, for any $k \in \mathbb{N}_0$, and $\Re(\rho+n) \geq \Re(\rho) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$, and applying (5.1) with ρ replaced by $\rho+n$, we obtain

$$\begin{aligned}\mathcal{I} &= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\\ &\quad\times\frac{\Gamma(\rho+n)\Gamma(\rho+\beta'-\alpha'+n)\Gamma(\rho+\gamma-\alpha-\beta-\alpha'+n)}{\Gamma(\rho+\beta'+n)\Gamma(\rho+\gamma-\alpha-\alpha'+n)\Gamma(\rho+\gamma-\beta-\alpha'+n)}\frac{(ax)^n}{n!}\\ &= x^{\rho+\gamma-\alpha-\alpha'-1}\frac{\Gamma(\rho)\Gamma(\rho+\beta'-\alpha')\Gamma(\rho+\gamma-\alpha-\beta-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\beta-\alpha')}\\ &\quad\times\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{(\rho)_n(\rho+\beta'-\alpha')_n(\rho+\gamma-\alpha-\beta-\alpha')_n}{(\rho+\beta')_n(\rho+\gamma-\alpha-\alpha')_n(\rho+\gamma-\beta-\alpha')_n}\frac{(ax)^n}{n!}.\end{aligned}$$

This, in view of (2.15), proves (6.37).

Similarly (6.38) can be proved. □

Taking $\alpha' = 0$ in Theorem 6.33, we get the Saigo hypergeometric fractional image formulas of the generalized incomplete hypergeometric type functions ${}_p\Gamma_q[z]$ and ${}_p\gamma_q[z]$ as in the following corollary.

Corollary 6.34. *Under the modified conditions, the following formulas hold true*:

$$\begin{aligned}\left(I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\left[t^{\rho-1}\,{}_p\gamma_q(at)\right]\right)(x) &= x^{\rho+\gamma-\alpha-1}\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\beta)}{\Gamma(\rho+\gamma-\alpha)\Gamma(\rho+\gamma-\beta)}\\ &\times{}_{p+2}\,\gamma_{q+2}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho),(\rho+\gamma-\alpha-\beta);\\ b_1,\ldots,b_q,(\rho+\gamma-\alpha),(\rho+\gamma-\beta)\,;\end{matrix}ax\right]\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}\left(I_{0,x}^{\gamma,\alpha-\gamma,-\beta}\left[t^{\rho-1}\,{}_p\Gamma_q(at)\right]\right)(x) &= x^{\rho+\gamma-\alpha-1}\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\beta)}{\Gamma(\rho+\gamma-\alpha)\Gamma(\rho+\gamma-\beta)}\\ &\times{}_{p+2}\,\Gamma_{q+2}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho),(\rho+\gamma-\alpha-\beta);\\ b_1,\ldots,b_q,(\rho+\gamma-\alpha),(\rho+\gamma-\beta)\,;\end{matrix}ax\right].\end{aligned} \tag{6.41}$$

Setting $\alpha' = 0$ and $\alpha = 0$ in Theorem 6.33, we get the Riemann-Liouville fractional image formulas of the generalized incomplete hypergeometric type functions ${}_p\Gamma_q[z]$ and ${}_p\gamma_q[z]$ as in the following corollary.

Corollary 6.35. *Under the modified conditions, the following formulas hold true*:

$$\begin{aligned}&\left(\mathcal{R}_{0,x}^{\gamma}\left[t^{\rho-1}\,{}_p\gamma_q(at)\right]\right)(x)\\ &= x^{\rho+\gamma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\gamma)}\times{}_{p+1}\,\gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho);\\ b_1,\ldots,b_q,(\rho+\gamma);\end{matrix}ax\right]\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}&\left(\mathcal{R}_{0,x}^{\gamma}\left[t^{\rho-1}\,{}_p\Gamma_q(at)\right]\right)(x)\\ &= x^{\rho+\gamma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\gamma)}\times{}_{p+1}\,\Gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho);\\ b_1,\ldots,b_q,(\rho+\gamma);\end{matrix}ax\right].\end{aligned} \tag{6.43}$$

We present formulas for the right-hand sided Marichev-Saigo-Maeda fractional integration (3.3) of the generalized incomplete hypergeometric functions ${}_p\Gamma_q[z]$ and ${}_p\gamma_q[z]$ asserted by the following theorem.

Theorem 6.36. *Let $x > 0$ and α, α', β, β', γ, ρ, $a \in \mathbb{C}$ such that*

$$\min\{\Re(\gamma),\, \Re(\rho)\} > 0 \text{ and}$$
$$\Re(\rho) < 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}.$$

Then the following formulas hold true:

$$(6.44)\quad \left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\,{}_p\gamma_q\left(\frac{a}{t}\right)\right]\right)(x) = x^{\rho+\gamma-\alpha-\alpha'-1} \times \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho-\gamma+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\gamma+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} \times {}_{p+3}\gamma_{q+3}\left[\begin{matrix}(a_1,x), a_2,\ldots,a_p, (1-\rho-\beta), (1-\rho-\gamma+\alpha+\alpha'), (1-\rho-\gamma+\alpha+\beta'); \\ b_1,\ldots,b_q, (1-\rho), (1-\rho-\gamma+\alpha+\alpha'+\beta'), (1-\rho+\alpha-\beta);\end{matrix}\ \frac{a}{x}\right]$$

and

$$(6.45)\quad \left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\rho-1}\,{}_p\Gamma_q\left(\frac{a}{t}\right)\right]\right)(x) = x^{\rho+\gamma-\alpha-\alpha'-1} \times \frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho-\gamma+\alpha+\beta')}{\Gamma(1-\rho)\Gamma(1-\rho-\gamma+\alpha+\alpha'+\beta')\Gamma(1-\rho+\alpha-\beta)} \times {}_{p+3}\Gamma_{q+3}\left[\begin{matrix}(a_1,x), a_2,\ldots,a_p, (1-\rho-\beta), (1-\rho-\gamma+\alpha+\alpha'), (1-\rho-\gamma+\alpha+\beta'); \\ b_1,\ldots,b_q, (1-\rho), (1-\rho-\gamma+\alpha+\alpha'+\beta'), (1-\rho+\alpha-\beta);\end{matrix}\ \frac{a}{x}\right].$$

Proof. A similar argument as in the proof of Theorem 6.33, here applying (2.15) and using (3.3), will establish the results in Theorem 6.36. So the detailed account of its proof is omitted. □

Setting $\alpha' = 0$ in Theorem 6.36, we get the right-sided Saigo hypergeometric fractional image formulas of the generalized incomplete hypergeometric type functions ${}_p\Gamma_q[z]$ and ${}_p\gamma_q[z]$ as in the following corollary.

Corollary 6.37. *Under the modified conditions, the following formulas hold true*:

$$(6.46)\quad \left(I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta}\left[t^{\rho-1}\,{}_p\gamma_q\left(\frac{a}{t}\right)\right]\right)(x) = x^{\rho+\gamma-\alpha-1}\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha-\beta)} \times {}_{p+2}\gamma_{q+2}\left[\begin{matrix}(a_1,x), a_2,\ldots,a_p, (1-\rho-\beta), (1-\rho-\gamma+\alpha); \\ b_1,\ldots,b_q, (1-\rho), (1-\rho+\alpha-\beta);\end{matrix}\ \frac{a}{x}\right]$$

and

$$(6.47)\quad \left(I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta}\left[t^{\rho-1}\,{}_p\Gamma_q\left(\frac{a}{t}\right)\right]\right)(x) = x^{\rho+\gamma-\alpha-1}\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha-\beta)}$$

$$\times {}_{p+2}\Gamma_{q+2}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(1-\rho-\beta),(1-\rho-\gamma+\alpha);\\ b_1,\ldots,b_q,(1-\rho),(1-\rho+\alpha-\beta);\end{matrix}\ \frac{a}{x}\right].$$

Setting $\alpha'=0$ and $\alpha=0$ in Theorem 6.36, we obtain the Riemann-Liouville fractional image formulas of the generalized incomplete hypergeometric type functions ${}_p\Gamma_q[z]$ and ${}_p\gamma_q[z]$ as in the following corollary.

Corollary 6.38. *Under the modified conditions, the following formulas hold true*:

$$\left(\mathcal{R}^{\gamma}_{x,\infty}\left[t^{\rho-1}\,{}_p\gamma_q\left(\frac{a}{t}\right)\right]\right)(x) \tag{6.48}$$
$$= x^{\rho+\gamma-1}\frac{\Gamma(1-\rho-\gamma)}{\Gamma(1-\rho)}\times {}_{p+1}\gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(1-\rho-\gamma);\\ b_1,\ldots,b_q,(1-\rho);\end{matrix}\ \frac{a}{x}\right]$$

and

$$\left(\mathcal{R}^{\gamma}_{x,\infty}\left[t^{\rho-1}\,{}_p\Gamma_q\left(\frac{a}{t}\right)\right]\right)(x) \tag{6.49}$$
$$= x^{\rho+\gamma-1}\frac{\Gamma(1-\rho-\gamma)}{\Gamma(1-\rho)}\times {}_{p+1}\Gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(1-\rho-\gamma);\\ b_1,\ldots,b_q,(1-\rho);\end{matrix}\ \frac{a}{x}\right].$$

A similar argument as above will establish the following formulas in Corollaries 6.39 and 6.40 whose proofs are left to the interested readers.

Corollary 6.39. *Let* $x>0$, α, γ, $\rho\in\mathbb{C}$ *with* $\min\{\Re(\gamma),\Re(\rho)\}>0$. *Then the following formulas hold true*:

$$\left(\mathcal{E}^{\alpha,\gamma}_{0,x}\left[t^{\rho-1}\,{}_p\gamma_q(at)\right]\right)(x) \tag{6.50}$$
$$= x^{\rho-1}\frac{\Gamma(\rho+\gamma)}{\Gamma(\rho+\gamma+\alpha)}\times {}_{p+1}\gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho+\gamma);\\ b_1,\ldots,b_q,(\rho+\gamma+\alpha);\end{matrix}\ ax\right]$$

and

$$\left(\mathcal{E}^{\alpha,\gamma}_{0,x}\left[t^{\rho-1}\,{}_p\Gamma_q(at)\right]\right)(x) \tag{6.51}$$
$$= x^{\rho-1}\frac{\Gamma(\rho+\gamma)}{\Gamma(\rho+\gamma+\alpha)}\times {}_{p+1}\Gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(\rho+\gamma);\\ b_1,\ldots,b_q,(\rho+\gamma+\alpha);\end{matrix}\ ax\right].$$

Corollary 6.40. *Let* $x>0$, α, γ, ρ, $a\in\mathbb{C}$ *with* $\min\{\Re(\gamma),\Re(\rho)\}>0$. *Then the following formulas hold true*:

$$\left(\mathcal{K}^{\alpha,\gamma}_{x,\infty}\left[t^{\rho-1}\,{}_p\gamma_q\left(\frac{a}{t}\right)\right]\right)(x) \tag{6.52}$$
$$= x^{\rho-1}\frac{\Gamma(1-\rho+\gamma)}{\Gamma(1-\rho+\gamma+\alpha)}\times {}_{p+1}\gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(1-\rho+\gamma);\\ b_1,\ldots,b_q,(1-\rho+\gamma+\alpha);\end{matrix}\ \frac{a}{x}\right]$$

and

(6.53)

$$\begin{aligned}&\left(\mathcal{K}_{x,\infty}^{\alpha,\gamma}\left[t^{\rho-1}\,{}_p\Gamma_q\left(\frac{a}{t}\right)\right]\right)(x)\\&=x^{\rho-1}\frac{\Gamma(1-\rho+\gamma)}{\Gamma(1-\rho+\gamma+\alpha)}\times{}_{p+1}\Gamma_{q+1}\left[\begin{matrix}(a_1,x),a_2,\ldots,a_p,(1-\rho+\gamma);\ \dfrac{a}{x}\\ b_1,\ldots,b_q,(1-\rho+\gamma+\alpha);\ \dfrac{a}{x}\end{matrix}\right].\end{aligned}$$

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