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DISCRETE EVOLUTION EQUATIONS ON NETWORKS AND A UNIQUE IDENTIFIABILITY OF THEIR WEIGHTS

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Abstract. In this paper, we first discuss a representation of solutions to the initial value problem and the initial-boundary value problem for discrete evolution equations

$$\sum_{n=0}^{l} c_n \partial_t^n u(x,t) - \rho(x)\Delta_\omega u(x,t) = H(x,t),$$

defined on networks, i.e. on weighted graphs. Secondly, we show that the weight of each link of networks can be uniquely identified by using their Dirichlet data and Neumann data on the boundary, under a monotonicity condition on their weights.

1. Introduction

Over recent years, studying the structure of networks has attracted great attention from many researchers in various fields. Among these studies, solving forward and inverse problems for equations by means of an elliptic operator, called an ω-Laplacian Δ_ω on networks, which can be interpreted as a diffusion equation on graphs modeled by electric networks, have been investigated by a lot of authors, because of their applications to many practical examples such as identification of conductivity or finding perturbation of electric networks. See, for example, [4], [5], [8], [9], [10], [12] and [13].

Recently, the author and C. A. Berenstein published a paper [6], which offered another approach, so called the *partial differential equations on networks*, on studying inverse problems for the ω-Laplace operator Δ_ω on networks. In their paper, they defined discrete analogues of some notions on vector calculus such as integration, directional derivative, gradient and so on, and showed that some fundamental properties on vector calculus, for example Green's theorem, are nicely behaved on networks. By using these properties, they proved the solvability of direct problems such as Dirichlet and Neumann boundary value problems for the ω-harmonic equations on networks and then, based on the

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results, proved the global uniqueness of the inverse problem for the equation under the monotonicity condition.

On the other hand, in the paper [7], by using the operators $\frac{\partial}{\partial t}-\Delta_\omega$ and $\frac{\partial^2}{\partial t^2}-\Delta_\omega$, the author, Y.-S. Chung and J.-H. Kim introduced ω-diffusion equations and ω-elastic equations, which are mathematical models of flowing heat (or energy etc.) through a network and vibration of molecules, respectively, and discussed their direct problems, such as Cauchy problems and Dirichlet boundary value problems.

In this paper, motivated by [6] and [7] we discuss direct and inverse problems for equations on networks of the type

$$\sum_{n=0}^{l} c_n \partial_t^n u(x,t) - \rho(x)\Delta_\omega u(x,t) = H(x,t), \quad (x,t) \in V \times [0,T)$$

where V is the set of vertices of the given network, $\rho : V \to \mathbb{R}$ and $H : V \times [0,T) \to \mathbb{R}$ are given functions, $c_0, c_1, \ldots, c_n \in \mathbb{R}$ are given constants and $u(x,t)$ is the unknown, which are generalizations of the ω-*diffusion equations* and ω-*elastic equations* on networks. The main concern of this paper is to solve the Cauchy problems and the Dirichlet boundary problems for the above equations and, based on these results, to study an inverse conductivity problem of identifying conductivities on edges of a given network under the monotonicity condition. This paper is organized as follows. In Chapter 2, we study vector calculus on networks, by recalling the paper [6]. In Chapter 3, we introduce the *evolution equations* on networks, which are generalizations of the ω-diffusion equations and the ω-elastic equations on networks discussed in the paper [7]. The existence and the uniqueness of the solution of the Cauchy problems and the Dirichlet boundary value problems for the equations are discussed. Finally in Section 4, based on the results of these direct problems discussed in Section 3, the main result of this paper - a global uniqueness of the conductivity on edges under monotonicity condition - is proved by applying Laplace transform and a discrete version of Dirichlet principle for a certain type of nonlinear Poisson equations on networks, called the Schrödinger equations on networks.

2. Preliminaries

By a *graph* $G = G(V,E)$ we mean a finite set V of *vertices* with a set E of two-element subsets of V (whose elements are called *edges*). A graph G is said to be *simple* if it has neither multiple edges nor loops, and G is said to be *connected* if for every pair of vertices x and y there exist a sequence of vertices $x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_n = y$ where $x \sim y$ means that two vertices x and y are connected (adjacent) by an edge in E. A *weighted* (*undirected*) *graph* is a graph $G(V,E)$ associated with a *weight* function $\omega : V \times V \to [0,\infty)$ satisfying

(i) $\omega(x,y) = \omega(y,x)$, $x, y \in V$,
(ii) $\omega(x,y) = 0$ if and only if $\{x,y\} \notin E$.

In particular, a weight ω satisfying $\omega(x,y)=1,\ x\sim y$, is called the *standard* weight on G. From now on, the term *network* denotes a finite, connected, simple and weighted graph. A network $S=S(V',E')$ is said to be a *subnetwork* of $G(V,E)$ if $V'\subset V$ and $E'\subset E$. If E' consists of all the edges from E which connect the vertices of V' in its host network G, then S is called an *induced* subnetwork. Throughout this paper, all the subnetworks in our concern are assumed to be induced subnetworks. For a subnetwork S of a network $G=G(V,E)$, the (vertex) *boundary* ∂S of S is the set of all vertices $z\in V$ not in S but adjacent to some vertices in S, i.e., $\partial S:=\{z\in V\,|\,z\sim y \text{ for some } y\in S\}$. Also, by $\overline{S}$ we denote a network whose vertices and edges are in S and vertices in ∂S.

The *integration* of a function $f:V\to\mathbb{R}$ on a network $G=G(V,E)$ is defined by

$$\int_G f(x)\,dx \quad \left(\text{or } \int_G f\right):=\sum_{x\in V} f(x).$$

For the *directional derivative* of a function $f:V\to\mathbb{R}$ to the direction y, we mean

$$D_{\omega,y}f(x):=[f(y)-f(x)]\sqrt{\omega(x,y)},\quad x,y\in V$$

and the *gradient* ∇_ω of a function f is defined to be a vector

$$\nabla_\omega f(x):=\big(D_{\omega,y}f(x)\big)_{y\in V}.$$

The (outward) *normal derivative* $\frac{\partial f}{\partial_\omega n}(z)$ at $z\in\partial S$ is defined to be

$$\frac{\partial f}{\partial_\omega n}(z):=\sum_{y\in S}[f(z)-f(y)]\omega(z,y).$$

The ω-*Laplacian* Δ_ω of a function $f:G\to\mathbb{R}$ on a network G is defined by

$$\Delta_\omega f(x):=-\sum_{y\in V} D_{\omega,y}\left(D_{\omega,y}f(x)\right)=\sum_{y\in V}[f(y)-f(x)]\omega(x,y),\quad x\in V.$$

Remark 2.1. From now on, it is assumed that for each z_1 and z_2 in ∂S, $\omega(z_1,z_2)=0$, which means that every pair of vertices in the boundary does not connected by an edge.

The next theorem can be proved easily by following the proof of Theorem 1.2 and Corollary 1.3 in the paper [6] by S.-Y. Chung and C. A. Berenstein.

Theorem 2.2. *Let S be a subnetwork of a host network G. Then for any pair of functions $f:\overline{S}\to\mathbb{R}$ and $h:\overline{S}\to\mathbb{R}$, we have*

(i)
$$2\int_{\overline{S}} h(-\Delta_\omega f)=\int_{\overline{S}}\nabla_\omega h\cdot\nabla_\omega f,$$

(ii)
$$\int_{\overline{S}} h\Delta_\omega f=\int_{\overline{S}} f\Delta_\omega h.$$

3. Evolution equations on networks and its direct problems

Let a network $G = G(V, E)$ and a weight ω be given. Consider a function $u : V \times [0, T) \to \mathbb{R}$, where $u(x, t)$ represents the temperature at each vertex $x \in V$ at time $t \in [0, T)$, where T is a positive real number or infinity. Assume that heat flows from a vertex x to its adjacent vertex y through edges. Then the velocity of flowing heat from x to y is proportional to (i) the difference of the temperature of two vertices x and y and (ii) the heat conductivity $\omega(x, y)$ of the edge between x and y. Thus it is easy to see that the function u satisfy the equation

$$\partial_t u(x,t) = \sum_{y \in V} [u(y,t) - u(x,t)]\, \omega(x,y), \quad (x,t) \in V \times [0,T),$$

or equivalently,

$$\partial_t u(x,t) - \Delta_\omega u(x,t) = 0, \quad (x,t) \in V \times [0,T). \tag{3.1}$$

In this paper, as a generalization of the equation (3.1), we deal with the following type of evolution equations

$$\sum_{n=0}^{l} c_n \partial_t^n u(x,t) - \rho(x) \Delta_\omega u(x,t) = H(x,t), \quad (x,t) \in V \times [0,T),$$

where $c_0, c_1, \ldots, c_l$ be given real constants and $\rho(x)$ and $H(x, t)$ be given functions.

In what follows, T always denotes a given positive number or infinity. For a network $G(V, E)$ and an interval $I \subset \mathbb{R}$, we say that a function $f : V \times I \to \mathbb{R}$ belongs to $\mathcal{C}^n(V \times I)$ if for each $x \in V$, the function $f(x, \cdot)$ is a $\mathcal{C}^n$-function on I.

We now discuss the existence and the uniqueness of the Cauchy problems for the evolution equations. For a function $f : V \to \mathbb{R}$, with $|V| = N$, we may consider it as an N-dimensional vector. By the same sense, the ω-Laplacian operator Δ_ω also can be considered as a matrix defined by

$$\Delta_\omega(x,y) = \begin{cases} -d_\omega x, & \text{if } x = y \\ \omega(x,y), & \text{otherwise.} \end{cases}$$

For a given positive valued function $\rho : V \to (0, \infty)$, let D_ρ denotes the diagonal matrix with (x, x)-th entry having the value $\rho(x)$ for each $x \in V$ and define $\mathcal{L}_{\rho,\omega} := \big(D_\rho^{1/2}\big) \Delta_\omega \big(D_\rho^{1/2}\big)$. Then it is easy to see that,

$$\mathcal{L}_{\rho,\omega} f(x) = \sqrt{\rho(x)} \sum_{y \in V} \Big[\sqrt{\rho(y)} f(y) - \sqrt{\rho(x)} f(x)\Big] \omega(x,y), \quad x \in V.$$

Moreover, $-\mathcal{L}_{\rho,\omega}$ is a nonnegative definite symmetric matrix, so that it has the eigenvalues $0 \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1}$, and corresponding eigenfunctions $\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_{N-1}$, which are orthonormal in the sense that for each pair of distinct i and j, $\int_V \Phi_i(x) \Phi_j(x) dx = 0$, while, for all j, $\int_V |\Phi_j(x)|^2 dx = 1$. Moreover, it is easy to show that $\lambda_0 = 0$ and $\lambda_1 > 0$.

Theorem 3.1. *Let $G(V,E)$ be a network with a weight ω, ρ be a positive valued function defined in V and constants $c_0, c_1, \dots, c_l \in \mathbb{R}$ be given, where at least one of c_j among $j = 1, 2, \dots, l$ is not zero. For given functions $H \in \mathcal{C}\big(V \times [0,T)\big)$ and $f_n : V \to \mathbb{R}$, $n = 0, 1, \dots, l-1$, the following Cauchy problem for the evolution equation*

$$\left\{ \begin{array}{ll} \big(\sum_{n=0}^{l} c_n \partial_t^n - \rho(x)\Delta_\omega\big) u(x,t) = H(x,t), & (x,t) \in V \times (0,T), \\ \partial_t^n u(x,0) = f_n(x),\ n = 0, 1, \dots, l-1, & x \in V \end{array} \right. \tag{3.2}$$

has a unique solution in $\mathcal{C}^{l-1}\big(V \times [0,T)\big)$ represented by using the eigenvalues $\lambda_0 \le \lambda_1 \le \cdots \lambda_{|V|-1}$ of $\mathcal{L}_{\rho,\omega}$ and their corresponding orthonormal eigenfunctions $\Phi_0, \Phi_1, \dots, \Phi_{|V|-1}$ as

$$u(x,t) = \sqrt{\rho(x)} \sum_{j=0}^{|V|-1} a_j(t)\Phi_j(x), \quad (x,t) \in V \times [0,T) \tag{3.3}$$

where $a_j(t)$ is the solution in $\mathcal{C}^{l-1}[0,T)$ of the following initial value problem

$$\begin{aligned} \sum_{n=0}^{l} c_n a_j^{(n)}(t) + \lambda_j a_j(t) &= \int_G \frac{1}{\sqrt{\rho(y)}} H(y,t)\Phi_j(y)dy, \quad t \in [0,T), \\ a_j^{(n)}(0) &= \int_G \frac{1}{\sqrt{\rho(y)}} f_n(y)\Phi_j(y)dy, \quad n = 0, 1, \dots, l-1, \end{aligned} \tag{3.4}$$

for each $j = 0, 1, \dots, |V|-1$.

Proof. Let N denote $|V|$. Suppose that the equation (3.2) has a solution $u(x,t) \in \mathcal{C}^{l-1}\big(V \times [0,T)\big)$. Consider the following expansion

$$\frac{1}{\sqrt{\rho(x)}} u(x,t) = \sum_{j=0}^{N-1} a_j(t)\Phi_j(x), \quad (x,t) \in V \times (0,T),$$

where $a_j(t) = \int_G D_\rho^{-1/2} u(y,t)\ \Phi_j(y)dy$, $j = 0, 1, \dots, N-1$. Since $\mathcal{L}_{\rho,\omega}\ D_\rho^{-1/2} = D_\rho^{1/2}\ \Delta_\omega$, we have for $t \in (0,T)$,

$$\begin{aligned} -\lambda_j a_j(t) &= \int_G D_\rho^{-1/2} u(y,t)\ \mathcal{L}_{\rho,\omega}\Phi_j(y)dy \\ &= \int_G D_\rho^{1/2} \Delta_\omega u(y,t)\ \Phi_j(y)dy \\ &= \sum_{n=1}^{l} c_n a_j^{(n)}(t) - \int_G \frac{1}{\sqrt{\rho(y)}} H(y,t)\ \Phi_j(y)dy. \end{aligned}$$

Therefore if $(x,t) \in V \times (0,T)$, then $u(x,t)$ satisfies

$$u(x,t) = \sqrt{\rho(x)} \sum_{j=0}^{N-1} a_j(t)\Phi_j(x), \quad (x,t) \in V \times (0,T) \tag{3.5}$$

where $a_j(t)$ is a solution of the equation

$$\sum_{n=0}^{l} c_n a_j^{(n)}(t) + \lambda_j a_j(t) = \int_G \frac{1}{\sqrt{\rho(y)}} H(y,t)\Phi_j(y)dy, \quad t \in (0,T),$$

for each $j = 0, 1, \ldots, N-1$. We now extend the domain of $u(x,t)$ in (3.5) to $V \times [0,T)$ continuously. Take any $n = 0, 1, \ldots, l-1$. Since $u(x,\cdot) \in \mathcal{C}^{l-1}[0,T),\ x \in V$, we have

$$f_n(x) = \partial_t^n u(x,0) = \lim_{t\to 0} \sqrt{\rho(x)} \sum_{j=0}^{N-1} a_j^{(n)}(t)\Phi_j(x)$$

and hence for each $j = 0, 1, \ldots, N-1$, we obtain

$$\int_G \frac{1}{\sqrt{\rho(y)}} f_n(y)\Phi_j(y)dy = \lim_{t\to 0} a_k^{(n)}(t) \sum_{k=0}^{N-1} \int_G \Phi_k(y)\Phi_j(y)dy = \lim_{t\to 0} a_j^{(n)}(t).$$

Thus, we finally conclude that for each $(x,t) \in V \times [0,T)$, $u(x,t)$ satisfies (3.3), where $a_j(t)$ is the (unique) solution in $\mathcal{C}^{l-1}[0,T)$ of the initial value problem (3.4), for each $j = 0, 1, \ldots, N-1$. Now, it is easy to verify by substitution that such $u(x,t)$ is the solution of the equation (3.2). $\square$

Let us now turn to the boundary value problems. For a subnetwork S of a network G with a weight ω and for a given function $\rho : \overline{S} \to (0,\infty)$, the *Dirichlet eigenvalues* of $-\mathcal{L}_{\rho,\omega} = -\big(D_\rho^{1/2}\big)\Delta_\omega\big(D_\rho^{1/2}\big)$ are defined to be the eigenvalues $\nu_1 \le \nu_2 \le \cdots \le \nu_n$ of the matrix $-\mathcal{L}_{\rho,\omega,S}$ where $\mathcal{L}_{\rho,\omega,S}$ is a submatrix of $\mathcal{L}_{\rho,\omega}$ with rows and columns restricted to those indexed by vertices in S and $n = |S|$. It is well known that there is an eigenfunctions $\phi_1, \phi_2, \ldots, \phi_n$ corresponding to $\nu_1, \nu_2, \ldots, \nu_n$ which are orthonormal in the sense that for each pair of distinct i and j, $\int_S \phi_i(x) \cdot \phi_j(x)dx = 0$, while, for all j, $\int_S |\phi_j(x)|^2 dx = 1$. As usual, it is easy to show that the first eigenvalue $\nu_1 > 0$.

We are now ready to solve the Dirichlet boundary value problems of the evolution equations on networks.

Theorem 3.2. *Let S be a subnetwork of a network G with a weight ω with $\partial S \neq \emptyset$, ρ be a positive valued function defined in $\overline{S}$ and constants $c_0, c_1, \ldots, c_l \in \mathbb{R}$, where at least one of c_j among $j = 1, 2, \ldots, l$ is not zero, be given. For $\sigma \in \mathcal{C}\big(\partial S \times [0,T)\big)$, $H \in \mathcal{C}\big(S \times [0,T)\big)$ and $f_n : S \to \mathbb{R},\ n = 0, 1, \ldots, l-1$, the following Dirichlet boundary value problem for the evolution equation*

$$\text{(3.6)} \quad \begin{cases} \big(\sum_{n=0}^{l} c_n \partial_t^n - \rho(x)\Delta_\omega\big)u(x,t) = H(x,t), & (x,t) \in S \times (0,T) \\ u(z,t) = \sigma(z,t), & (z,t) \in \partial S \times [0,T) \\ \partial_t^n u(x,0) = f_n(x),\ n = 0, 1, \ldots, l-1, & x \in S \end{cases}$$

has a unique solution in $\mathcal{C}^{l-1}(S \times [0,T))$ represented by using the Dirichlet eigenvalues $\nu_1 \le \nu_2 \le \cdots \le \nu_{|S|}$ of $\mathcal{L}_{\rho,\omega}$ and their corresponding orthonormal *eigenfunctions* $\phi_1, \ldots, \phi_{|S|}$ *as*

$$u(x,t) = \sqrt{\rho(x)} \sum_{j=0}^{|S|} a_j(t)\phi_j(x), \quad (x,t) \in S \times [0,T), \tag{3.7}$$

where $a_j(t)$ *is the solution in* $\mathcal{C}^{l-1}[0,T)$ *of the following initial value problem*

$$\begin{aligned} \sum_{n=0}^{l} c_n a_j^{(n)}(t) + \lambda_j a_j(t) &= \int_S \frac{1}{\sqrt{\rho(y)}} \left[B_{\rho,\omega,\sigma}(y,t) + H(y,t)\right] \phi_j(y)dy, \\ a_j^{(n)}(0) &= \int_S \frac{1}{\sqrt{\rho(y)}} f_n(y)\phi_j(y)dy, \quad n = 0,1,\ldots,l-1, \end{aligned} \tag{3.8}$$

$t \in [0,\infty)$, *for* $j = 0,1,\ldots,|S|$.
Here, $B_{\rho,\omega,\sigma}(y,t) := \int_{\partial S} \sigma(z,t)\omega(y,z)\rho(y)dz$.

Proof. Let $\widetilde{\phi}_j : \overline{S} \to \mathbb{R}$ be the function satisfying $\widetilde{\phi}_j(x) = \phi_j(x),\ x \in S$ and $\widetilde{\phi}_j(x) = 0,\ x \in \partial S$. Suppose that the equation (3.6) has a solution $u(x,t) \in \mathcal{C}^{l-1}\big(\overline{S} \times [0,T)\big)$. Consider the following expansion

$$\frac{1}{\sqrt{\rho(x)}} u(x,t) = \sum_{j=0}^{|S|} a_j(t)\phi_j(x), \quad (x,t) \in S \times (0,T),$$

where $a_j(t) = \int_S D_\rho^{-1/2} u(y,t)\ \phi_j(y)dy$, for $j = 0,1,2,\ldots,|S|$. Since $\mathcal{L}_{\rho,\omega}\ D_\rho^{-1/2} = D_\rho^{1/2}\Delta_\omega$, we have for $t \in (0,T)$,

$$\begin{aligned} &-\lambda_j a_j(t) \\ &= \int_S D_\rho^{-1/2} u(y,t)\ \mathcal{L}_{\rho,\omega,S}\phi_j(y)dy \\ &= \int_{\overline{S}} D_\rho^{-1/2} u(y,t)\ \mathcal{L}_{\rho,\omega}\widetilde{\phi}_j(y)dy - \int_{\partial S} D_\rho^{-1/2} u(z,t)\ \mathcal{L}_{\rho,\omega}\widetilde{\phi}_j(z)dz \\ &= \int_S D_\rho^{1/2}\Delta_\omega u(y,t)\phi_j(y)dy - \int_S \int_{\partial S} \sigma(z,t)\omega(y,z)\rho(y)dz\ \frac{1}{\sqrt{\rho(y)}}\widetilde{\phi}_j(y)dy \\ &= \sum_{n=1}^{l} c_n a_j^{(n)}(t) - \int_S \frac{1}{\sqrt{\rho(y)}} \left[B_{\rho,\omega,\sigma}(y,t) + H(y,t)\right] \phi_j(y)dy. \end{aligned}$$

Therefore if $(x,t) \in S \times (0,T)$, then $u(x,t)$ satisfies

$$u(x,t) = \sqrt{\rho(x)} \sum_{j=0}^{|S|} a_j(t)\phi_j(x), \tag{3.9}$$

where $a_j(t)$ is a solution of the equation

$$\sum_{n=0}^{l} c_n a_j^{(n)}(t) + \lambda_j a_j(t) = \int_S \frac{1}{\sqrt{\rho(y)}} \left[B_{\rho,\omega,\sigma}(y,t) + H(y,t)\right] \phi_j(y)dy$$

for each $j = 0, 1, \dots, |S|$. By extending the domain of $u(x,t)$ in (3.9) to $S\times[0,T)$ continuously just as we have done in the proof of Theorem 3.1, we conclude that for each $(x,t) \in S\times[0,T)$, $u(x,t)$ satisfies (3.7), where $a_j(t)$ is the (unique) solution in $\mathcal{C}^{l-1}[0,T)$ of the initial value problem (3.8), for each $j = 0, 1, \dots, |S|$. Now, it is easy to verify by substitution that such $u(x,t)$ is the solution of the equation (3.6). □

Remark 3.3. In the previous theorems, the function ρ is assumed to be positive valued. But the condition of ρ can be weaken as '$\rho(x) \neq 0,$ for each x', if we extend the codomain of all functions in this section to the set of complex numbers.

The following corollary is a special case of Theorem 3.2.

Corollary 3.4. *Let $H \in \mathcal{C}\big(S \times [0,T)\big)$ and $\sigma \in \mathcal{C}\big(\partial S \times [0,T)\big)$ be given the same as above and $f : S \to \mathbb{R}$ be a function. The following Dirichlet boundary value problems for the discrete diffusion equation*

$$\begin{cases} \partial_t u(x,t) - \Delta_\omega u(x,t) = H(x,t), & (x,t) \in S \times (0,T), \\ u(z,t) = \sigma(z,t), & (z,t) \in \partial S \times [0,T) \\ u(x,0) = f(x), & x \in S \end{cases}$$

has a unique solution in $\mathcal{C}\big(S \times [0,T)\big)$ represented by

$$u(x,t) = \int_S E_{\omega,S}(x,y,t)\ f(y)dy + \int_0^t \int_S E_{\omega,S}(x,y,t-\tau)\ \big[B_{\omega,\sigma}(y,\tau) + H(y,\tau)\big]dy\ d\tau$$

for $(x,t) \in S\times[0,T)$. Here, $E_{\omega,S}(x,y,t) := \sum_{j=1}^{|S|} e^{-\nu_j t}\phi_j(x)\phi_j(y)$, where $\nu_1 \leq \cdots \leq \nu_{|S|}$ and $\phi_1, \dots, \phi_{|S|}$ are Dirichlet eigenvalues and their corresponding eigenfunctions of Δ_ω, respectively, and $B_{\omega,\sigma}(y,t) := \int_{\partial S} \sigma(y,t)\omega(y,z)dy$.

Proof. Solve the initial value problem of the ODE (3.8) in Theorem 3.2 with $\rho \equiv 1,\ l = 1,\ c_0 = 0$ and $c_1 = 1$ to get the result. □

4. An inverse conductivity problem

By the direct problems we discussed in the last section, we have for a given function $\sigma : \partial S \to \mathbb{R}$ with $\sigma \in \mathcal{C}^{l-1}\big(\partial S \times [0,\infty)\big)$, the Dirichlet boundary value problem for the equation

$$\begin{cases} \sum_{n=0}^{l} c_n \partial_t^n u(x,t) - \rho(x)\Delta_\omega u(x,t) = 0, & (x,t) \in S \times (0,\infty), \\ u(z,t) = \sigma(z,t), & (z,t) \in \partial S \times [0,\infty) \\ \partial_t^n u(x,0) = 0,\ n = 0, 1, \dots, l-1, & x \in S \end{cases} \tag{4.1}$$

where $c_0, c_1, \dots, c_l \in \mathbb{R}$ be given constants, has a unique solution in $\mathcal{C}^{l-1}(S \times [0,\infty))$. Thus if we give a Dirichlet data $\sigma(z,t)$ on the boundary of a network, then the Neumann data $\frac{\partial u}{\partial_\omega n}(z,t) = \sum_{y \in S}[u(z,t) - u(y,t)]\omega(z,y)$ is uniquely obtained in $\mathcal{C}^{l-1}(\overline{S} \times [0,\infty))$.

In this section, we discuss an inverse conductivity problem on networks with nonempty boundary. The main concern is related to the problem of recovering the conductivity (or weight) ω of the network by the Neumann data induced by the Dirichlet data with one boundary measurement. Since, in many practical examples, we can handle the weight of edges near to the boundary, so it is natural to assume that $u|_{\partial S}$ and $\omega|_{\partial S\times S}$ are given, and $\frac{\partial u}{\partial_\omega n}$ are known by measurement.

But even though we are given all these data on the boundary, the uniqueness of the conductivity ω is still not guaranteed, that means, there can be different conductivities on edges which induce the same boundary conditions. To avoid this difficulty and guarantee the uniqueness of the conductivity, we need to impose some more assumption than $u|_{\partial S}$, $\frac{\partial u}{\partial_\omega n}$ and $\omega|_{\partial S\times S}$ on the structure of network, or on the conductivity. We impose the additional constraint so called the *monotonicity condition* on conductivity on edges, following the paper [6] and other literatures in the continuous case such as [1] and [11]. The main result of this section shows that there is no different conductivities ω_1 and ω_2 on edges satisfying $\omega_1 \le \omega_2$, in $\partial S\times S$ which induce the same boundary conditions.

In the paper [6], the problem of unique identifiability of conductivity for the ω-harmonic equations under monotonicity condition was proved by using the *Dirichlet principle* for ω-harmonic equations, which characterizes the solutions of the Dirichlet boundary value problems for ω-harmonic equations as the minimizer of an appropriate functional. But it is not easy to apply this method directly to the problems for the evolution equations of the form (4.1), for it is not an easy problem to find an appropriate functional whose minimizer is the solution of the problem (4.1). To overcome this difficulty, we use Laplace transform to transfer the system (4.1) to the system of the equations of the following type,

$$q(x)u(x) - \Delta_\omega u(x) = 0, \quad x \in V$$

which are said to be the *Schrödinger equations* on networks. We first discuss the Dirichlet boundary value problem of the equations. For a subnetwork S of a network G with a weight ω and for a given function $q : \overline{S} \to \mathbb{R}$, the *Dirichlet eigenvalues* of $D_q - \Delta_\omega$ are defined as the eigenvalues $\nu_1 \le \nu_2 \le \cdots \le \nu_{|S|}$ of the matrix $(D_q - \Delta_\omega)_S$, which is a submatrix of $D_q - \Delta_\omega$ with rows and columns restricted to those indexed by vertices in S. Orthonormal functions $\phi_1, \ldots, \phi_{|S|}$ satisfying

$$\big(D_q - \Delta_\omega\big)_S\, \phi_j = \nu_j \phi_j, \quad j = 1, 2, \ldots, |S|$$

are said to be the corresponding orthonormal eigenfunctions of $\nu_1, \ldots, \nu_{|S|}$, respectively. Note that it is not, in general, true that the operator $D_q - \Delta_\omega$ is positive definite, and therefore the first eigenvalue ν_1 is not always greater than 0.

We now discuss the existence and the uniqueness of the solutions of the Dirichlet boundary value problems for the Schrödinger equations on networks.

Theorem 4.1. *Let $\overline{S}$ be a graph with $\partial S \neq \emptyset$ and $q : \overline{S} \to \mathbb{R}$ be given. For $\sigma : \partial S \to \mathbb{R}$ and $H : S \to \mathbb{R}$, the Dirichlet boundary value problem for the following Schrödinger equation*

$$\begin{cases} q(x)u(x) - \Delta_\omega u(x) = H(x), & x \in S, \\ u(z) = \sigma(z), & z \in \partial S. \end{cases} \tag{4.2}$$

(i) *has a unique solution, if 0 is not a Dirichlet eigenvalue of $D_q - \Delta_\omega$,*

(ii) *has infinitely many solutions, if 0 is a Dirichlet eigenvalue of $D_q - \Delta_\omega$ and σ satisfies that for each $j \in A$,*

$$\int_S \left[B_{\omega,\sigma}(y) + H(y)\right] \phi_j(y)dy = 0,$$

(iii) *has no solution, if 0 is a Dirichlet eigenvalue of $D_q - \Delta_\omega$ and σ satisfies that there exists $j \in A$ such that*

$$\int_S \left[B_{\omega,\sigma}(y) + H(y)\right] \phi_j(y)dy \neq 0,$$

where $\phi_1, \ldots, \phi_{|S|}$ are corresponding orthonormal eigenfunctions of Dirichlet eigenvalues $\nu_1 \leq \cdots \leq \nu_{|S|}$ of $D_q - \Delta_\omega$, $A := \{i \in \mathbb{N} \mid \nu_i = 0\}$ and $B_{\omega,\sigma}(y) = \int_{\partial S} \sigma(z,t)\omega(y,z)$. Moreover, in the case of (i) we have the following explicit solution

$$u(x) = \sum_{j=1}^{|S|} \frac{1}{\nu_j} \int_S \phi_j(x)\phi_j(y) \left[B_{\omega,\sigma}(y) + H(y)\right] dy, \tag{4.3}$$

$x \in S$ and in the case of (ii), the solutions are given by

$$u(x) = \sum_{j\in A} a_j\phi_j + \sum_{j\in A^c} \frac{1}{\nu_j} \int_S \phi_j(x)\phi_j(y) \left[B_{\omega,\sigma}(y) + H(y)\right] dy, \tag{4.4}$$

$x \in S$ where $a_j \in \mathbb{R}$, $j \in A$ is chosen arbitrary and A^c denotes the set $\{1, 2, \ldots, |S|\} \setminus A$.

Proof. Let $\widetilde{\phi}_j : \overline{S} \to \mathbb{R}$ be the function satisfying $\widetilde{\phi}_j(x) = \phi_j(x)$, $x \in S$ and $\widetilde{\phi}_j(x) = 0$, $x \in \partial S$, for $j = 1, 2, \ldots, |S|$. Suppose that there exists a solution $u(x)$ of the equation (4.2) and consider the expansion

$$u(x) = \sum_{j=1}^{|S|} a_j\phi_j(x), \quad x \in S,$$

where $a_j = \int_S u(y)\phi_j(y)dy$. Then we have

$$\begin{aligned} \nu_j a_j &= \int_{\overline{S}} u(y)\big(D_q - \Delta_\omega\big)\widetilde{\phi}_j(y)dy - \int_{\partial S} u(z)\big(D_q - \Delta_\omega\big)\widetilde{\phi}_j(z)dz \\ &= \int_S \big(D_q - \Delta_\omega\big)u(y)\phi_j(y)dy + \int_S \left[\int_{\partial S} \sigma(z)\omega(y,z)dz\right] \phi_j(y)dy \\ &= \int_S \left[B_{\omega,\sigma}(y) + H(y)\right] \phi_j(y)dy \end{aligned} \tag{4.5}$$

for $j = 1, 2, \ldots, |S|$. From (4.5), the case (iii) is proved immediately and in the cases of (i) and (ii), we have if $\nu_j \neq 0$, then

$$a_j = \frac{1}{\nu_j} \int_S \left[B_{\omega,\sigma}(y) + H(y)\right] \phi_j(y) dy$$

and if $\nu_j = 0$, then a_j can be chosen arbitrary. Now it is a simple calculation to verify (4.3) and (4.4) are solutions of the equation (4.2) in the cases of (i) and (ii), respectively. □

Note that the previous theorem guarantees the existence and the uniqueness of the solution of every equation (4.2) with a nonnegative real-valued function q (see [14]). Here, we characterize its solution as a minimizer of an appropriate functional. This characterization can be called the discrete version of *Dirichlet principle* for ω-Schrödinger equations on networks.

For a given function $q : \overline{S} \to [0, \infty)$, we define a functional $I_{\omega,q}$ by

$$I_{\omega,q}[v] := \int_{\overline{S}} \frac{1}{4} |\nabla_\omega v(x)|^2 + \frac{q(x)}{2} v^2(x)\, dx \tag{4.6}$$

for $v : \overline{S} \to \mathbb{R}$. For a given $\sigma : \partial S \to \mathbb{R}$, we define an admissible set

$$A_\sigma = \{v : \overline{S} \to \mathbb{R} \mid v(z) = \sigma(z),\ z \in \partial S\}.$$

Theorem 4.2. *Let S be a subnetwork of a network G with a weight ω with $\partial S \neq \emptyset$ and functions $q : \overline{S} \to [0, \infty)$ and $\sigma : \partial S \to \mathbb{R}$ be given. Assume that u is the solution of the following equation*

$$\begin{cases} q(x)u(x) - \Delta_\omega u(x) = 0, & x \in S \\ u(z) = \sigma(z), & z \in \partial S. \end{cases} \tag{4.7}$$

Then we have,

$$I_{\omega,q}[u] = \min_{v \in A_\sigma} I_{\omega,q}[v]. \tag{4.8}$$

Conversely, if $u \in A_\sigma$ satisfies (4.8)*, then u is the solution of the equation* (4.7)*.*

Proof. ($\Rightarrow$) Take any $v \in A_\sigma$. Since

$$\begin{aligned} 0 &= \int_{\overline{S}} \big[-\Delta_\omega u(x) + q(x)u(x)\big] \big[u(x) - v(x)\big]\, dx \\ &= \int_{\overline{S}} \frac{1}{2} \nabla_\omega u(x) \cdot \nabla_\omega \big[u(x) - v(x)\big] + q(x)u(x)\big[u(x) - v(x)\big]\, dx \\ &= \frac{1}{2} \int_{\overline{S}} |\nabla_\omega u(x)|^2 + qu^2(x)\, dx - \int_{\overline{S}} \nabla_\omega u(x) \cdot \nabla_\omega v(x) + quv(x)\, dx, \end{aligned}$$

we have

$$\int_{\overline{S}} \frac{1}{2} |\nabla_\omega u(x)|^2 + q(x)[u(x)]^2\, dx$$

$$\leq \int_{\overline{S}} \frac{1}{4}|\nabla_\omega u(x)|^2 + \frac{1}{4}|\nabla_\omega v(x)|^2 + \frac{1}{2}q(x)[u(x)]^2 + \frac{1}{2}q(x)[v(x)]^2 \; dx,$$

by using the inequality $|ab| \leq \frac{1}{2}(a^2+b^2),\ a, b \in \mathbb{R}$. Therefore we finally get the result.

($\Leftarrow$) Take any $x_0 \in S$ and define a continuous function $i(\tau) := I_{\omega,q}[u + \tau\delta_{x_0}],\ \tau \in \mathbb{R}$, where δ_{x_0} is a function on $\overline{S}$ defined by $\delta_{x_0}(x) = 1$ if $x = x_0$ and $\delta_{x_0}(x) = 0$ otherwise. Then it follows from the fact that $u + \tau\delta_{x_0} \in A_\sigma,\ \tau \in \mathbb{R}$ and the assumption (4.8) that $i(\tau)$ has a minimum at $\tau = 0$, which implies

$$i'(0) = 0. \tag{4.9}$$

Since we have

$$\begin{aligned} i(\tau) = &\int_{\overline{S}} \frac{1}{4}|\nabla_\omega u(x) + \tau\nabla_\omega\delta_{x_0}(x)|^2 + \frac{t}{2}\big[u(x) + \tau\delta_{x_0}(x)\big]^2 \; dx \\ = &\frac{1}{4}\int_{\overline{S}} |\nabla_\omega u(x)|^2 + 2\tau\nabla_\omega u(x)\cdot\nabla\delta_{x_0}(x) + \tau^2|\nabla_\omega\delta_{x_0}(x)|^2 \; dx \\ &+ \frac{t}{2}\int_{\overline{S}} \big[u(x)\big]^2 + 2\tau u(x)\delta_{x_0}(x) + \tau^2[\delta_{x_0}(x)]^2 \; dx, \end{aligned}$$

it follows from (4.9) that

$$\begin{aligned} 0 &= \int_{\overline{S}} \frac{1}{2}\nabla_\omega u(x)\cdot\nabla_\omega\delta_{x_0} + qu(x)\delta_{x_0}(x) \; dx \\ &= \int_{\overline{S}} \delta_{x_0}(x)\big[-\Delta_\omega u(x) + qu(x)\big] \; dx \\ &= -\Delta_\omega u(x_0) + q(x_0)u(x_0). \end{aligned}$$

Since $x_0 \in S$ is chosen arbitrarily, we get the result. □

Remark 4.3. On preparing this paper, authors found a paper [2] and a preprint [3] by Bendito, Carmona, Encinas and Gesto which also deal with the Schrödinger equations on networks and generalize Theorem 4.2 to allow more general conditions on the function q than the condition given in Theorem 4.2.

Now we are in a position to state and to prove the main result of this paper. In the following theorem, the positive valued function ρ on S and constants $c_0, c_1, \ldots, c_l$, where at least one of c_j among $j = 1, 2, \ldots, l$ is not zero, are assumed to be given.

Theorem 4.4. *Let S be a subnetwork of a network G with $\partial S \neq \emptyset$ and ω_1 and ω_2 be weights on the same network $\overline{S}$ with $\omega_1(x,y) \leq \omega_2(x,y),\ (x,y) \in \overline{S}\times\overline{S}$. Suppose constants $c_0, c_1, \ldots, c_n$ satisfy $\sum_{n=0}^{l} c_n t^n > 0,\ t \in (0,\infty)$. Let $u_j : \overline{S} \to \mathbb{R},\ j = 1, 2$ be functions in $\mathcal{C}^{l-1}\big(\overline{S}\times[0,\infty)\big)$ satisfying*

$$\begin{cases} \big(\sum_{n=0}^{l} c_n\partial_t^n - \rho(x)\Delta_{\omega_j}\big)u_j(x,t) = 0, & (x,t) \in S\times(0,\infty), \\ u_j(z,t) = \sigma(z,t), & (z,t) \in \partial S\times[0,\infty) \\ \partial_t^n u_j(x,0) = 0,\ n = 0, 1, \ldots, l, & x \in S, \end{cases} \tag{4.10}$$

for $j = 1, 2$. *If we assume that*

(i) $\omega_1(z, y) = \omega_2(z, y), \quad (z, y) \in \partial S \times S,$
(ii) $\frac{\partial u_1}{\partial_{\omega_1} n} \equiv \frac{\partial u_2}{\partial_{\omega_2} n} \in \mathcal{C}^{l-1}\big(\partial S \times [0, \infty)\big),$
(iii) *for each* $z \in \partial S$, *there exists* $\gamma > 0$ *such that* $\sigma(z, t) = \mathbf{O}(e^{\gamma t})$,

then we have

(i) $u_1 = u_2$ *on* $\overline{S} \times [0, \infty)$,
(ii) $\omega_1(x, y) = \omega_2(x, y)$ *if* $u(x, t) \neq u(y, t)$ *for some* $t > 0$, *where* $u := u_1 = u_2$.

Proof. Let $\psi : \partial S \times [0, \infty) \to \mathbb{R}$ be the function defined by $\psi(z, t) := \frac{\partial u_1}{\partial_{\omega_1} n}(z, t) = \frac{\partial u_2}{\partial_{\omega_2} n}(z, t)$, $(z, t) \in \partial S \times [0, \infty)$. By (3.7) and (3.8) in Theorem 3.2 and the condition of σ in the assumption (iii), it is easy to see that there exist $L > 0$ such that $u(x, t) = \mathbf{O}(e^{Lt}),\ x \in \overline{S}$. Now take Laplace transform $\hat{f}(s) := \int_0^\infty f(t)e^{-st}dt$ with respect to the variable t of the equation (4.10) to get the following equation

$$\begin{cases} \rho^{-1}(x)P(s)\hat{u_j}(x, s) - \Delta_{\omega_j}\hat{u_j}(x, s) = 0, & (x, s) \in S \times (L, \infty), \\ \hat{u_j}(z, s) = \hat{\sigma}(z, s), (z, s) \in \partial S \times (L, \infty) \end{cases} \tag{4.11}$$

for $j = 1, 2$ with

$$\frac{\partial \hat{u_1}}{\partial_{\omega_1} n}(z, s) = \frac{\partial \hat{u_2}}{\partial_{\omega_2} n}(z, s) = \hat{\psi}(z, s),\ (z, s) \in \partial S \times (L, \infty),$$

where $P(s) = \sum_{n=0}^{l} c_n s^n$. For each (fixed) $s > L$, (4.11) is a Dirichlet boundary value problem for a Schrödinger equation with $q(x) = \rho^{-1}(x)P(s)$. Note that, from *Remark* 2.1 and the coincidence of the Neumann data, we have

$$\Delta_{\omega_1}\hat{u_1}(z, s) = -\hat{\psi}(z, s) = \Delta_{\omega_2}\hat{u_2}(z, s), \quad (z, s) \in \partial S \times (0, \infty). \tag{4.12}$$

By virtue of the condition $\omega_1 \leq \omega_2$, we have for each $v : \overline{S} \to \mathbb{R}$,

$$\begin{aligned} |\nabla_{\omega_1} v(x)|^2 &= \sum_{y \in \overline{S}} \big[v(x) - v(y)\big]^2 \omega_1(x, y) \\ &\leq \sum_{y \in \overline{S}} \big[v(x) - v(y)\big]^2 \omega_2(x, y) = |\nabla_{\omega_2} v(x)|^2 \end{aligned} \tag{4.13}$$

for $x \in \overline{S}$. In what follows, since there is no worry of confusion, the notation $I_{\omega, \rho^{-1}P(s)}$ is denoted by $I_{\omega, s}$ for simplicity. It follows from (4.12), (4.13) and

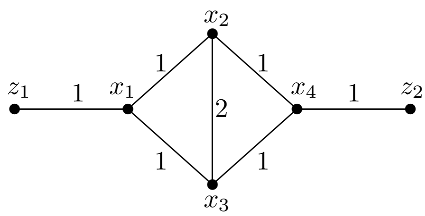

FIGURE 1

Theorem 2.2 that for each $s > L$, we have

(4.14)

$$\begin{aligned}
I_{\omega_1,s}[\hat{u_1}(\cdot,s)] &= \frac{1}{2}\int_{\overline{S}} \hat{u_1}(x,s)\big[-\Delta_{\omega_1}\hat{u_1}(x,s)+\rho^{-1}(x)P(s)U_1(x,s)\big]\ dx\\
&= \frac{1}{2}\int_{\partial S} \hat{u_2}(x,s)\big[-\Delta_{\omega_2}\hat{u_2}(x,s)+\rho^{-1}(x)P(s)U_2(x,s)\big]\ dx\\
&= \frac{1}{2}\int_{\overline{S}} \hat{u_2}(x,s)\big[-\Delta_{\omega_2}\hat{u_2}(x,s)+\rho^{-1}(x)P(s)U_2(x,s)\big]\ dx\\
&= \int_{\overline{S}} \frac{1}{4}|\nabla_{\omega_2}\hat{u_2}(x,s)|^2+\frac{1}{2}\rho^{-1}(x)P(s)\big[\hat{u_2}(x,s)\big]^2\ dx\\
&\geq I_{\omega_1,s}[\hat{u_2}(\cdot,s)].
\end{aligned}$$

Since $\hat{u_2}(\cdot,s)\in A_{\hat{\sigma}(\cdot,s)},\ s>L$, by virtue of Theorem 4.2, we have

$$\hat{u_1}\equiv\hat{u_2}\quad \text{ in } \overline{S}\times(L,\infty).$$

Let $\hat{u}:=\hat{u_1}=\hat{u_2}$ on $\overline{S}\times(L,\infty)$. Since a calculation in (4.14) shows

$$I_{\omega_1,s}[\hat{u}(\cdot,s)]=I_{\omega_2,s}[\hat{u}(\cdot,s)],\quad s>L,$$

we have, from an easy calculation, the following

$$\frac{1}{4}\sum_{x\in\overline{S}}\sum_{y\in\overline{S}}\big[\hat{u}(x,s)-\hat{u}(y,s)\big]^2\big[\omega_2(x,y)-\omega_1(x,y)\big]=0,\quad s>L,$$

which implies, $\omega_1(x,y)=\omega_2(x,y)$ whenever $u(x,s)\neq u(y,s)$, for some $s>L$. This completes the proof. $\square$

In the previous theorem, the unique identifiability of the weight ω is guaranteed only if for each $x\sim y\in\overline{S}$, there exist at least one $t>0$ such that $u(x,t)\neq u(y,t)$. We give an example which illustrates that this condition should not be omitted.

Consider a network $(\overline{S},E)$, $S=\{x_1,x_2,x_3,x_4\}$ with $\partial S=\{z_1,z_2\}$ which are connected by edges as in the Figure 1. Suppose that ω_1 is the standard weight and ω_2 is the weight given by $\omega_1=\omega_2$ except for $\omega_2(x_2,x_3)=2$ (See

Figure 1). A calculation shows that the eigenvalues and the eigenfunctions of the operators $\Delta_{\omega_j},\ j=1,2$ are given by

$$\nu_1=\frac{1}{2}(5-\sqrt{17}),\ \nu_2=3,\ \nu_3=4,\ \nu_4=\frac{1}{2}(5+\sqrt{17}),$$

$$\begin{aligned}
\phi_1 &= \left(\frac{1}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}+\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}+\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{1}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}}\right),\\
\phi_2 &= \left(-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right),\quad \phi_3=\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\\
\phi_4 &= \left(\frac{1}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}-\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}-\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{1}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}}\right),
\end{aligned}$$

and

$$\nu_1=\frac{1}{2}(5-\sqrt{17}),\ \nu_2=3,\ \nu_3=\frac{1}{2}(5+\sqrt{17}),\ \nu_4=6,$$

$$\begin{aligned}
\phi_1 &= \left(\frac{1}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}+\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}+\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}},\frac{1}{\sqrt{\frac{17}{4}+\frac{\sqrt{17}}{4}}}\right),\\
\phi_2 &= \left(-\frac{1}{\sqrt{2}},0,0,\frac{1}{\sqrt{2}}\right),\\
\phi_3 &= \left(\frac{1}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}-\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{\frac{1}{4}-\frac{\sqrt{17}}{4}}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}},\frac{1}{\sqrt{\frac{17}{4}-\frac{\sqrt{17}}{4}}}\right),\\
\phi_4 &= \left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),
\end{aligned}$$

respectively. Then it follows from the result in Corollary 3.4 with a little calculation that for any Dirichlet data $\sigma(z,t)\in\mathcal{C}\big(\overline{S}\times[0,\infty)\big)$ of the following evolution equations

$$\begin{cases}
\partial_t u_j(x,t)-\Delta_{\omega_j}u_j(x,t)=0, & (x,t)\in S\times(0,\infty),\\
u_j(z,t)=\sigma(z,t),(z,t)\in\partial S\times[0,\infty)\\
u_j(x,0)=0,x\in S,
\end{cases}$$

for $j=1,2$, we have $u_j(x_2,t)=u_j(x_3,t),\ t>0$, for $j=1,2$. Now, although $\omega_1\le\omega_2$, their Neumann data $\frac{\partial u_1}{\partial_{\omega_1}n}$ and $\frac{\partial u_2}{\partial_{\omega_2}n}$ are calculated to be the same as

$$\begin{aligned}
\tfrac{\partial u_1}{\partial_{\omega_1}n}(z_1,t)= \quad & -\textstyle\int_0^t\left(\tfrac{1}{2}e^{3(t-\tau)}+\dfrac{e^{\frac{1}{2}(5-\sqrt{17})(t-\tau)}}{\frac{17}{4}+\frac{\sqrt{17}}{4}}+\dfrac{e^{\frac{1}{2}(5+\sqrt{17})(t-\tau)}}{\frac{17}{4}-\frac{\sqrt{17}}{4}}\right)\sigma(z_1,\tau)d\tau\\
& -\textstyle\int_0^t\left(-\tfrac{1}{2}e^{3(t-\tau)}+\dfrac{e^{\frac{1}{2}(5-\sqrt{17})(t-\tau)}}{\frac{17}{4}+\frac{\sqrt{17}}{4}}+\dfrac{e^{\frac{1}{2}(5+\sqrt{17})(t-\tau)}}{\frac{17}{4}-\frac{\sqrt{17}}{4}}\right)\sigma(z_2,\tau)d\tau\\
& +\sigma(z_1,t)=\tfrac{\partial u_2}{\partial_{\omega_2}n}(z_1,t)
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{\partial u_1}{\partial_{\omega_1}n}(z_2,t)= \quad & -\textstyle\int_0^t\left(-\tfrac{1}{2}e^{3(t-\tau)}+\dfrac{e^{\frac{1}{2}(5-\sqrt{17})(t-\tau)}}{\frac{17}{4}+\frac{\sqrt{17}}{4}}+\dfrac{e^{\frac{1}{2}(5+\sqrt{17})(t-\tau)}}{\frac{17}{4}-\frac{\sqrt{17}}{4}}\right)\sigma(z_1,\tau)d\tau\\
& -\textstyle\int_0^t\left(\tfrac{1}{2}e^{3(t-\tau)}+\dfrac{e^{\frac{1}{2}(5-\sqrt{17})(t-\tau)}}{\frac{17}{4}+\frac{\sqrt{17}}{4}}+\dfrac{e^{\frac{1}{2}(5+\sqrt{17})(t-\tau)}}{\frac{17}{4}-\frac{\sqrt{17}}{4}}\right)\sigma(z_2,\tau)d\tau\\
& +\sigma(z_2,t)=\tfrac{\partial u_2}{\partial_{\omega_2}n}(z_2,t).
\end{aligned}$$

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