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ON THE THEORY OF LORENTZ SURFACES WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR FIELD IN PSEUDO-EUCLIDEAN 4-SPACE

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Abstract. We develop an invariant local theory of Lorentz surfaces in pseudo-Euclidean 4-space by use of a linear map of Weingarten type. We find a geometrically determined moving frame field at each point of the surface and obtain a system of geometric functions. We prove a fundamental existence and uniqueness theorem in terms of these functions. On any Lorentz surface with parallel normalized mean curvature vector field we introduce special geometric (canonical) parameters and prove that any such surface is determined up to a rigid motion by three invariant functions satisfying three natural partial differential equations. In this way we minimize the number of functions and the number of partial differential equations determining the surface, which solves the Lund-Regge problem for this class of surfaces.

1. Introduction

In pseudo-Euclidean spaces there are two types of surfaces according to their induced metric – Riemannian or Lorentz metric. In the present paper we study Lorentz surfaces in pseudo-Euclidean space $\mathbb{E}^4_2$.

Recently, many classification results for Lorentz surfaces in pseudo-Euclidean spaces have been obtained imposing some extra conditions on the mean curvature vector, the Gauss curvature, or the second fundamental form.

In [15] B.-Y. Chen obtained several classification results for minimal Lorentz surfaces in indefinite space forms. In particular, he completely classified all minimal Lorentz surfaces in a pseudo-Euclidean space $\mathbb{E}^m_s$ with arbitrary dimension m and arbitrary index s.

A natural extension of minimal surfaces are quasi-minimal (or marginally trapped) surfaces - these are surfaces whose mean curvature vector is lightlike at each point of the surface. Quasi-minimal surfaces in pseudo-Euclidean space

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have been very actively studied in the last few years. In [4] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}^4_2$ and gave a complete classification of biharmonic Lorentz surfaces in $\mathbb{E}^4_2$ with lightlike mean curvature vector. Several other families of quasi-minimal surfaces have also been classified. For example, quasi-minimal surfaces with constant Gauss curvature in $\mathbb{E}^4_2$ were classified in [5, 21]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [16] and [19]. The classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ is obtained in [18]. In [25] the classification of quasi-minimal rotational surfaces of elliptic, hyperbolic or parabolic type is given. For an up-to-date survey on quasi-minimal surfaces, see also [6].

A Lorentz surface of an indefinite space form is called parallel if its second fundamental form is parallel with respect to the Van der Waerden-Bortolotti connection. Parallel surfaces are important in differential geometry as well as in physics since extrinsic invariants of such surfaces do not change from point to point. Parallel Lorentz surfaces in four-dimensional Lorentzian space forms were studied by B.-Y. Chen and J. Van der Veken in [20]. An explicit classification of parallel Lorentz surfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$, in the pseudo-hyperbolic space $\mathbb{H}^4_2(-1)$, and in the neutral pseudo-sphere $\mathbb{S}^4_2(1)$ is given by Chen et al. in [9], [10] and [17], respectively. The complete classification of parallel Lorentz surfaces in a pseudo-Euclidean space with arbitrary codimension and arbitrary index is obtained in [11].

Another basic class of surfaces in Riemannian and pseudo-Riemannian geometry are the surfaces with parallel mean curvature vector field, since they are critical points of some natural functionals and play important role in differential geometry, the theory of harmonic maps, as well as in physics. Surfaces with parallel mean curvature vector field in Riemannian space forms were classified in the early 1970s by Chen [2] and Yau [32]. Recently, spacelike surfaces with parallel mean curvature vector field in arbitrary indefinite space forms were classified in [7] and [8]. A complete classification of Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{E}^m_s$ is given in [12] and [22]. A survey on classical and recent results concerning submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [13].

A natural extension of the class of surfaces with parallel mean curvature vector field are surfaces with parallel normalized mean curvature vector field. A surface M in a Riemannian manifold is said to have parallel normalized mean curvature vector field if the mean curvature vector H is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [3]. The condition to have parallel normalized mean curvature vector field is weaker than the condition to have parallel mean curvature vector field. It is known that every surface in the Euclidean 3-space has parallel normalized mean curvature vector field but in the 4-dimensional Euclidean space, there exist abundant examples of surfaces which lie fully in $\mathbb{E}^4$ with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field.

In [3] it is proved that every analytic surface with parallel normalized mean curvature vector in the Euclidean space $\mathbb{E}^m$ must either lie in a 4-dimensional space $\mathbb{E}^4$ or in a hypersphere of $\mathbb{E}^m$ as a minimal surface. Spacelike submanifolds with parallel normalized mean curvature vector field in a general de Sitter space are studied in [30]. It is shown that compact spacelike submanifolds whose mean curvature does not vanish and whose corresponding normalized vector field is parallel, must be, under some suitable geometric assumptions, totally umbilical.

In the present paper we study the local theory of Lorentz surfaces with parallel normalized mean curvature vector field in the pseudo-Euclidean space $\mathbb{E}^4_2$. Our approach to the study of these surfaces is based on the introduction of canonical parameters.

In Section 3 we develop an invariant theory of Lorentz surfaces in $\mathbb{E}^4_2$ similarly to the theory of surfaces in the Euclidean space $\mathbb{E}^4$ and the theory of spacelike surfaces in the Minkowski space $\mathbb{E}^4_1$. We introduce an invariant linear map γ of Weingarten-type in the tangent plane at any point of the surface, which generates two invariant functions $k = \det \gamma$ and $\varkappa = -\frac{1}{2}\,\mathrm{tr}\gamma$. In the case $\varkappa^2 - k > 0$ we introduce principal lines and a geometrically determined moving frame field at each point of the surface. Writing derivative formulas of Frenet-type for this frame field, we obtain a system of geometric functions and prove a fundamental existence and uniqueness theorem, stating that these functions determine the surface up to a rigid motion in $\mathbb{E}^4_2$.

The basic geometric classes of surfaces in $\mathbb{E}^4_2$ such as quasi-minimal surfaces, surfaces with flat normal connection, surfaces with constant Gauss curvature or constant normal curvature, surfaces with parallel mean curvature vector field, etc., are characterized by conditions on their geometric functions.

We focus our attention on the class of surfaces with parallel normalized mean curvature vector field. We introduce canonical parameters on each such surface that allow us to formulate the fundamental existence and uniqueness theorem in terms of three invariant functions. Our main result states that any Lorentz surface with parallel normalized mean curvature vector field is determined up to a rigid motion in $\mathbb{E}^4_2$ by three invariant functions satisfying a system of three natural partial differential equations (Theorem 6.4). This theorem solves the Lund-Regge problem for the class of surfaces with parallel normalized mean curvature vector field in $\mathbb{E}^4_2$.

2. Preliminaries

We consider the pseudo-Euclidean 4-dimensional space $\mathbb{E}^4_2$ endowed with the canonical pseudo-Euclidean metric of index 2 given in local coordinates by

$$g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}^4_2$. As usual, we denote by $\langle\,,\rangle$ the indefinite inner scalar product with respect to g_0.

A vector v is said to be *spacelike* (respectively, *timelike*) if $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A vector v is called *lightlike* if it is nonzero and satisfies $\langle v, v\rangle = 0$.

A surface M^2_1 in $\mathbb{E}^4_2$ is called *Lorentz* if the induced metric g on M^2_1 is Lorentzian. So, at each point $p \in M^2_1$ we have the following decomposition

$$\mathbb{E}^4_2 = T_pM^2_1 \oplus N_pM^2_1$$

with the property that the restriction of the metric onto the tangent space $T_pM^2_1$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_pM^2_1$ is of signature $(1, 1)$.

We denote by ∇ and ∇' the Levi Civita connections of M^2_1 and $\mathbb{E}^4_2$, respectively. For vector fields x, y tangent to M^2_1 and a vector field ξ normal to M^2_1, the formulas of Gauss and Weingarten, giving a decomposition of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components, are given respectively by [2]:

$$\nabla'_x y = \nabla_x y + \sigma(x, y);$$
$$\nabla'_x \xi = -A_\xi x + D_x \xi.$$

These formulas define the second fundamental form σ, the normal connection D, and the shape operator A_ξ with respect to ξ. For each normal vector field ξ, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_pM^2_1$ at $p \in M^2_1$. In general, A_ξ is not diagonalizable. It is well known that the shape operator and the second fundamental form are related by the formula

$$\langle \sigma(x, y), \xi\rangle = \langle A_\xi x, y\rangle$$

for x, y tangent to M^2_1 and ξ normal to M^2_1.

The mean curvature vector field H of M^2_1 in $\mathbb{E}^4_2$ is defined as $H = \frac{1}{2}\operatorname{tr}\sigma$. The surface M^2_1 is called *minimal* if its mean curvature vector vanishes identically, i.e., $H = 0$. A natural extension of minimal surfaces are quasi-minimal surfaces. The surface M^2_1 is called *quasi-minimal* (or *pseudo-minimal*) if its mean curvature vector is lightlike at each point, i.e., $H \neq 0$ and $\langle H, H\rangle = 0$ [29]. Obviously, quasi-minimal surfaces are always non-minimal.

A normal vector field ξ on M^2_1 is called *parallel in the normal bundle* (or simply *parallel*) if $D\xi = 0$ holds identically [14]. The surface M^2_1 is said to have *parallel mean curvature vector field* if its mean curvature vector H satisfies $DH = 0$ identically.

Surfaces for which the mean curvature vector H is non-zero and there exists a parallel unit vector field b in the direction of the mean curvature vector H, such that b is parallel in the normal bundle, are called surfaces with *parallel normalized mean curvature vector field* [3]. It is easy to see that if M^2_1 is a surface with non-zero parallel mean curvature vector field H (i.e., $DH = 0$), then M^2_1 is a surface with parallel normalized mean curvature vector field, but the converse is not true in general. It is true only in the case $\|H\| = const$.

A submanifold M^2_1 of a pseudo-Riemannian manifold is called *totally geodesic* if the second fundamental form σ of M^2_1 vanishes identically. It is called *totally umbilical* if its second fundamental form satisfies $\sigma(x, y) = \langle x, y\rangle H$ for arbitrary vector fields x, y tangent to M^2_1.

3. Weingarten map of a Lorentz surface in $\mathbb{E}^4_2$

Let $M^2_1 : z = z(u, v)$, $(u, v) \in \mathcal{D}$ $(\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on a Lorentz surface in $\mathbb{E}^4_2$. The tangent space T_pM^2 at an arbitrary point $p = z(u, v)$ of M^2 is spanned by the vector fields z_u and z_v. We assume that $\langle z_u, z_u\rangle > 0$ and $\langle z_v, z_v\rangle < 0$. Hence the coefficients of the first fundamental form of M^2_1 are $E = \langle z_u, z_u\rangle$, $F = \langle z_u, z_v\rangle$, $G = \langle z_v, z_v\rangle$, where $E > 0$ and $G < 0$. We denote $W = \sqrt{|EG - F^2|}$.

We consider a normal frame field $\{n_1, n_2\}$ satisfying the conditions $\langle n_1, n_1\rangle = 1$, $\langle n_1, n_2\rangle = 0$, $\langle n_2, n_2\rangle = -1$. Then we have the following derivative formulas:

$$\begin{aligned}
\nabla'_{z_u} z_u &= z_{uu} = \Gamma^1_{11} z_u - \Gamma^2_{11} z_v + c^1_{11} n_1 - c^2_{11} n_2,\\
\nabla'_{z_u} z_v &= z_{uv} = \Gamma^1_{12} z_u - \Gamma^2_{12} z_v + c^1_{12} n_1 - c^2_{12} n_2,\\
\nabla'_{z_v} z_v &= z_{vv} = \Gamma^1_{22} z_u - \Gamma^2_{22} z_v + c^1_{22} n_1 - c^2_{22} n_2,
\end{aligned} \tag{1}$$

where Γ^k_{ij} are the Christoffel's symbols and the functions $c^k_{ij},\ i, j, k = 1, 2$ are given by

$$\begin{aligned}
c^1_{11} &= \langle z_{uu}, n_1\rangle; \quad c^2_{11} = \langle z_{uu}, n_2\rangle;\\
c^1_{12} &= \langle z_{uv}, n_1\rangle; \quad c^2_{12} = \langle z_{uv}, n_2\rangle;\\
c^1_{22} &= \langle z_{vv}, n_1\rangle; \quad c^2_{22} = \langle z_{vv}, n_2\rangle.
\end{aligned}$$

Obviously, if $c^k_{ij} = 0,\ i, j, k = 1, 2$, then M^2_1 is totally geodesic and hence, the surface M^2_1 is an open part of a pseudo-Euclidean linear subspace of $\mathbb{E}^4_2$, i.e., M^2_1 is part of a 2-plane. So, we assume that at least one of the coefficients c^k_{ij} is not zero.

It follows from (1) that

$$\begin{aligned}
\sigma(z_u, z_u) &= c^1_{11} n_1 - c^2_{11} n_2;\\
\sigma(z_u, z_v) &= c^1_{12} n_1 - c^2_{12} n_2;\\
\sigma(z_v, z_v) &= c^1_{22} n_1 - c^2_{22} n_2.
\end{aligned} \tag{2}$$

Using c^k_{ij} we introduce the following functions:

$$\Delta_1 = \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^2_{11} & c^2_{12} \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} c^1_{11} & c^1_{22} \\ c^2_{11} & c^2_{22} \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} c^1_{12} & c^1_{22} \\ c^2_{12} & c^2_{22} \end{vmatrix};$$

$$L(u,v) = \frac{2\Delta_1}{W}; \qquad M(u,v) = \frac{\Delta_2}{W}; \qquad N(u,v) = \frac{2\Delta_3}{W}.$$

Let

$$u = u(\bar u, \bar v); \quad v = v(\bar u, \bar v), \quad (\bar u, \bar v) \in \bar{\mathcal{D}},\ \bar{\mathcal{D}} \subset \mathbb{R}^2,$$

be a smooth change of the parameters (u,v) on M^2 with $J = u_{\bar u}\, v_{\bar v} - u_{\bar v}\, v_{\bar u} \neq 0$. Then

$$\begin{aligned} z_{\bar u} &= z_u\, u_{\bar u} + z_v\, v_{\bar u},\\ z_{\bar v} &= z_u\, u_{\bar v} + z_v\, v_{\bar v}. \end{aligned} \tag{3}$$

If $\overline{E} = \langle z_{\bar u}, z_{\bar u}\rangle$, $\overline{F} = \langle z_{\bar u}, z_{\bar v}\rangle$ and $\overline{G} = \langle z_{\bar v}, z_{\bar v}\rangle$, then we get

$$\begin{aligned} \overline{E} &= u_{\bar u}^2\, E + 2\, u_{\bar u} v_{\bar u}\, F + v_{\bar u}^2\, G,\\ \overline{F} &= u_{\bar u} u_{\bar v}\, E + (u_{\bar u} v_{\bar v} + v_{\bar u} u_{\bar v})\, F + v_{\bar u} v_{\bar v}\, G,\\ \overline{G} &= u_{\bar v}^2\, E + 2\, u_{\bar v} v_{\bar v}\, F + v_{\bar v}^2\, G \end{aligned}$$

and hence, $\overline{E}\,\overline{G} - \overline{F}^2 = J^2\,(EG - F^2)$, $\overline{W} = \varepsilon J\, W$, where $\varepsilon = \operatorname{sign} J$.

Further we calculate the functions $\bar c_{ij}^k$ which take part in the following equalities:

$$\begin{aligned} \sigma(z_{\bar u}, z_{\bar u}) &= \bar c_{11}^1\, n_1 - \bar c_{11}^2\, n_2;\\ \sigma(z_{\bar u}, z_{\bar v}) &= \bar c_{12}^1\, n_1 - \bar c_{12}^2\, n_2;\\ \sigma(z_{\bar v}, z_{\bar v}) &= \bar c_{22}^1\, n_1 - \bar c_{22}^2\, n_2. \end{aligned} \tag{4}$$

Using (2), (3), and (4) we find

$$\begin{aligned} \bar c_{11}^k &= u_{\bar u}^2\, c_{11}^k + 2u_{\bar u}\, v_{\bar u}\, c_{12}^k + v_{\bar u}^2\, c_{22}^k,\\ \bar c_{12}^k &= u_{\bar u}\, u_{\bar v}\, c_{11}^k + (u_{\bar u}\, v_{\bar v} + u_{\bar v}\, v_{\bar u})\, c_{12}^k + v_{\bar u}\, v_{\bar v}\, c_{22}^k, \qquad (k = 1,2),\\ \bar c_{22}^k &= u_{\bar v}^2\, c_{11}^k + 2u_{\bar v}\, v_{\bar v}\, c_{12}^k + v_{\bar v}^2\, c_{22}^k, \end{aligned}$$

and hence

$$\begin{aligned} \bar\Delta_1 &= J\left(u_{\bar u}^2\, \Delta_1 + u_{\bar u}\, v_{\bar u}\, \Delta_2 + v_{\bar u}^2\, \Delta_3\right);\\ \bar\Delta_2 &= J\left(2u_{\bar u}\, u_{\bar v}\, \Delta_1 + (u_{\bar u}\, v_{\bar v} + u_{\bar v}\, v_{\bar u})\, \Delta_2 + 2v_{\bar u}\, v_{\bar v}\, \Delta_3\right);\\ \bar\Delta_3 &= J\left(u_{\bar v}^2\, \Delta_1 + u_{\bar v}\, v_{\bar v}\, \Delta_2 + v_{\bar v}^2\, \Delta_3\right). \end{aligned}$$

Thus we find that the functions $\overline{L}$, $\overline{M}$, $\overline{N}$ are expressed as follows:

$$\begin{aligned} \overline{L} &= \varepsilon(u_{\bar u}^2\, L + 2\, u_{\bar u} v_{\bar u}\, M + v_{\bar u}^2\, N),\\ \overline{M} &= \varepsilon(u_{\bar u} u_{\bar v}\, L + (u_{\bar u} v_{\bar v} + v_{\bar u} u_{\bar v})\, M + v_{\bar u} v_{\bar v}\, N),\\ \overline{N} &= \varepsilon(u_{\bar v}^2\, L + 2\, u_{\bar v} v_{\bar v}\, M + v_{\bar v}^2\, N). \end{aligned}$$

Hence, the functions L, M, N change in the same way as the coefficients of the first fundamental form E, F, G under any change of the parameters on M_1^2.

If we take a vector field $X \in T_pM^2_1$, such that $X = \lambda z_u + \mu z_v = \bar\lambda z_{\bar u} + \bar\mu z_{\bar v}$, then $\lambda = u_{\bar u}\bar\lambda + u_{\bar v}\bar\mu$, $\mu = v_{\bar u}\bar\lambda + v_{\bar v}\bar\mu$. So, for each two tangent vector fields $X_1 = \lambda_1 z_u + \mu_1 z_v$ and $X_2 = \lambda_2 z_u + \mu_2 z_v$ we can consider the quadratic form with coefficients L, M, N and we have the following equality:

$$\overline{L}\bar\lambda_1\bar\lambda_2 + \overline{M}(\bar\lambda_1\bar\mu_2 + \bar\mu_1\bar\lambda_2) + \overline{N}\bar\mu_1\bar\mu_2 = \varepsilon\,(L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \mu_1\lambda_2) + N\mu_1\mu_2).$$

The last formula allows us to define second fundamental form II of the surface M^2_1 at $p \in M^2_1$ as follows. Let $X = \lambda z_u + \mu z_v$, $(\lambda,\mu) \neq (0,0)$ be a tangent vector at a point $p \in M^2_1$. Then

$$II(\lambda,\mu) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda,\mu \in \mathbb{R}.$$

Further we will show that the functions L, M, N do not depend on the choice of the normal frame of the surface. Let $\{\widetilde n_1, \widetilde n_2\}$ be another normal frame field of M^2_1, such that $\langle \widetilde n_1, \widetilde n_1\rangle = 1$, $\langle \widetilde n_1, \widetilde n_2\rangle = 0$, $\langle \widetilde n_2, \widetilde n_2\rangle = -1$. The relation between the two normal frame fields is given by

$$n_1 = \varepsilon'(\cosh\theta\,\widetilde n_1 + \sinh\theta\,\widetilde n_2); \quad n_2 = \varepsilon'(\sinh\theta\,\widetilde n_1 + \cosh\theta\,\widetilde n_2), \quad \varepsilon' = \pm 1$$

for some smooth function θ, and the relation between the corresponding functions c^k_{ij} and $\widetilde c^k_{ij}$, $i,j,k = 1,2$ is

$$\widetilde c^1_{ij} = \varepsilon'(\cosh\theta\, c^1_{ij} - \sinh\theta\, c^2_{ij}); \quad \widetilde c^2_{ij} = \varepsilon'(-\sinh\theta\, c^1_{ij} + \cosh\theta\, c^2_{ij}), \quad i,j = 1,2.$$

Thus, $\widetilde\Delta_i = \Delta_i$, $i = 1,2,3$, and $\widetilde L = L$, $\widetilde M = M$, $\widetilde N = N$. So, the functions L, M, N do not depend on the normal frame of the surface.

Hence, the second fundamental form II is invariant up to the orientation of the tangent space or the normal space of the surface.

Such a bilinear form has been considered for an arbitrary 2-dimensional surface in a 4-dimensional affine space $\mathbb{A}^4$ (see for example [1, 27, 31]). Here we use this form for Lorentz surfaces in $\mathbb{E}^4_2$ and taking into consideration also the first fundamental form we develop the theory of Lorentz surfaces similar to the theory of surfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_1$.

It follows from formulas (2) that the condition $L(u,v) = M(u,v) = N(u,v) = 0$, $(u,v) \in \mathcal{D}$ characterizes points at which the space $\{\sigma(x,y) : x,y \in T_pM^2_1\}$ is one-dimensional. We call such points *flat points* of the surface since they are analogous to flat points in the theory of surfaces in $\mathbb{R}^3$. In [27] and [28] such points are called inflection points. E. Lane has shown that every point of a surface is an inflection point if and only if the surface is either developable or lies in a 3-dimensional space [27]. So, further we consider surfaces free of flat points, i.e., we assume that $(L,M,N) \neq (0,0,0)$.

The second fundamental form II determines a map of Weingarten-type $\gamma : T_pM^2_1 \to T_pM^2_1$ at any point of M^2_1 in the standard way:

$$\gamma(z_u) = \gamma^1_1 z_u + \gamma^2_1 z_v,$$
$$\gamma(z_v) = \gamma^1_2 z_u + \gamma^2_2 z_v,$$

where

$$\gamma_1^1 = \frac{FM - GL}{EG - F^2}, \quad \gamma_1^2 = \frac{FL - EM}{EG - F^2}, \quad \gamma_2^1 = \frac{FN - GM}{EG - F^2}, \quad \gamma_2^2 = \frac{FM - EN}{EG - F^2}.$$

The linear map γ is invariant under changes of the parameters of the surface and changes of the normal frame field. In general, γ is not diagonalizable.

As in the classical differential geometry of surfaces the map γ generates the following invariants:

$$k := \det \gamma = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa := -\frac{1}{2} \operatorname{tr} \gamma = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

The functions k and $\varkappa$ are invariant under changes of the parameters of the surface and changes of the normal frame field. Next we shall prove the following:

Proposition 3.1. *The function $\varkappa$ is the curvature of the normal connection of the surface M^2_1.*

Proof. The curvature tensor $R^\perp$ of the normal connection D is given by

$$R^\perp(x, y)n = D_x D_y n - D_y D_x n - D_{[x,y]} n,$$

where x, y are tangent vector fields and n is a normal vector field of M^2_1. The curvature of the normal connection at a point $p \in M^2_1$ is defined by $\langle R^\perp(x, y)n_2, n_1\rangle$, where $\{x, y, n_1, n_2\}$ is a right oriented orthonormal quadruple.

Without loss of generality we assume that $F = 0$ and take x, y to be the unit vector fields in the direction of z_u and z_v, i.e., $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{-G}}$. Let $\{n_1, n_2\}$ be a normal frame field such that $\langle n_1, n_1\rangle = 1$, $\langle n_1, n_2\rangle = 0$, $\langle n_2, n_2\rangle = -1$. Denote by A_1 (resp. A_2) the shape operator corresponding to n_1 (resp. n_2).

Since the curvature tensor R' of the connection ∇' is zero, we have

$$\nabla'_x \nabla'_y n_1 - \nabla'_y \nabla'_x n_1 - \nabla'_{[x,y]} n_1 = 0.$$

Therefore the tangent component and the normal component of $R'(x, y)n_1$ are both zero. The normal component is $D_x D_y n_1 - D_y D_x n_1 - D_{[x,y]} n_1 - \sigma(x, A_1(y)) + \sigma(y, A_1(x))$. Hence,

$$D_x D_y n_1 - D_y D_x n_1 - D_{[x,y]} n_1 = \sigma(x, A_1(y)) - \sigma(y, A_1(x)).$$

The left-hand side of the last equality is $R^\perp(x, y)n_1$. Then

$$\langle R^\perp(x, y)n_1, n_2\rangle = \langle (A_2 \circ A_1 - A_1 \circ A_2)(y), x\rangle. \tag{5}$$

Further we consider the operator $A_2 \circ A_1 - A_1 \circ A_2$. For the second fundamental tensor σ we have:

$$\sigma(x,x) = \frac{c_{11}^1}{E}\; n_1 \;-\; \frac{c_{11}^2}{E}\; n_2,$$
$$\sigma(x,y) = \frac{c_{12}^1}{\sqrt{-EG}}\, n_1 - \frac{c_{12}^2}{\sqrt{-EG}}\, n_2, \tag{6}$$
$$\sigma(y,y) = -\frac{c_{22}^1}{G}\; n_1 \;+\; \frac{c_{22}^2}{G}\; n_2.$$

Using the last formulas we calculate the vector fields $A_1(x)$, $A_1(y)$, $A_2(x)$, $A_2(y)$ and obtain:

$$A_1(x) = \frac{c_{11}^1}{E}\, x - \frac{c_{12}^1}{\sqrt{-EG}}\, y, \qquad A_2(x) = \frac{c_{11}^2}{E}\, x - \frac{c_{12}^2}{\sqrt{-EG}}\, y, \tag{7}$$
$$A_1(y) = \frac{c_{12}^1}{\sqrt{-EG}}\, x + \frac{c_{22}^1}{G}\, y, \qquad A_2(y) = \frac{c_{12}^2}{\sqrt{-EG}}\, x + \frac{c_{22}^2}{G}\, y.$$

Taking in mind (7) we get:

$$(A_2 \circ A_1 - A_1 \circ A_2)(x) = -\left(\frac{c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1}{E\sqrt{-EG}} + \frac{c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1}{G\sqrt{-EG}}\right) y;$$
$$(A_2 \circ A_1 - A_1 \circ A_2)(y) = -\left(\frac{c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1}{E\sqrt{-EG}} + \frac{c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1}{G\sqrt{-EG}}\right) x.$$

Hence

$$(A_2 \circ A_1 - A_1 \circ A_2)(x) = -\varkappa\, y;$$
$$(A_2 \circ A_1 - A_1 \circ A_2)(y) = -\varkappa\, x.$$

The last equalities imply that:

$$\langle (A_2 \circ A_1 - A_1 \circ A_2)(x), y\rangle = \varkappa; \tag{8}$$
$$\langle (A_2 \circ A_1 - A_1 \circ A_2)(y), x\rangle = -\varkappa.$$

Note that $A_2 \circ A_1 - A_1 \circ A_2$ is an invariant skew-symmetric operator in the tangent space, i.e., it does not depend on the choice of the orthonormal tangent frame field $\{x, y\}$.

Finally, from (5) and (8) we get that $\langle R^\perp(x,y)n_1, n_2\rangle = -\varkappa$. Consequently,

$$\langle R^\perp(x,y)n_2, n_1\rangle = \varkappa. \qquad \square$$

The second fundamental form determines conjugate tangents at a point p of M_1^2 in the same way as in the classical differential geometry of surfaces.

Definition. Two tangents $g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v$ and $g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v$ are said to be *conjugate tangents*, if

$$L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\mu_1\mu_2 = 0.$$

Asymptotic tangents and principal tangents can be defined in the standard way:

Definition. A tangent $g : X = \lambda z_u + \mu z_v$ is said to be *asymptotic*, if it is self-conjugate.

Definition. A tangent $g : X = \lambda z_u + \mu z_v$ is said to be *principal*, if it is perpendicular to its conjugate.

The equation of the asymptotic tangents at a point $p \in M^2_1$ is

$$L\lambda^2 + 2M\lambda\mu + N\mu^2 = 0.$$

The equation of the principal tangents at a point $p \in M^2_1$ is

$$(EM - FL)\lambda^2 + (EN - GL)\lambda\mu + (FN - GM)\mu^2 = 0. \tag{9}$$

A line $c : u = u(q),\ v = v(q);\ q \in J \subset \mathbb{R}$ on M^2_1 is said to be an *asymptotic line*, respectively a *principal line*, if its tangent at any point is asymptotic, respectively principal.

In the theory of surfaces in $\mathbb{E}^4$ and the theory of spacelike surfaces in $\mathbb{E}^4_1$ equation (9) always has solutions since the discriminant of (9) is greater or equal to zero. So, at each point of a surface in $\mathbb{E}^4$ (or a spacelike surface in $\mathbb{E}^4_1$) there exist principal tangents. For a Lorentz surface in $\mathbb{E}^4_2$ the existence of solutions of equation (9) depends on the sign of the invariant $\varkappa^2 - k$. Indeed, it can easily be seen that the discriminant D of (9) is expressed as

$$D = 4(EG - F^2)^2(\varkappa^2 - k).$$

In the case $\varkappa^2 - k > 0$ at each point of the surface there exist two principal tangents. This case corresponds to the case when the map γ is diagonalizable. If $\varkappa^2 - k > 0$ we can assume that the parametric lines of the surface are principal. It is clear that M^2_1 is parameterized by principal lines if and only if $F = 0,\ M = 0$.

Further we study Lorentz surfaces in $\mathbb{E}^4_2$ for which $\varkappa^2 - k > 0$ at each point.

4. Lorentz surfaces free of flat points

4.1. Minimal Lorentz surfaces

Let M^2_1 be a Lorentz surface free of flat points, i.e., $(L, M, N) \neq (0, 0, 0)$. Using the terminology from the classical differential geometry of surfaces, we call a point $p \in M^2_1$ *umbilical* if the coefficients of the first and the second fundamental forms at p are proportional, i.e., $L = \rho\, E,\ M = \rho\, F,\ N = \rho\, G$ for some $\rho \in \mathbb{R}$.

The normal mean curvature vector field of the surface is given by the formula $H = \frac{1}{2}\,\mathrm{tr}\sigma = \frac{1}{2}\,(\sigma(x, x) - \sigma(y, y))$. Recall that M^2_1 is said to be *minimal* if the mean curvature vector $H = 0$. We shall prove the following characterization of minimal Lorentz surfaces in $\mathbb{E}^4_2$.

Proposition 4.1. *Let M^2_1 be a Lorentz surface in $\mathbb{E}^4_2$ free of flat points. Then M^2_1 is minimal if and only if M^2_1 consists of umbilical points.*

Proof. Without loss of generality we assume that $F = 0$ and take x, y to be the unit vector fields defined by $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{-G}}$. It follows from formulas (6) that $\sigma(x,x) - \sigma(y,y) = 0$ if and only if

$$\left(\frac{c^1_{11}}{E} + \frac{c^1_{22}}{G}\right) n_1 - \left(\frac{c^2_{11}}{E} + \frac{c^1_{22}}{G}\right) n_2 = 0,$$

or equivalently,

$$c^1_{22} = -\frac{G}{E}\, c^1_{11};$$
$$c^2_{22} = -\frac{G}{E}\, c^2_{11}. \tag{10}$$

Now, if $H = 0$, then (10) implies

$$\Delta_2 = 0;$$
$$\frac{\Delta_3}{G} = \frac{\Delta_1}{E}.$$

Therefore

$$L = \rho\, E; \quad M = \rho\, F; \quad N = \rho\, G,$$

where ρ is a function on M^2_1. Hence, all points of M^2_1 are umbilical.

Conversely, if $L = \rho\, E$; $M = \rho\, F$; $N = \rho\, G$, $\rho \neq 0$, then the condition $F = 0$ implies that $M = 0$. Hence, $\begin{vmatrix} c^1_{11} & c^1_{22} \\ c^2_{11} & c^2_{22} \end{vmatrix} = 0$, and so there exists a function $\widetilde{\rho}$ such that $c^1_{22} = \widetilde{\rho} c^1_{11}$, $c^2_{22} = \widetilde{\rho} c^2_{11}$. Further, the equality $\frac{L}{E} = \frac{N}{G}$ implies that $\widetilde{\rho} = -\frac{G}{E}$. Hence, equalities (10) hold. Consequently, $\operatorname{tr}\sigma = 0$, i.e., $H = 0$. □

The meaning of the above proposition is that in $\mathbb{E}^4_2$ the Lorentz surfaces consisting of umbilical points are exactly the minimal surfaces. The complete classification of minimal surfaces in an arbitrary indefinite pseudo-Eucliden space $\mathbb{E}^m_s$ is given by B.-Y. Chen in the next theorem.

Theorem 4.2 ([15])**.** *A Lorentz surface in a pseudo-Euclidean m-space $\mathbb{E}^m_s$ is minimal if and only if, locally the surface is parametrized by*

$$z(u,v) = \alpha(u) + \beta(v),$$

where α and β are null curves satisfying $\langle \alpha'(u), \beta'(v)\rangle \neq 0$.

Further, we consider surfaces free of umbilical (minimal) points. Note that if $p \in M^2_1$ is a non-umbilical point and $\varkappa^2 - k > 0$, then there exist exactly two principal tangents passing through p. We assume that M^2_1 is parameterized by principal lines, i.e., $F = 0$ and $M = 0$, and suppose that the principal tangents z_u and z_v are spacelike and timelike, respectively. Consider the unit tangent vector fields x and y defined by $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{-G}}$, i.e., x and y are collinear with the principal directions. The condition $M = 0$ implies that $\sigma(x,x)$ and $\sigma(y,y)$ are collinear. So, there exists a geometrically determined normal vector field n such that

$$\sigma(x,x) = \nu_1 n$$
$$\sigma(y,y) = \nu_2 n,$$

where ν_1 and ν_2 are functions on M^2_1. Hence, the mean curvature vector field H is expressed as follows:

$$H = \frac{\nu_1 - \nu_2}{2}\, n.$$

We have the following possibilities for the mean curvature vector field:

- H is *spacelike*, i.e., $\langle H, H \rangle > 0$;
- H is *timelike*, i.e., $\langle H, H \rangle < 0$;
- H is *lightlike*, i.e., $\langle H, H \rangle = 0$.

If the mean curvature vector is lightlike at each point, i.e., $H \neq 0$ and $\langle H, H \rangle = 0$, the surface M^2_1 is *quasi-minimal*. In what follows we study surfaces free of minimal points and such that the mean curvature vector H is either spacelike or timelike at each point. We call such surfaces *Lorentz surfaces of general type*.

4.2. Lorentz surfaces whose mean curvature vector at each point is a non-zero spacelike vector

In this subsection we consider the case $\langle H, H \rangle > 0$. In order to obtain a geometrically determined frame field at each point p of M^2_1, we denote by b the unit normal vector field defined by $b = \frac{H}{\sqrt{\langle H, H \rangle}}$. Note that $\langle b, b \rangle = 1$ and b is collinear with $\sigma(x,x)$ and $\sigma(y,y)$. Further we consider the unit normal vector field l such that $\{x, y, b, l\}$ is a positively oriented orthonormal frame field in $\mathbb{E}^4_2$. So, $\langle l, l \rangle = -1$. In such a way we obtain a geometrically determined moving frame field at each point of the surface. With respect to this frame field the following formulas are true:

$$\sigma(x,x) = \nu_1\, b;$$
$$\sigma(x,y) = \lambda\, b - \mu\, l;$$
$$\sigma(y,y) = \nu_2\, b,$$

where ν_1, ν_2, λ, μ are functions on M^2_1 determined by the geometric frame field as follows: $\nu_1 = \langle \sigma(x,x), b \rangle$, $\nu_2 = \langle \sigma(y,y), b \rangle$, $\lambda = \langle \sigma(x,y), b \rangle$, $\mu = \langle \sigma(x,y), l \rangle$.

Now with respect to the geometric frame field $\{x, y, b, l\}$ we have the following Frenet-type derivative formulas of M^2_1:

$$\begin{array}{llll} & \nabla'_x x = \quad\;\; -\gamma_1\, y + \nu_1\, b, & & \nabla'_x b = -\nu_1\, x + \lambda\, y \quad\;\; - \beta_1\, l, \\ & \nabla'_x y = -\gamma_1\, x \quad\;\; + \lambda\, b - \mu\, l, & & \nabla'_y b = -\lambda\, x + \nu_2\, y \quad\;\; - \beta_2\, l, \\ (11) & \nabla'_y x = \quad\;\; -\gamma_2\, y + \lambda\, b - \mu\, l, & & \nabla'_x l = \quad\;\; \mu\, y - \beta_1\, b, \\ & \nabla'_y y = -\gamma_2\, x \quad\;\; + \nu_2\, b, & & \nabla'_y l = -\mu\, x \quad\;\; - \beta_2\, b, \end{array}$$

where $\gamma_1 = -y(\ln\sqrt{E}) = \langle \nabla'_x x, y\rangle$, $\gamma_2 = -x(\ln\sqrt{-G}) = \langle \nabla'_y x, y\rangle$, $\beta_1 = \langle \nabla'_x b, l\rangle$ and $\beta_2 = \langle \nabla'_y b, l\rangle$.

The mean curvature vector field is expressed as

$$H = \frac{\nu_1 - \nu_2}{2}\, b.$$

4.3. Lorentz surfaces whose mean curvature vector at each point is a timelike vector

In this subsection we consider the case $\langle H, H\rangle < 0$. Now we set the unit normal vector field b to be $b = -\frac{H}{\sqrt{-\langle H,H\rangle}}$. In this case we have $\langle b, b\rangle = -1$. Taking the normal unit vector field l in such a way that $\{x, y, l, b\}$ be a positively oriented orthonormal frame field in $\mathbb{E}^4_2$, we get a geometrically determined orthonormal frame field at each point of the surface. Note that $\langle l, l\rangle = 1$. The Frenet-type derivative formulas of the surface look as follows:

$$\begin{array}{llll} & \nabla'_x x = \quad\;\; -\gamma_1\, y \quad\;\; - \nu_1\, b, & & \nabla'_x l = \quad\;\; \mu\, y \quad\;\; + \beta_1\, b, \\ & \nabla'_x y = -\gamma_1\, x \quad\;\; + \mu\, l - \lambda\, b, & & \nabla'_y l = -\mu\, x \quad\;\; + \beta_2\, b, \\ (12) & \nabla'_y x = \quad\;\; -\gamma_2\, y + \mu\, l - \lambda\, b, & & \nabla'_x b = -\nu_1\, x + \lambda\, y + \beta_1\, l, \\ & \nabla'_y y = -\gamma_2\, x \quad\;\; - \nu_2\, b, & & \nabla'_y b = -\lambda\, x + \nu_2\, y + \beta_2\, l. \end{array}$$

where ν_1, ν_2, λ, μ, γ_1, γ_2, β_1, β_2 are determined by the geometric moving frame field in the same way as in the previous case. In this case the mean curvature vector field H is expressed by the formula

$$H = -\frac{\nu_1 - \nu_2}{2}\, b.$$

4.4. Fundamental theorem for Lorentz surfaces of general type

The general fundamental existence and uniqueness theorems for submanifolds of pseudo-Riemannian manifolds are formulated in terms of tensor fields and connections on vector bundles (e.g. [14], Theorem 2.4 and Theorem 2.5). In [23] we formulated and proved a fundamental theorem in terms of the geometric functions of a spacelike surface in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. In [24] we proved the fundamental existence and uniqueness theorem for marginally trapped surfaces in terms of their geometric functions. Here we shall formulate and prove the fundamental existence and uniqueness theorem for Lorentz surfaces in $\mathbb{E}^4_2$ of the general class. This theorem is a special case of the general fundamental theorem but in the present form it is more appropriate and easier to apply.

The Frenet-type formulas given in (11) and (12) for the case of spacelike and timelike mean curvature vector field, respectively, can be unified as follows:

$$(13)\qquad \begin{array}{ll} \nabla'_x x = -\gamma_1\, y + \varepsilon\nu_1\, b, & \nabla'_x b = -\nu_1\, x + \lambda\, y - \varepsilon\beta_1\, l, \\ \nabla'_x y = -\gamma_1\, x + \varepsilon\lambda\, b - \varepsilon\mu\, l, & \nabla'_y b = -\lambda\, x + \nu_2\, y - \varepsilon\beta_2\, l, \\ \nabla'_y x = -\gamma_2\, y + \varepsilon\lambda\, b - \varepsilon\mu\, l & \nabla'_x l = \mu\, y - \varepsilon\beta_1\, b, \\ \nabla'_y y = -\gamma_2\, x + \varepsilon\nu_2\, b, & \nabla'_y l = -\mu\, x - \varepsilon\beta_2\, b, \end{array}$$

where $\varepsilon = 1$ in the case $\langle H, H\rangle > 0$ and $\varepsilon = -1$ in the case $\langle H, H\rangle < 0$. In both cases $b = \frac{\varepsilon H}{\sqrt{\varepsilon\langle H,H\rangle}}$; $\langle b, b\rangle = \varepsilon$; $\langle l, l\rangle = -\varepsilon$; $\langle b, l\rangle = 0$.

Taking into account that $R'(x, y, x) = 0$, $R'(x, y, y) = 0$, $R'(x, y, b) = 0$, and using derivative formulas (13) we get the following integrability conditions:

$$(14)\qquad \begin{array}{l} 2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2 = x(\mu); \\ 2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2 = y(\mu); \\ 2\lambda\,\gamma_2 - \varepsilon\,\mu\,\beta_1 - (\nu_1 + \nu_2)\,\gamma_1 = x(\lambda) - y(\nu_1); \\ 2\lambda\,\gamma_1 - \varepsilon\,\mu\,\beta_2 - (\nu_1 + \nu_2)\,\gamma_2 = -x(\nu_2) + y(\lambda); \\ \gamma_1\,\beta_1 - \gamma_2\,\beta_2 + (\nu_1 + \nu_2)\,\mu = -x(\beta_2) + y(\beta_1); \\ \varepsilon(\lambda^2 - \mu^2 - \nu_1\,\nu_2) = x(\gamma_2) - y(\gamma_1) + (\gamma_1)^2 - (\gamma_2)^2. \end{array}$$

Having in mind that $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{-G}}$, we can write the integrability conditions in the following way:

$$\begin{array}{l} 2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2 = \dfrac{1}{\sqrt{E}}\,\mu_u; \\ 2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2 = \dfrac{1}{\sqrt{-G}}\,\mu_v; \\ 2\lambda\,\gamma_2 - \varepsilon\,\mu\,\beta_1 - (\nu_1 + \nu_2)\,\gamma_1 = \dfrac{1}{\sqrt{E}}\,\lambda_u - \dfrac{1}{\sqrt{-G}}\,(\nu_1)_v; \\ 2\lambda\,\gamma_1 - \varepsilon\,\mu\,\beta_2 - (\nu_1 + \nu_2)\,\gamma_2 = -\dfrac{1}{\sqrt{E}}\,(\nu_2)_u + \dfrac{1}{\sqrt{-G}}\,\lambda_v; \\ \gamma_1\,\beta_1 - \gamma_2\,\beta_2 + (\nu_1 + \nu_2)\,\mu = -\dfrac{1}{\sqrt{E}}\,(\beta_2)_u + \dfrac{1}{\sqrt{-G}}\,(\beta_1)_v; \\ \varepsilon(\lambda^2 - \mu^2 - \nu_1\,\nu_2) = \dfrac{1}{\sqrt{E}}\,(\gamma_2)_u - \dfrac{1}{\sqrt{-G}}\,(\gamma_1)_v + (\gamma_1)^2 - (\gamma_2)^2. \end{array}$$

It is clear that if $\mu_u\,\mu_v \neq 0$, then we can express the functions $\sqrt{E}$ and $\sqrt{-G}$ in the following way:

$$\begin{array}{l} \sqrt{E} = \dfrac{\mu_u}{2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2}; \\ \sqrt{-G} = \dfrac{\mu_v}{2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2}. \end{array}$$

The condition $\mu_u\,\mu_v \neq 0$ is equivalent to

$$(2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2)(2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2) \neq 0.$$

Now we shall prove the following Bonnet-type theorem for Lorentz surfaces in $\mathbb{E}^4_2$.

Theorem 4.3. *Let γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{array}{l} \dfrac{\mu_u}{2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2} > 0; \\ \dfrac{\mu_v}{2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2} > 0; \\ -\gamma_1\sqrt{E}\sqrt{-G} = (\sqrt{E})_v; \\ -\gamma_2\sqrt{E}\sqrt{-G} = (\sqrt{-G})_u; \\ 2\lambda\,\gamma_2 - \varepsilon\,\mu\,\beta_1 - (\nu_1+\nu_2)\,\gamma_1 = \dfrac{1}{\sqrt{E}}\,\lambda_u - \dfrac{1}{\sqrt{-G}}\,(\nu_1)_v; \\ 2\lambda\,\gamma_1 - \varepsilon\,\mu\,\beta_2 - (\nu_1+\nu_2)\,\gamma_2 = -\dfrac{1}{\sqrt{E}}\,(\nu_2)_u + \dfrac{1}{\sqrt{-G}}\,\lambda_v; \\ \gamma_1\,\beta_1 - \gamma_2\,\beta_2 + (\nu_1+\nu_2)\,\mu = -\dfrac{1}{\sqrt{E}}\,(\beta_2)_u + \dfrac{1}{\sqrt{-G}}\,(\beta_1)_v; \\ \varepsilon(\lambda^2 - \mu^2 - \nu_1\,\nu_2) = \dfrac{1}{\sqrt{E}}\,(\gamma_2)_u - \dfrac{1}{\sqrt{-G}}\,(\gamma_1)_v + (\gamma_1)^2 - (\gamma_2)^2, \end{array} \tag{15}$$

where

$$\sqrt{E} = \frac{\mu_u}{2\mu\,\gamma_2 - \varepsilon\,\lambda\,\beta_1 + \varepsilon\,\nu_1\,\beta_2}, \quad \sqrt{-G} = \frac{\mu_v}{2\mu\,\gamma_1 + \varepsilon\,\nu_2\,\beta_1 - \varepsilon\,\lambda\,\beta_2}.$$

Let $\{x_0,\, y_0,\, b_0,\, l_0\}$ be an orthonormal frame at a point $p_0 \in \mathbb{E}^4_2$ (with $\langle b_0, b_0\rangle = \varepsilon$; $\langle l_0, l_0\rangle = -\varepsilon$; $\langle b_0, l_0\rangle = 0$). Then there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique Lorentz surface $M^2_1 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$, whose mean curvature vector at any point is a non-zero spacelike vector if $\varepsilon = 1$, or timelike vector if $\varepsilon = -1$. Moreover, M^2_1 passes through p_0, the functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 are the geometric functions of M^2_1 and $\{x_0,\, y_0,\, b_0,\, l_0\}$ is the geometric frame of M^2_1 at point p_0.

Proof. We consider the following system of partial differential equations for the unknown vector functions $x = x(u,v)$, $y = y(u,v)$, $b = b(u,v)$, $l = l(u,v)$ in

$\mathbb{E}^4_2$:

$$\begin{array}{ll}
x_u = (-\gamma_1\, y + \varepsilon\, \nu_1\, b)\sqrt{E} & x_v = (-\gamma_2\, y + \varepsilon\, \lambda\, b - \varepsilon\, \mu\, l)\sqrt{-G}\\
y_u = (-\gamma_1\, x + \varepsilon\, \lambda\, b - \varepsilon\, \mu\, l)\sqrt{E} & y_v = (-\gamma_2\, x + \varepsilon\, \nu_2\, b)\sqrt{-G}\\
b_u = (-\nu_1\, x + \lambda\, y - \varepsilon\, \beta_1\, l)\sqrt{E} & b_v = (-\lambda\, x + \nu_2\, y - \varepsilon\, \beta_2\, l)\sqrt{-G}\\
l_u = (\mu\, y - \varepsilon\, \beta_1\, b)\sqrt{E} & l_v = (-\mu\, x - \varepsilon\, \beta_2\, b)\sqrt{-G}
\end{array} \tag{16}$$

We denote

$$Z = \begin{pmatrix} x \\ y \\ b \\ l \end{pmatrix}; \quad A = \sqrt{E}\begin{pmatrix} 0 & -\gamma_1 & \varepsilon\,\nu_1 & 0 \\ -\gamma_1 & 0 & \varepsilon\,\lambda & -\varepsilon\,\mu \\ -\nu_1 & \lambda & 0 & -\varepsilon\,\beta_1 \\ 0 & \mu & -\varepsilon\,\beta_1 & 0 \end{pmatrix};$$

$$B = \sqrt{-G}\begin{pmatrix} 0 & -\gamma_2 & \varepsilon\,\lambda & -\varepsilon\,\mu \\ -\gamma_2 & 0 & \varepsilon\,\nu_2 & 0 \\ -\lambda & \nu_2 & 0 & -\varepsilon\,\beta_2 \\ -\mu & 0 & -\varepsilon\,\beta_2 & 0 \end{pmatrix}.$$

Using matrices A and B we can rewrite system (16) in the form:

$$\begin{array}{l} Z_u = A\,Z; \\ Z_v = B\,Z. \end{array} \tag{17}$$

The integrability conditions of system (17) are

$$Z_{uv} = Z_{vu},$$

i.e.,

$$\frac{\partial a_i^k}{\partial v} - \frac{\partial b_i^k}{\partial u} + \sum_{j=1}^{4} (a_i^j\, b_j^k - b_i^j\, a_j^k) = 0, \quad i, k = 1, \dots, 4, \tag{18}$$

where a_i^j and b_i^j are the elements of the matrices A and B. Using (15) we obtain that equalities (18) are fulfilled. Hence, there exist a subset $\mathcal{D}_1 \subset \mathcal{D}$ and unique vector functions $x = x(u,v)$, $y = y(u,v)$, $b = b(u,v)$, $l = l(u,v)$, $(u,v) \in \mathcal{D}_1$, which satisfy system (16) and the initial conditions

$$x(u_0, v_0) = x_0, \quad y(u_0, v_0) = y_0, \quad b(u_0, v_0) = b_0, \quad l(u_0, v_0) = l_0.$$

We shall prove that the vectors $x(u,v)$, $y(u,v)$, $b(u,v)$, $l(u,v)$ form an orthonormal frame in $\mathbb{E}^4_2$ for each $(u,v) \in \mathcal{D}_1$. Let us consider the following functions:

$$\begin{array}{lll}
\varphi_1 = \langle x, x\rangle - 1; & \varphi_5 = \langle x, y\rangle; & \varphi_8 = \langle y, b\rangle;\\
\varphi_2 = \langle y, y\rangle + 1; & \varphi_6 = \langle x, b\rangle; & \varphi_9 = \langle y, l\rangle;\\
\varphi_3 = \langle b, b\rangle - \varepsilon; & \varphi_7 = \langle x, l\rangle; & \varphi_{10} = \langle b, l\rangle;\\
\varphi_4 = \langle l, l\rangle + \varepsilon; & &
\end{array}$$

defined for $(u,v) \in \mathcal{D}_1$. Since $x(u,v)$, $y(u,v)$, $b(u,v)$, $l(u,v)$ satisfy (16), for the derivatives of φ_i we obtain the following system:

$$\frac{\partial \varphi_i}{\partial u} = \alpha_i^j\, \varphi_j, \quad \frac{\partial \varphi_i}{\partial v} = \beta_i^j\, \varphi_j; \qquad i = 1, \dots, 10,$$

where α_i^j, β_i^j, $i,j = 1,\dots,10$ are functions of $(u,v) \in \mathcal{D}_1$. This is a linear system of partial differential equations for the functions $\varphi_i(u,v)$, $i = 1,\dots,10$, $(u,v) \in \mathcal{D}_1$, satisfying the initial conditions $\varphi_i(u_0,v_0) = 0$, $i = 1,\dots,10$. Hence, $\varphi_i(u,v) = 0$, $i = 1,\dots,10$ for each $(u,v) \in \mathcal{D}_1$. Consequently, the vector functions $x(u,v)$, $y(u,v)$, $b(u,v)$, $l(u,v)$ form an orthonormal frame in $\mathbb{E}^4_2$ for each $(u,v) \in \mathcal{D}_1$ (with $\langle b,b\rangle = \varepsilon$; $\langle l,l\rangle = -\varepsilon$; $\langle b,l\rangle = 0$).

Now we consider the following system of partial differential equations for the vector function $z = z(u,v)$:

$$\begin{aligned} z_u &= \sqrt{E}\,x \\ z_v &= \sqrt{-G}\,y \end{aligned} \tag{19}$$

By the use of (15) and (16) we get that the integrability conditions $z_{uv} = z_{vu}$ of system (19) are fulfilled. Hence, there exist a subset $\mathcal{D}_0 \subset \mathcal{D}_1$ and a unique vector function $z = z(u,v)$, defined for $(u,v) \in \mathcal{D}_0$ and satisfying $z(u_0,v_0) = p_0$.

Consequently, the surface $M^2_1 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ satisfies the assertion of the theorem. $\square$

5. Basic classes of Lorentz surfaces characterized in terms of their geometric functions

Further we shall give expressions for the Gauss curvature K and the invariant functions k and $\varkappa$.

We use that for pseudo-Riemannian submanifolds the Gauss curvature is given by the following formula:

$$K = \frac{\langle \sigma(x,x), \sigma(y,y)\rangle - \langle \sigma(x,y), \sigma(x,y)\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2}.$$

Since $\langle x,x\rangle = 1$, $\langle y,y\rangle = -1$, $\langle x,y\rangle = 0$, from (13) we get the following expression for the Gauss curvature:

$$K = \varepsilon\,(\lambda^2 - \mu^2 - \nu_1\nu_2). \tag{20}$$

The curvature of the normal connection is expressed in terms of the geometric functions μ, ν_1, and ν_2 by the next formula:

$$\varkappa = \mu(\nu_1 + \nu_2). \tag{21}$$

The invariant function k, generated by the Weingarten map γ according to the formula $k = \det\gamma = \frac{LN-M^2}{EG-F^2}$, is expressed as

$$k = 4\mu^2\nu_1\nu_2. \tag{22}$$

Since we consider Lorentz surfaces for which $\varkappa^2 - k > 0$, equalities (21) and (22) imply that $\mu \neq 0$ and $\nu_1 \neq \nu_2$.

The next statements follow directly from formulas (20) and (21).

Proposition 5.1. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 is flat if and only if $\lambda^2 - \mu^2 = \nu_1\nu_2$.*

Proposition 5.2. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 has constant Gauss curvature if and only if $\lambda^2 - \mu^2 - \nu_1\nu_2 = const.$*

Proposition 5.3. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 is of flat normal connection if and only if $\nu_1 + \nu_2 = 0$.*

Proposition 5.4. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 has constant normal curvature $\varkappa$ if and only if $\mu(\nu_1 + \nu_2) = const.$*

Now we shall characterize some basic classes of surfaces satisfying special conditions on the mean curvature vector field. It follows from (13) that the mean curvature vector field H is expressed by the formula:

$$H = \frac{\varepsilon(\nu_1 - \nu_2)}{2}\, b. \tag{23}$$

Hence, the length of the mean curvature vector field is:

$$\|H\| = \varepsilon\sqrt{|\langle H, H\rangle|} = \varepsilon\frac{|\nu_1 - \nu_2|}{2}.$$

Proposition 5.5. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 has non-zero constant mean curvature if and only if $\nu_1 - \nu_2 = const \neq 0$.*

Using (23) and (13) we get that

$$D_x H = \varepsilon x\left(\frac{\nu_1 - \nu_2}{2}\right) b - \frac{\nu_1 - \nu_2}{2}\,\beta_1\, l;$$
$$D_y H = \varepsilon y\left(\frac{\nu_1 - \nu_2}{2}\right) b - \frac{\nu_1 - \nu_2}{2}\,\beta_2\, l.$$

The last equalities imply that H is parallel in the normal bundle (i.e., $DH = 0$ holds identically) if and only if $\beta_1 = \beta_2 = 0$ and $\nu_1 - \nu_2 = const.$ So, the next statement holds true.

Proposition 5.6. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 has parallel mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$ and $\nu_1 - \nu_2 = const.$*

The geometric meaning of the invariant λ is connected with the notion of allied mean curvature vector field defined by B.-Y. Chen for submanifolds of Riemannian manifolds. Let M be an n-dimensional submanifold of $(n+m)$-dimensional Riemannian manifold $\widetilde{M}$ and ξ be a normal vector field of M. The *allied vector field* $a(\xi)$ of ξ is defined by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^{m} \{\mathrm{tr}(A_1 \circ A_k)\} \xi_k,$$

where $\{\xi_1 = \frac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M, and $A_i = A_{\xi_i}$, $i = 1, \dots, m$ is the shape operator with respect to ξ_i. In particular, the allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M in $\widetilde{M}$ [2]. B.-Y. Chen defined the $\mathcal{A}$-submanifolds to be those submanifolds of $\widetilde{M}$ for which $a(H)$ vanishes identically. The $\mathcal{A}$-submanifolds are also called *Chen submanifolds.* It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial $\mathcal{A}$-submanifolds.

The notion of allied mean curvature vector field is extended by S. Haesen and M. Ortega to the case when the normal space is a two-dimensional Lorentz space [26].

In the following proposition we characterize non-trivial Chen surfaces in $\mathbb{E}^4_2$.

Proposition 5.7. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 is a non-trivial Chen surface if and only if $\lambda = 0$.*

Proof. Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$ and $\{b, l\}$ be the geometric normal frame field defined in Section 4. The allied mean curvature vector field is given by the formula

$$a(H) = \frac{\|H\|}{2} \mathrm{tr}(A_b \circ A_l)\; l, \tag{24}$$

where A_b and A_l are the shape operators corresponding to b and l, respectively. Using equalities (13) we obtain

$$\begin{aligned} A_b(x) &= \nu_1 x - \lambda y; & A_l(x) &= -\mu y; \\ A_b(y) &= \lambda x - \nu_2 y; & A_l(y) &= \mu x. \end{aligned}$$

Hence, $\mathrm{tr}(A_b \circ A_l) = -2\lambda\mu$. Now applying (24) and using (23), we get

$$a(H) = -\varepsilon \frac{|\nu_1 - \nu_2|}{2} \lambda\, \mu\, l.$$

Since $\mu \neq 0$ and $\nu_1 \neq \nu_2$, we conclude that $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant λ: M^2_1 is a non-trivial Chen surface if and only if the geometric function λ is zero. □

6. Lorentz surfaces with parallel normalized mean curvature vector field

In this section we shall focus our attention on a special class of Lorentz surfaces in $\mathbb{E}^4_2$, namely surfaces with parallel normalized mean curvature vector field.

Let M^2_1 be a Lorentz surface of general type. In the previous section we proved that the mean curvature vector field H is parallel if and only if $\beta_1 = \beta_2 = 0$ and $\nu_1 - \nu_2 = const$. Now we shall consider the class of surfaces satisfying the conditions $\beta_1 = \beta_2 = 0$. It follows from (13) that $\beta_1 = \beta_2 = 0$ if and only if $D_x b = D_y b = 0$ (or equivalently, $D_x l = D_y l = 0$). Since b is a unit normal vector field in the direction of the mean curvature vector field, the conditions $\beta_1 = \beta_2 = 0$ characterize surfaces with parallel normalized mean curvature vector field. Hence, we have the following statement.

Proposition 6.1. *Let M^2_1 be a Lorentz surface of general type in $\mathbb{E}^4_2$. Then M^2_1 has parallel normalized mean curvature vector field if and only if $\beta_1 = \beta_2 = 0$.*

Note that the condition to have parallel normalized mean curvature vector field is weaker than the condition to have parallel mean curvature vector field. In what follows we shall study Lorentz surfaces with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field, i.e., $\beta_1 = \beta_2 = 0$, $\|H\| \neq const$. For these surfaces we shall introduce canonical parameters.

Definition. Let M^2_1 be a Lorentz surface of general type with parallel normalized mean curvature vector field. The parameters (u, v) of M^2_1 are said to be *canonical*, if

$$E = \frac{1}{|\mu|}; \qquad F = 0; \qquad G = -\frac{1}{|\mu|}.$$

Theorem 6.2. *Each Lorentz surface of general type with parallel normalized mean curvature vector field in $\mathbb{E}^4_2$ locally admits canonical parameters.*

Proof. Using the Gauss and Codazzi equations, from (13) we obtain that the geometric functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of the surface satisfy the integrability conditions given in formulas (14). Putting $\beta_1 = \beta_2 = 0$ in (14), we get

$$\begin{aligned}
&2\mu\,\gamma_2 = x(\mu);\\
&2\mu\,\gamma_1 = y(\mu);\\
&2\lambda\,\gamma_2 - (\nu_1 + \nu_2)\,\gamma_1 = x(\lambda) - y(\nu_1);\\
&2\lambda\,\gamma_1 - (\nu_1 + \nu_2)\,\gamma_2 = -x(\nu_2) + y(\lambda);\\
&(\nu_1 + \nu_2)\,\mu = 0;\\
&\varepsilon(\lambda^2 - \mu^2 - \nu_1\,\nu_2) = x(\gamma_2) - y(\gamma_1) + (\gamma_1)^2 - (\gamma_2)^2.
\end{aligned} \tag{25}$$

The first and second equalities of (25) imply $\gamma_1 = \frac{1}{2} y(\ln|\mu|)$, $\gamma_2 = \frac{1}{2} x(\ln|\mu|)$. On the other hand, $\gamma_1 = -y(\ln\sqrt{E})$, $\gamma_2 = -x(\ln\sqrt{-G})$. Hence, $x(\ln|\mu|(-G)) = 0$ and $y(\ln|\mu|E) = 0$, which imply that $E|\mu|$ does not depend on v, and $G|\mu|$ does not depend on u. Hence, there exist functions $\varphi(u) > 0$ and $\psi(v) > 0$, such that

$$E|\mu| = \varphi(u); \qquad -G|\mu| = \psi(v).$$

Under the following change of the parameters:

$$\overline{u} = \int_{u_0}^{u} \sqrt{\varphi(u)}\, du + \overline{u}_0, \quad \overline{u}_0 = const$$

$$\overline{v} = \int_{v_0}^{v} \sqrt{\psi(v)}\, dv + \overline{v}_0, \quad \overline{v}_0 = const$$

we obtain

$$\overline{E} = \frac{1}{|\mu|}; \qquad \overline{F} = 0; \qquad \overline{G} = -\frac{1}{|\mu|},$$

which imply that the parameters $(\overline{u}, \overline{v})$ are canonical. □

It is clear that the canonical parameters are determined locally up to an orientation.

Proposition 6.3. *Each Lorentz surface of general type with parallel normalized mean curvature vector field in $\mathbb{E}^4_2$ is a surface with flat normal connection.*

Proof. If M^2_1 is a Lorentz surface of general type with parallel normalized mean curvature vector field, then formulas (25) hold. Since $\mu \neq 0$, from the fifth equality of (25) we get that $\nu_1 + \nu_2 = 0$. Hence, the normal connection of the surface is $\varkappa = 0$, i.e., M^2_1 is a surface with flat normal connection. □

Now let $M^2_1 : z = z(u, v),\ (u, v) \in \mathcal{D}$ be a Lorentz surface of general type with parallel normalized mean curvature vector field and (u, v) be canonical parameters. Since $\nu_1 + \nu_2 = 0$, we denote $\nu := \nu_1 = -\nu_2$. The functions γ_1 and γ_2 are expressed as follows:

$$\gamma_1 = \frac{1}{2} y(\ln|\mu|) = \left(\sqrt{|\mu|}\right)_v; \qquad \gamma_2 = \frac{1}{2} x(\ln|\mu|) = \left(\sqrt{|\mu|}\right)_u.$$

The third and fourth equalities of (25) imply the following partial differential equations for the functions λ, μ, and ν:

$$\nu_u = -\lambda_v + \lambda(\ln|\mu|)_v;$$
$$\nu_v = \lambda_u - \lambda(\ln|\mu|)_u.$$

The last equality of (25) implies

$$\varepsilon(\lambda^2 - \mu^2 + \nu^2) = \frac{1}{2}|\mu| \Delta^h \ln|\mu|,$$

where $\Delta^h = \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}$ is the hyperbolic Laplace operator.

The fundamental existence and uniqueness theorem for the class of Lorentz surfaces of general type with parallel normalized mean curvature vector field can be formulated in terms of canonical parameters as follows.

Theorem 6.4. *Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned}
&\mu \neq 0, \quad \nu \neq const;\\
&\nu_u = -\lambda_v + \lambda(\ln|\mu|)_v;\\
&\nu_v = \lambda_u - \lambda(\ln|\mu|)_u;\\
&\varepsilon(\lambda^2 - \mu^2 + \nu^2) = \frac{1}{2}|\mu|\Delta^h \ln|\mu|.
\end{aligned}$$

If $\{x_0, y_0, b_0, l_0\}$ is an orthonormal frame at a point $p_0 \in \mathbb{E}^4_2$ (with $\langle b_0, b_0\rangle = \varepsilon$; $\langle l_0, l_0\rangle = -\varepsilon$; $\langle b_0, l_0\rangle = 0$), then there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique Lorentz surface $M^2_1 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ of general type with parallel normalized mean curvature vector field, such that M^2_1 passes through p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u,v)$ are the geometric functions of M^2 and $\{x_0, y_0, b_0, l_0\}$ is the geometric frame of M^2_1 at the point p_0. Furthermore, (u,v) are canonical parameters of M^2.

The meaning of Theorem 6.4 is that any Lorentz surface of general type with parallel normalized mean curvature vector field is determined up to a rigid motion in $\mathbb{E}^4_2$ by three invariant functions λ, μ, ν satisfying the following system of three natural partial differential equations:

$$\begin{aligned}
&\nu_u = -\lambda_v + \lambda(\ln|\mu|)_v;\\
&\nu_v = \lambda_u - \lambda(\ln|\mu|)_u;\\
&\varepsilon(\lambda^2 - \mu^2 + \nu^2) = \frac{1}{2}|\mu|\Delta^h \ln|\mu|.
\end{aligned}$$

Remarks. (1) The introducing of canonical parameters on a Lorentz surface with parallel normalized mean curvature vector field allows us to minimize the number of invariants and the number of partial differential equations which determine the surface. This solves the problem of Lund-Regge for the class of Lorentz surfaces with parallel normalized mean curvature vector field.

(2) With respect to canonical parameters the coefficients of the first fundamental form and the coefficients of the second fundamental tensor are expressed in terms of the invariants of the surface.

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