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ON ϕ-SCHREIER RINGS

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Abstract. Let R be a ring in which $Nil(R)$ is a divided prime ideal of R. Then, for a suitable property X of integral domains, we can define a ϕ-X-ring if $R/Nil(R)$ is an X-domain. This device was introduced by Badawi [8] to study rings with zero divisors with a homomorphic image a particular type of domain. We use it to introduce and study a number of concepts such as ϕ-Schreier rings, ϕ-quasi-Schreier rings, ϕ-almost-rings, ϕ-almost-quasi-Schreier rings, ϕ-GCD rings, ϕ-generalized GCD rings and ϕ-almost GCD rings as rings R with $Nil(R)$ a divided prime ideal of R such that $R/Nil(R)$ is a Schreier domain, quasi-Schreier domain, almost domain, almost-quasi-Schreier domain, GCD domain, generalized GCD domain and almost GCD domain, respectively.

We study some generalizations of these concepts, in light of generalizations of these concepts in the domain case, as well. Here a domain D is pre-Schreier if for all $x, y, z \in D\backslash 0$, $x \mid yz$ in D implies that $x = rs$ where $r \mid y$ and $s \mid z$. An integrally closed pre-Schreier domain was initially called a Schreier domain by Cohn in [15] where it was shown that a GCD domain is a Schreier domain.

1. Introduction

We assume throughout that all rings are commmutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Nil(R)$ denotes the set of nilpotent elements of R and $Z(R)$ denotes the set of zero-divisors of R. Recall that a nonzerodivisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. A ring R is called a Prüfer ring, in the sense of [20], if every finitely generated regular ideal of R is invertible, i.e., if I is finitely generated regular ideal of R and $I^{-1} = \{x \in T(R) \mid xI \subset R\}$, then $II^{-1} = R$ [5]. Recall from [16] and [9], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [8], [10], [11], [12] and [13], the scond-named author investigated the class of rings $\mathcal{H} = \{R \mid R \text{ is a commutative ring with } 1 \neq$

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0 and $Nil(R)$ is a divided prime ideal of $R\}$. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$. Recall from [8] that for a ring $R \in H$ with total quotient ring $T(R)$, if $a \in R$ and $b \in R \setminus Z(R)$, then $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. A nonnil ideal I of R is said to be a ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil finitely generated ideal of R is ϕ-invertible, then we say that R is a ϕ-Prüfer ring [5]. In [5, Lemma 2.5], it is shown that, if $R \in H$ and P an ideal of R, then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. A ring $R \in H$ is called ϕ-integrally closed if $\phi(R)$ is integrally closed in $T(\phi(R)) = R_{Nil(R)}$. It is shown that R is ϕ-integrally closed if and only if $R/NilR$ is integrally closed if and only if $\phi(R)/Nil(\phi(R))$ is integrally closed.

Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subset Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$ and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. Therefore we have $x \in R \setminus Nil(R)$ if and only if $\phi(x) \in \phi(R) \setminus Z(\phi(R))$. Let $R \in \mathcal{H}$. Then I is a finitely generated nonnil ideal of R if and only if $\phi(I)$ is a finitely generated regular ideal of $\phi(R)$ [5, Lemma 2.1]. Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$ and let I be an ideal of R. Then I is an invertible ideal of R if and only if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$ [5, Lemma 2.3]. Let $R \in \mathcal{H}$ and let I be an ideal od R. Then I is a finitely generated nonnil ideal of R if and only if $I/Nil(R)$ is a finitely generated nonzero ideal of $R/Nil(R)$ [5, Lemma 2.4].

An element x of a ring R is called primal if whenever $x \mid y_1y_2$, with $x, y_1, y_2 \in R$, then $x = z_1z_2$ where $z_1 \mid y_1$ and $z_2 \mid y_2$. P. M. Cohen in [15] introduced the concept of Schreier domain. A domain D is called a pre-Schreier domain if every nonzero element of D is primal. If in addition D is integrally closed, then D is called a Schreier domain. The study of Schreier domains was continued in MacAdam and Rush [22] and M. Zafrullah [24]. In [18] and [7], an extension of the class of pre-Schreier domains was studied. A domain D as called quasi-Schreier domain if whenever I, J_1, J_2 are invertible ideals of D and $I \supseteq J_1J_2$, then $I = I_1I_2$ for some invertible ideals I_1, I_2 of D with $I_i \supseteq J_i$ for $i = 1, 2$. In [17], another generalization of the pre-Schreier domains was studied. A domain D was called an almost-Schreier domain (AS domain) if whenever a, b_1, b_2 are nonzero elements of D and $a \mid b_1b_2$, there exist an integer $k \geq 1$ and nonzero elements a_1, a_2 of D such that $a^k = a_1a_2$ and $a_i \mid b_i^k$ for $i = 1, 2$. In [1], Z. Ahmad and T. Dumitrescu introduced another generalization of the pre-Schreier domains which includes the pre-Schreier domains and the AS domains. They called this domain almost quasi-Schreier domain. A domain D is called almost-quasi-Schreier domain (AQS domain) if whenever I, J_1, J_2 are nonzero invertible ideals of D such that $I \supseteq J_1J_2$ there exist an integer $k \geq 1$ and nonzero invertible ideals I_1, I_2 of D such that $I^k = I_1I_2$ and $I_i \supseteq J_i^k$ for $i = 1, 2$. A GCD domain is a domain in which every two elements have greatest common divisor. Anderson [3] and Anderson and Anderson [4] introduced and investigated a class of domains called generalized GCD domains. A generalized GCD domain (GGCD domain) is a domain in which every intersection of two invertible nonzero ideals is an invertible ideal, [4]. An almost GCD domain (AGCD domain) D is a domain in which for every two principal ideals I, J of D there exists some $k \geq 1$ such that $I^k \cap J^k$ is a principal ideal of D, [23].

Now we generalize above concepts. A ring R is called a pre-Schreier ring if every regular element of R is primal. If in addition R is integrally closed, then R is called a Schreier ring. A ring R is called a quasi-Schreier ring if whenever I, J_1, J_2 are regular invertible ideals of R and $I \supseteq J_1J_2$, then $I = I_1I_2$ for regular invertible ideals I_1, I_2 of R with $I_i \supseteq J_i$ for $i = 1, 2$. We say that a ring R is almost-Schreier ring (AS ring) if whenever a, b_1, b_2 are regular elements of R and $a \mid b_1b_2$, there exist an integer $k \geq 1$ and regular elements a_1, a_2 of R such that $a^k = a_1a_2$ and $a_i \mid b_i^k$ for $i = 1, 2$. A ring R is called almost-quasi-Schreier ring (AQS ring) if whenever I, J_1, J_2 are regular invertible ideals of R such that $I \supseteq J_1J_2$ there exist an integer $k \geq 1$ and regular invertible ideals I_1, I_2 of R such that $I^k = I_1I_2$ and $I_i \supseteq J_i^k$ for $i = 1, 2$. A ring R is called a GCD ring if every two regular elements of R have a greatest common divisor. We say that a ring R is a generalized GCD ring ($GGCD$ ring) if R is a ring in which every intersection of two regular invertible ideals is an invertible ideal of R. A generalized GCD ring (GGCD ring) was introduced by M. M. Ali and D. J. Smith in [2]. They called a ring R a GGCD ring if in R the intersection of every two finitely generated faithful multiplication ideals is a finitely generated faithful multiplication ideal. In fact two above definitions for GGCD ring are equivalent. For proof, it is sufficient to consider the following remarks in [19]:

Remark 1. Every invertible ideal is finitely generated ideal.

Remark 2. Every invertible ideal is a multiplication ideal.

Remark 3. An ideal I is multiplication if and only if I is locally principal.

Remark 4. Let I be a finitely generated ideal of R. Then I is an invertible ideal if and only if I is locally principal.

Here an ideal I of a ring R is called a multiplication ideal if every ideal contained in I is a multiple of I [19].

An almost GCD ring ($AGCD$ ring) is a ring in which for every two regular principal ideals I and J of R, there exists some $k \geq 1$ such that $I^k \cap J^k$ is a regular principal ideal of R.

In this paper, we define a ϕ-primal element. We say that a nonnil element $x \in R$ is ϕ-primal if and only if $\phi(x)$ is primal in $\phi(R)$. In Lemma 2.2, we show that $x \in R$ is ϕ-primal if and only if $x + Nil(R)$ is primal in $R/Nil(R)$. A ring R is called a ϕ-pre-Schreier ring if every element of R is ϕ-primal. In addition if R is ϕ-integrally closed, then we say that R is a ϕ-Schreier ring. A ring R called a ϕ-quasi-Schreier ring if whenever I, J_1, J_2 are nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$, then $I = I_1I_2$ for nonnil ϕ-invertible ideals I_1, I_2 of R with

$I_i \supseteq J_i$ for $i = 1, 2$. we show that a ϕ-pre-Schreier ring is a ϕ-quasi-Schreier ring, Corollary 2.18. We say that a regular element x of a ring R is strongly primal, whenever $x \mid ab$ for regular elements a, b of R, then there exists $k \geq 1$ and regular elements a', b' of R such that $x^k = a'b'$ with $a' \mid a^k$ and $b' \mid b^k$. An element x of R is called ϕ-strongly primal in R if and only if $\phi(x)$ is strongly primal in $\phi(R)$. In Lemma 2.22, we show that $x \in R$ is a ϕ-strongly primal element if and only if $x + Nil(R)$ is a strongly primal element in $R/Nil(R)$. A ring R is called a ϕ-almost-Schreier ring (ϕ-AS ring) if every nonnil element of R is ϕ-strongly primal. In Corollary 2.29, we show that a ϕ-pre-Schreier ring is a ϕ-AS ring. A ring R called a ϕ-almost-quasi-Schreier ring (ϕ-AQS ring) if whenever I, J_1, J_2 are nonnil ϕ-invertible ideals of R such that $I \supseteq J_1J_2$ there exist an integer $k \geq 1$ and nonnil ϕ-invertible ideals I_1, I_2 of R such that $I^k = I_1I_2$ and $I_i \supseteq J_i^k$ for $i = 1, 2$. In Corollary 2.36, we show that a ϕ-AS ring is a ϕ-AQS ring and in Corollary 2.37, we prove that a ϕ-quasi-Schreier ring is a ϕ-AQS ring. A ring R is called a ϕ-GCD ring is a ring in which every two nonnil elements of R have a greatest common divisor. A ϕ-GCD ring is a ϕ-Schreier ring, Corollary 3.7. A ϕ-generalized GCD ring (ϕ-$GGCD$ ring) is a ring in which every intersection of two nonnil principal ideals is a ϕ-invertible ideal. In Corollary 3.13, we show that a ϕ-GCD ring is a ϕ-$GGCD$ ring and in Corollary 3.14, we prove that a ϕ-Prüfer ring is a ϕ-$GGCD$ ring. A ϕ-$GGCD$ ring is a ϕ-quasi-Schreier ring, Corollary 3.15. A ϕ-almost GCD ring (ϕ-$AGCD$ ring) is a ring in which for every two nonnil principal ideals I and J of R, there exists some $k \geq 1$ such that $I^k \cap J^k$ is a nonnil principal ideal of R. In Corollary 3.22, we show that a ϕ-$AGCD$ ring is a ϕ-AS ring.

2. ϕ-Schreier rings

Definition 2.1. Let $R \in \mathcal{H}$. A nonnil element $x \in R$ is said to be ϕ-primal if and only if $\phi(x)$ is primal in $\phi(R)$.

Lemma 2.2. *Let $R \in \mathcal{H}$ and $x \in R$. Then x is ϕ-primal in R if and only if $x + Nil(R)$ is primal in $R/Nil(R)$.*

Proof. Let x be ϕ-primal in R and let $x+Nil(R) \mid (a+Nil(R))(b+Nil(R)) = ab + Nil(R)$ in $R/Nil(R)$ for nonnil elements x, a, b of R. Then there exists $y + Nil(R)$ in $R/Nil(R)$ such that $ab + Nil(R) = xy + Nil(R)$. So $ab - xy \in Nil(R)$. Thus $ab = xy + w$ for some $w \in Nil(R)$. Since $Nil(R)$ is a divided prime ideal. Then $Nil(R) \subseteq (x)$. So $w = xz$ for some $z \in Nil(R)$. Therefore $ab = xy + xz = x(y + z)$. Hence $x \mid ab$ and so $x = a'b'$ for nonnil elements a', b' of R with $a' \mid a$ and $b' \mid b$ in R. So $x + Nil(R) = a'b' + Nil(R) = (a' + Nil(R))(b'+Nil(R))$ with $a'+Nil(R) \mid a+Nil(R)$ and $b'+Nil(R) \mid b+Nil(R)$. Thus $x + Nil(R)$ is primal in $R/Nil(R)$. Conversely, let $x + Nil(R)$ be primal in $R/Nil(R)$. Since, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is ring-homomorphic to $R/Nil(R)$, then $\phi(x) + Nil(\phi(R))$ is primal in $\phi(R)/Nil(\phi(R))$. Now, let $x \mid ab$ for nonnil elements of R. Then $\phi(x)+Nil(\phi(R)) \mid \phi(a)+Nil(\phi(R))\phi(b)+ Nil(\phi(R))$ and so $\phi(x)+Nil(\phi(R)) = (\phi(a')+Nil(\phi(R)))(\phi(b')+Nil(\phi(R))) =$

$\phi(a')\phi(b')+Nil(\phi(R))$ for nonzero elements $\phi(a')+Nil(\phi(R)), \phi(b')+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ with $\phi(a') + Nil(\phi(R)) \mid \phi(a) + Nil(\phi(R))$ and $\phi(b') + Nil(\phi(R)) \mid \phi(b) + Nil(\phi(R))$. Therefore $\phi(x) - \phi(a')\phi(b') \in Nil(\phi(R))$. Since $Nil(\phi(R)) = Z(\phi(R)$, we conclude that $\phi(x) = \phi(a')\phi(b')$ with $\phi(a') \mid \phi(a)$ and $\phi(b') \mid \phi(b)$. So $\phi(x)$ is primal in $\phi(R)$ and by definition of ϕ-primal element, x is a ϕ-primal element of R. □

Proposition 2.3. *Let $R \in \mathcal{H}$. Any product of ϕ-primal elements in R is ϕ-primal.*

Proof. Let $p, q \in R \setminus Nil(R)$. Then $pq \in R \setminus Nil(R)$. Suppose that $pq \mid a_1a_2$ with $a_1, a_2 \in R\setminus Nil(R)$. So $a_1a_2 = pqs$ for some nonnil element s of R. Hence $p \mid a_1a_2$. Then $p = p_1p_2$ for some $p_1, p_2 \in R\setminus Nil(R)$ and $p_i \mid a_i$ in R for $i = 1, 2$. Writing $a_i = p_ir_i$ with $r_i \in R \setminus Nil(R)$. Thus, we have $a_1a_2 = p_1r_1p_2r_2 = pqs$. Hence $r_1r_2 = qs$, i.e., $q \mid r_1r_2$, whence $q = q_1q_2$ with $q_i \mid r_i$ and $q_i \in R\setminus Nil(R)$. Therefore $pq = p_1q_1p_2q_2$ and $p_iq_i \mid p_ir_i = a_i$ and $p_iq_i \in R \setminus Nil(R)$. So pq is ϕ-primal. by induction it follows that any product of ϕ-primal elements is again ϕ-primal. □

Definition 2.4. A ring R is called a ϕ-pre-Schreier ring if every element of R is ϕ-primal. In addition if R is ϕ-integrally closed, then we say that R is a ϕ-Schreier ring.

Theorem 2.5. *Let $R \in \mathcal{H}$. Then R is a ϕ-Schreier ring if and only if $R/Nil(R)$ is a Schreier domain.*

Proof. By [6], R is ϕ-integrally closed if and only if $R/Nil(R)$ is integrally closed. Then, by Lemma 2.2, R is a ϕ-Schreier ring if and only if $R/Nil(R)$ is a Schreier domain. □

Theorem 2.6. *Let $R \in \mathcal{H}$. Then R is a ϕ-Schreier ring if and only if $\phi(R)$ is a Schreier ring.*

Proof. Note that R is ϕ-integrally closed in $T(R)$ if and only if $\phi(R)$ is integrally closed in $T(\phi(R)) = R_{Nil(R)}$, by the definition of ϕ being integrally closed. Now, let R be a ϕ-Schreier ring, then by Theorem 2.5, $R/Nil(R)$ is a Schreier domain. So, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is a Schreier domain. Let $\phi(x) \mid \phi(a)\phi(b)$ for regular elements $\phi(x), \phi(a), \phi(b)$ of $\phi(R)$. Then $\phi(x) + Nil(\phi(R)) \mid (\phi(a) + Nil(\phi(R)))(\phi(b) + Nil(\phi(R)))$ for nonzero elements $\phi(x)+Nil(\phi(R)), \phi(a)+Nil(\phi(R)), \phi(b)+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Thus $\phi(x)+Nil(\phi(R)) = (\phi(a')+Nil(\phi(R)))(\phi(b')+Nil(\phi(R)))$ for nonzero elements $\phi(a')+Nil(\phi(R)), \phi(b')+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ with $\phi(a')+Nil(\phi(R)) \mid \phi(a) + Nil(\phi(R))$ and $\phi(b') + Nil(\phi(R)) \mid \phi(b) + Nil(\phi(R))$. Therefore $\phi(x) - \phi(a')\phi(b') \in Nil(\phi(R))$. Since $Nil(\phi(R)) = Z(\phi(R)$, we conclude that $\phi(x) = \phi(a')\phi(b')$ for regular elements $\phi(a')$, $\phi(b')$ of $\phi(R)$ with $\phi(a') \mid \phi(a)$ and $\phi(b') \mid \phi(b)$. Hence $\phi(R)$ is a Schreier ring. Conversely, let $\phi(R)$ be a Schreier ring. Let $x \mid ab$ for nonnil elements of R. Then $\phi(x) \mid \phi(a)\phi(b)$ for regular elements $\phi(x)$,

$\phi(a)$, $\phi(b)$ of $\phi(R)$. So $\phi(x)+Nil(\phi(R)) \mid (\phi(a)+Nil(\phi(R)))(\phi(b)+Nil(\phi(R)))$ for nonzero elements $\phi(x) + Nil(\phi(R))$, $\phi(a) + Nil(\phi(R))$, $\phi(b) + Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Since $\phi(x) = \phi(a')\phi(b')$ for regular elements $\phi(a'), \phi(b')$ of $\phi(R)$ with $\phi(a') \mid \phi(a)$ and $\phi(b') \mid \phi(b)$, so $\phi(x) + Nil(\phi(R)) = (\phi(a') + Nil(\phi(R)))(\phi(b')+Nil(\phi(R)))$ for nonzero elements $\phi(a')+Nil(\phi(R))$, $\phi(b')+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ with $\phi(a') + Nil(\phi(R)) \mid \phi(a) + Nil(\phi(R))$ and $\phi(b') + Nil(\phi(R)) \mid \phi(b) + Nil(\phi(R))$. Therefore $\phi(R)/Nil(\phi(R))$ is a Schreier domain. Since, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is ring-homomorphic to $R/Nil(R)$, then $R/Nil(R)$ is a Schreier domain. Hence, by Theorem 2.5, R is a ϕ-Schreier ring. □

Corollary 2.7. *Let $R \in \mathcal{H}$. The following are equivalent*:
(1) *R is a ϕ-Schreier ring*;
(2) *$\phi(R)$ is a Schreier ring*;
(3) *$R/Nil(R)$ is a Schreier domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a Schreier domain.*

Lemma 2.8. *Let $R \in \mathcal{H}$ and $x \in R$. If x is ϕ-primal, then x is primal.*

Proof. If x is ϕ-primal, then by Lemma 2.2, $x + Nil(R)$ is primal in $R/Nil(R)$. If $x \mid ab$ for regular elements x, a, b of R. Then with a same way of proof Theorem 2.6, we conclude that x is primal. □

Theorem 2.9. *Let $R \in \mathcal{H}$. If R is a ϕ-Schreier ring, then R is a Schreier ring.*

Proof. Let R be a ϕ-Schreier ring. Then, by Theorem 2.6, $\phi(R)$ is a Schreier ring. So $\phi(R)$ is integrally closed in $T(\phi(R))$. Thus, by [5, Lemma 2.13], R is integrally closed in $T(R)$. Now, by Lemma 2.8, it is clear that R is a Schreier ring. □

Theorem 2.10. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-Schreier ring if and only if R is a Schreier ring.*

Proof. Suppose that R is a Schreier ring. Then $\phi(R) = R$ is a Schreier ring. Hence, by Theorem 2.6, R is a ϕ-Schreier ring. The converse is clearly by Theorem 2.9. □

Note that the above results are satisfied for ϕ-pre-Schreier rings. Now, we define an extension of the class ϕ-pre-Schreier rings.

Definition 2.11. A ring R is called a ϕ-quasi-Schreier ring if whenever I, J_1, J_2 are nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$, then $I = I_1I_2$ for nonnil ϕ-invertible ideals I_1, I_2 of R with $I_i \supseteq J_i$ for $i = 1, 2$.

Lemma 2.12. *Let $R \in \mathcal{H}$ and I be a nonnil ideal of R. Then I is ϕ-invertible ideal of R if and only if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$.*

Proof. Let I be ϕ-invertible ideal of R. Then $\phi(I)$ is an invertible ideal of $\phi(R)$. So, by [14, Lemma 2.4], $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Conversely, if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$, then by [14, Lemma 2.4], $\phi(I)$ is an invertible ideal of $\phi(R)$. Hence, by definition of a ϕ-invertible ideal, I is ϕ-invertible ideal of R. □

Theorem 2.13. *Let $R \in \mathcal{H}$. Then R is a ϕ-quasi-Schreier ring if and only if $R/Nil(R)$ is a quasi-Schreier domain.*

Proof. Suppose that R is a ϕ-quasi-Schreier ring and let $I/Nil(R)$, $J_1/Nil(R)$ and $J_2/Nil(R)$ be nonzero invertible ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (J_1/Nil(R))(J_2/Nil(R))$. Then, by Lemma 2.12, I, J_1, J_2 are nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$. So $I = I_1I_2$ for ϕ-invertible ideals I_1, I_2 of R with $I_1 \supseteq J_1$ and $I_2 \supseteq J_2$. Therefore $I/Nil(R) = (I_1/Nil(R))(I_2/Nil(R))$ for invertible ideals $I_1/Nil(R), I_2/Nil(R)$ of $R/Nil(R)$ by Lemma 2.12, with $I_1/Nil(R) \supseteq J_1/Nil(R)$ and $I_2/Nil(R) \supseteq J_2/Nil(R)$. Hence, $R/Nil(R)$ is a quasi-Schreier domain. Conversely, suppose that $R/Nil(R)$ is a quasi-Schreier domain. Let I, J_1, J_2 be nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$. Then, by Lemma 2.12, $I/Nil(R), J_1/Nil(R), J_2/Nil(R)$ are nonzero invertible ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (J_1/Nil(R))(J_2/Nil(R))$. So

$$I/Nil(R) = (I_1/Nil(R))(I_2/Nil(R))$$

for invertible ideals $I_1/Nil(R)$ and $I_2/Nil(R)$ of $R/Nil(R)$ with $I_1/Nil(R) \supseteq J_1/Nil(R)$ and $I_2/Nil(R) \supseteq J_2/Nil(R)$. Therefore, $I = I_1I_2$ for ϕ-invertible ideals I_1, I_2 of R by Lemma 2.12, with $I_1 \supseteq J_1$ and $I_2 \supseteq J_2$. Therefore R is a ϕ-quasi-Schreier ring. □

Theorem 2.14. *Let $R \in \mathcal{H}$. Then R is a ϕ-quasi-Schreier ring if and only if $\phi(R)$ is a quasi-Schreier ring.*

Proof. Let R be a ϕ-quasi-Schreier ring, then by Theorem 2.13, $R/Nil(R)$ is a quasi-Schreier domain and so by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is a quasi-Schreier domain. Let $\phi(I) \supseteq \phi(J_1)\phi(J_2)$ for regular invertible ideals $\phi(I), \phi(J_1), \phi(J_2)$ of $\phi(R)$. Then

$$\phi(I)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))(\phi(J_2)/Nil(\phi(R)))$$

for invertible ideals $\phi(I)/Nil(\phi(R))$, $\phi(J_1)/Nil(\phi(R))$, $\phi(J_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4]. So

$$\phi(I)/Nil(\phi(R)) = (\phi(I_1)/Nil(\phi(R)))(\phi(I_2)/Nil(\phi(R)))$$

for invertible ideals $\phi(I_1)/Nil(\phi(R)), \phi(I_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ with $\phi(I_1)/Nil(\phi(R)) \supseteq \phi(J_1)/Nil(\phi(R))$ and $\phi(I_2)/Nil(\phi(R)) \supseteq \phi(J_2)/Nil(\phi(R))$. Thus $\phi(I) = \phi(I_1)\phi(I_2)$ for regular invertible ideals $\phi(I_1), \phi(I_2)$ of $\phi(R)$ by [14, Lemma 2.4], with $\phi(I_1) \supseteq \phi(J_1)$ and $\phi(I_2) \supseteq \phi(J_2)$. Therefore $\phi(R)$ is a quasi-Schreier ring. Conversely, let $\phi(R)$ be a quasi-Schreier ring. Let $I \supseteq J_1J_2$ for nonnil ϕ-invertible ideals I, J_1, J_2 of R. Then $\phi(I) \supseteq \phi(J_1)\phi(J_2)$ for regular invertible ideals $\phi(I)$, $\phi(J_1)$, $\phi(J_2)$ of $\phi(R)$. So

$$\phi(I)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))(\phi(J_2)/Nil(\phi(R)))$$

for invertible ideals $\phi(I)/Nil(\phi(R))$, $\phi(J_1)/Nil(\phi(R))$ and $\phi(J_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4]. Since $\phi(I) = \phi(I_1)\phi(I_2)$ for regular invertible ideals $\phi(I_1)$, $\phi(I_2)$ of $\phi(R)$ with $\phi(I_1) \supseteq \phi(J_1)$ and $\phi(I_2) \supseteq \phi(J_2)$, then

$$\phi(I)/Nil(\phi(R)) = (\phi(I_1)/Nil(\phi(R)))(\phi(I_2)/Nil(\phi(R)))$$

for invertible ideals $\phi(I_1)/Nil(\phi(R)), \phi(I_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4], in which

$$\phi(I_1)/Nil(\phi(R)) \supseteq \phi(J_1)/Nil(\phi(R)) \text{ and } \phi(I_2)/Nil(\phi(R)) \supseteq \phi(J_2)/Nil(\phi(R)).$$

Hence, $\phi(R)/Nil(\phi(R))$ is a quasi-Schreier domain and so by [5, Lemma 2.5], $R/Nil(R)$ is a quasi-Schreier domain. Therefore, by Theorem 2.13, R is a ϕ-quasi-Schreier ring. □

Corollary 2.15. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-quasi-Schreier ring*;
(2) *$\phi(R)$ is a quasi-Schreier ring*;
(3) *$R/Nil(R)$ is a quasi-Schreier domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a quasi-Schreier domain.*

Theorem 2.16. *Let $R \in \mathcal{H}$. If R is a ϕ-quasi-Schreier ring, then R is a quasi-Schreier ring.*

Proof. Suppose that R is a ϕ-quasi-Schreier ring. Then, by 2.13, $R/Nil(R)$ is a quasi-Schreier domain. Let I, J_1, J_2 be regular invertible ideals of R and $I \supseteq J_1J_2$. Then $I/Nil(R), J_1/Nil(R), J_2/Nil(R)$ are nonzero invertible ideals of $R/Nil(R)$ by [14, Lemma 2.4], and $I/Nil(R) \supseteq (J_1/Nil(R))(J_2/Nil(R))$. So, $I/Nil(R) = (I_1/Nil(R))(I_2/Nil(R))$ for some nonzero invertible ideals $I_1/Nil(R), I_2/Nil(R)$ of $R/Nil(R)$ with $I_i/Nil(R) \supseteq J_i/Nil(R)$ for $i = 1, 2$. Therefore $I = I_1I_2$ for some regular invertible ideals I_1, I_2 of R by [14, Lemma 2.4], with $I_i \supseteq J_i$ for $i = 1, 2$ and thus R is quasi-Schreier ring. □

Theorem 2.17. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-quasi-Schreier ring if and only if R is a quasi-Schreier ring.*

Proof. Suppose that R is a quasi-Schreier ring. Then $\phi(R) = R$ is a quasi-Schreier ring. Hence, by Theorem 2.14, R is a ϕ-quasi-Schreier ring. The converse is clear by Theorem 2.16. □

Corollary 2.18. *Let $R \in \mathcal{H}$. If R is a ϕ-pre-Schreier ring, then R is a ϕ-quasi-Schreier ring.*

Proof. Let R be a ϕ-pre-Schreier ring, then $R/Nil(R)$ is a pre-Schreier domain. So, by [18, Proposition 2.3], $R/Nil(R)$ is a quasi-Schreier domain and hence by Theorem 2.13, R is a ϕ-quasi-Schreier ring. □

Definition 2.19. We say that a regular element x of a ring R is strongly primal, whenever $x \mid ab$ for regular elements a, b of R, then there exist $k \geq 1$ and regular elements a', b' of R such that $x^k = a'b'$ with $a' \mid a^k$ and $b' \mid b^k$.

Definition 2.20. A nonnil element x of R is called ϕ-strongly primal in R if and only if $\phi(x)$ is strongly primal in $\phi(R)$.

Another generalization of the ϕ-pre-Schreier rings defined as follow.

Definition 2.21. A ring R is called a ϕ-almost-Schreier ring (ϕ-*AS* ring) if every nonnil element of R is ϕ-strongly primal.

Lemma 2.22. *Let $R \in \mathcal{H}$ and $x \in R$. Then x is a ϕ-strongly primal element in R if and only if $x + Nil(R)$ is a strongly primal element in $R/Nil(R)$.*

Proof. Let x be a ϕ-strongly primal element of R. Let $x + Nil(R) \mid (a + Nil(R))(b+Nil(R)) = ab+Nil(R)$ in $R/Nil(R)$ for nonnil elements x, a, b of R. Then there exists $y+Nil(R)$ in $R/Nil(R)$ such that $ab+Nil(R) = xy+Nil(R)$. So $ab - xy \in Nil(R)$. Thus $ab = xy + w$ for some $w \in Nil(R)$. Since $Nil(R)$ is a divided prime ideal. Then $Nil(R) \subseteq (x)$. So $w = xz$ for some $z \in Nil(R)$. Therefore $ab = xy+xz = x(y+z)$. Hence $x \mid ab$ in R for nonnil elements x, a, b of R. In this case $\phi(x) \mid \phi(ab) = \phi(a)\phi(b)$ in $\phi(R)$ for regular elements $\phi(x)$, $\phi(a)$, $\phi(b)$ of $\phi(R)$. Since x is ϕ-strongly primal in R, $\phi(x)$ is strongly primal in $\phi(R)$, by definition. So $\phi(x^k) = \phi(a')\phi(b') = \phi(a'b')$ for some regular elements $\phi(a')$, $\phi(b')$ of $\phi(R)$ and some integer $k \geq 1$ with $\phi(a') \mid \phi(a^k)$ and $\phi(b') \mid \phi(b^k)$. Hence $\langle\phi(x^k)\rangle = \langle\phi(a'b')\rangle$ with $\langle\phi(a^k)\rangle \subseteq \langle\phi(a')\rangle$ and $\langle\phi(b^k)\rangle \subseteq \langle\phi(b')\rangle$. Therefore, $\phi(\langle x^k\rangle) = \phi(\langle a'b'\rangle)$ with $\phi(\langle a^k\rangle) \subseteq \phi(\langle a'\rangle)$ and $\phi(\langle b^k\rangle) \subseteq \phi(\langle b'\rangle)$. Hence, by [14, Lemma 2.1], $\langle x^k\rangle = \langle a'b'\rangle$ with $\langle a^k\rangle \subseteq \langle a'\rangle$ and $\langle b^k\rangle \subseteq \langle b'\rangle$. So there exists a unit element u of R such that $x^k = ua'b'$ with $a' \mid a^k$ and $b' \mid b^k$. Since u is invertible, $u \notin Z(R)$ and so u is not a nonnil element of R. Hence $x^k + Nil(R) = ua'b' + Nil(R) = (ua' + Nil(R))(b' + Nil(R))$ with $ua' \mid (ua)^k$ and $b' \mid b^k$. Therefore $x + Nil(R)$ is strongly primal in $R/Nil(R)$. Conversely, let $x + Nil(R)$ be a strongly primal element of $R/Nil(R)$. Since, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is ring- homomorphic to $R/Nil(R)$, then $\phi(x)+Nil(\phi(R))$ is strongly primal in $\phi(R)/Nil(\phi(R))$. Now, let $x \mid ab$ for nonnil elements of R. Then $\phi(x) + Nil(\phi(R)) \mid \phi(a) + Nil(\phi(R))\phi(b) + Nil(\phi(R))$ and so $(\phi(x) + Nil(\phi(R)))^k = (\phi(a')+Nil(\phi(R)))(\phi(b')+Nil(\phi(R))) = \phi(a')\phi(b')+Nil(\phi(R))$ for nonzero elements $\phi(a') + Nil(\phi(R)), \phi(b') + Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ for some $k \geq 1$ with $\phi(a') + Nil(\phi(R)) \mid (\phi(a) + Nil(\phi(R))^k$ and $\phi(b') + Nil(\phi(R)) \mid (\phi(b) + Nil(\phi(R)))^k$. Therefore $\phi(x)^k - \phi(a')\phi(b') \in Nil(\phi(R))$. Since $Nil(\phi(R)) = Z(\phi(R)$, we conclude that $\phi(x)^k = \phi(a')\phi(b')$ for some $k \geq 1$ with $\phi(a') \mid (\phi(a))^k$ and $\phi(b') \mid (\phi(b))^k$. So $\phi(x)$ is strongly primal in $\phi(R)$ and by definition of ϕ-strongly primal element, x is a ϕ-strongly primal element of R. □

Theorem 2.23. *Let $R \in \mathcal{H}$. Then R is a ϕ-AS ring if and only if $R/Nil(R)$ is AS domain.*

Proof. It is clear by Lemma 2.22. □

Theorem 2.24. *Let $R \in \mathcal{H}$. Then R is a ϕ-AS ring if and only if $\phi(R)$ is an AS ring.*

Proof. Suppose that R is a ϕ-AS ring, then by Theorem 2.23, $R/Nil(R)$ is a AS domain. So, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is a AS domain. Let $\phi(x) \mid \phi(a)\phi(b)$ for regular elements $\phi(x)$, $\phi(a)$, $\phi(b)$ of $\phi(R)$. Then $\phi(x)+Nil(\phi(R)) \mid (\phi(a)+Nil(\phi(R)))(\phi(b)+Nil(\phi(R)))$ for nonzero elements $\phi(x)+Nil(\phi(R))$, $\phi(a)+Nil(\phi(R))$, $\phi(b)+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Thus $(\phi(x)+Nil(\phi(R)))^K = (\phi(a')+Nil(\phi(R)))(\phi(b')+Nil(\phi(R)))$ for nonzero elements $\phi(a')+Nil(\phi(R))$, $\phi(b')+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ and for some $k \geq 1$ with $\phi(a')+Nil(\phi(R)) \mid (\phi(a)+Nil(\phi(R)))^k$ and $\phi(b')+Nil(\phi(R)) \mid (\phi(b)+Nil(\phi(R)))^k$. Therefore $\phi(x)^k - \phi(a')\phi(b') \in Nil(\phi(R))$. Since $Nil(\phi(R)) = Z(\phi(R)$, we conclude that $\phi(x)^k = \phi(a')\phi(b')$ for regular elements $\phi(a')$, $\phi(b')$ of $\phi(R)$ with $\phi(a') \mid (\phi(a))^k$ and $\phi(b') \mid (\phi(b))^k$. Hence $\phi(R)$ is an AS ring. Conversely, let $\phi(R)$ be an AS ring. Let $x \mid ab$ for nonnil elements of R. Then $\phi(x) \mid \phi(a)\phi(b)$ for regular elements $\phi(x), \phi(a), \phi(b)$ of $\phi(R)$. So $\phi(x)+Nil(\phi(R)) \mid (\phi(a)+Nil(\phi(R)))(\phi(b)+Nil(\phi(R)))$ for nonzero elements $\phi(x)+Nil(\phi(R))$, $\phi(a)+Nil(\phi(R))$, $\phi(b)+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$. Since $\phi(x)^k = \phi(a')\phi(b')$ for regular elements $\phi(a'), \phi(b')$ of $\phi(R)$ and for some $k \geq 1$ with $\phi(a') \mid (\phi(a))^k$ and $\phi(b') \mid (\phi(b))^k$, so $(\phi(x))^k + Nil(\phi(R)) = (\phi(a')+Nil(\phi(R)))(\phi(b')+Nil(\phi(R)))$ for nonzero elements $\phi(a')+Nil(\phi(R))$, $\phi(b')+Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ with $\phi(a')+Nil(\phi(R)) \mid (\phi(a))^k + Nil(\phi(R))$ and $\phi(b')+Nil(\phi(R)) \mid (\phi(b))^k + Nil(\phi(R))$. Therefore $\phi(R)/Nil(\phi(R))$ is an AS domain. Since, by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is ring-homomorphic to $R/Nil(R)$, then $R/Nil(R)$ is an AS domain. Hence, by Theorem 2.23, R is a ϕ-AS ring. □

Corollary 2.25. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-AS ring*;
(2) *$\phi(R)$ is a AS ring*;
(3) *$R/Nil(R)$ is a AS domain*;
(4) *$\phi(R)/Nil(\phi(R)$ is a AS domain.*

Lemma 2.26. *Let $R \in \mathcal{H}$ and $x \in R$. If x is ϕ-strongly primal, then x is strongly primal.*

Proof. If x is ϕ-strongly primal, then by Lemma 2.22, $x + Nil(R)$ is strongly primal in $R/Nil(R)$. If $x \mid ab$ for regular elements x, a, b of R. Then with a same way of proof Theorem 2.24, we conclude that x is strongly primal. □

Theorem 2.27. *Let $R \in \mathcal{H}$. If R is a ϕ-AS ring, then R is a AS ring.*

Proof. It is obvious by Lemma 2.26. □

Theorem 2.28. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-AS ring if and only if R is a AS ring.*

Proof. Suppose that R is a AS ring. Then $\phi(R) = R$ is a AS ring. Hence, by Theorem 2.24, R is a ϕ-AS ring. The converse is clear by Theorem 2.27. □

Corollary 2.29. *Let $R \in \mathcal{H}$. If R is a ϕ-pre-Schreier ring, then R is a ϕ-AS ring.*

Proof. Let R be a ϕ-pre-Schreier ring, then $R/Nil(R)$ is a pre-Schreier domain. So $R/Nil(R)$ is a AS domain and hence R is a ϕ-AS ring. □

Definition 2.30. A ring R is called a ϕ-almost-quasi-Schreier ring (ϕ-AQS ring) if whenever I, J_1, J_2 are nonnil ϕ-invertible ideals of R such that $I \supseteq J_1J_2$ there exist an integer $k \geq 1$ and nonnil ϕ-invertible ideals I_1, I_2 of R such that $I^k = I_1I_2$ and $I_i \supseteq J_i^k$ for $i = 1, 2$.

Theorem 2.31. *Let $R \in \mathcal{H}$. Then R is a ϕ-AQS ring if and only if $R/Nil(R)$ is an AQS domain.*

Proof. Suppose that R is a ϕ-AQSring and let $I/Nil(R)$, $J_1/Nil(R)$ and $J_2/Nil(R)$ be nonzero invertible ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (J_1/Nil(R))(J_2/Nil(R))$. Then, by Lemma 2.12, I, J_1, J_2 are nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$. So $I^k = I_1I_2$ for ϕ-invertible ideals I_1, I_2 of R and for some $k \geq 1$ with $I_1 \supseteq J_1^k$ and $I_2 \supseteq J_2^k$. Therefore $I^k/Nil(R) = (I_1/Nil(R))(I_2/Nil(R))$ for invertible ideals $I_1/Nil(R)$ and $I_2/Nil(R)$ of $R/Nil(R)$ by Lemma 2.12 and for some $k \geq 1$, with $I_1/Nil(R) \supseteq J_1^k/Nil(R)$ and $I_2/Nil(R) \supseteq J_2^k/Nil(R)$. Hence, $R/Nil(R)$ is an AQS domain. Conversely, suppose that $R/Nil(R)$ is an AQS domain. Let I, J_1, J_2 be nonnil ϕ-invertible ideals of R and $I \supseteq J_1J_2$. Then, by Lemma 2.12, $I/Nil(R)$, $J_1/Nil(R)$, $J_2/Nil(R)$ are nonzero invertible ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq (J_1/Nil(R))(J_2/Nil(R))$. So $I^k/Nil(R) = (I_1/Nil(R))(I_2/Nil(R))$ for invertible ideals $I_1/Nil(R)$, $I_2/Nil(R)$ of $R/Nil(R)$ and for some $k \geq 1$ with $I_1/Nil(R) \supseteq J_1^k/Nil(R)$ and $I_2/Nil(R) \supseteq J_2^k/Nil(R)$. Therefore, $I^k = I_1I_2$ for ϕ-invertible ideals I_1, I_2 of R by Lemma 2.12 and for some $k \geq 1$, with $I_1 \supseteq J_1^k$ and $I_2 \supseteq J_2^k$. Therefore R is a ϕ-AQS ring. □

Theorem 2.32. *Let $R \in \mathcal{H}$. Then R is a ϕ-AQS ring if and only if $\phi(R)$ is a AQS ring.*

Proof. Let R be a ϕ-AQS ring, then by Theorem 2.31, $R/Nil(R)$ is a AQS domain and so by [5, Lemma 2.5], $\phi(R)/Nil(\phi(R))$ is a AQS domain. Let $\phi(I) \supseteq \phi(J_1)\phi(J_2)$ for regular invertible ideals $\phi(I)$, $\phi(J_1)$, $\phi(J_2)$ of $\phi(R)$. Then $\phi(I)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))(\phi(J_2)/Nil(\phi(R)))$ for invertible ideals $\phi(I)/Nil(\phi(R))$, $\phi(J_1)/Nil(\phi(R))$, $\phi(J_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4]. So $(\phi(I)/Nil(\phi(R)))^K = (\phi(I_1)/Nil(\phi(R)))(\phi(I_2)/Nil(\phi(R)))$ for invertible ideals $\phi(I_1)/Nil(\phi(R))$, $\phi(I_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ and for some $k \geq 1$ with $\phi(I_1)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))^k$ and $\phi(I_2)/Nil(\phi(R)) \supseteq (\phi(J_2)/Nil(\phi(R)))^k$. Thus $(\phi(I))^k = \phi(I_1)\phi(I_2)$ for regular invertible ideals $\phi(I_1), \phi(I_2)$ of $\phi(R)$ by [14, Lemma 2.4] and for some $k \geq 1$, with $\phi(I_1) \supseteq (\phi(J_1))^k$ and $\phi(I_2) \supseteq (\phi(J_2))^k$. Therefore $\phi(R)$ is an AQS ring. Conversely, let

$\phi(R)$ be an AQS ring. Let $I \supseteq J_1J_2$ for nonnil ϕ-invertible ideals I, J_1, J_2 of R. Then $\phi(I) \supseteq \phi(J_1)\phi(J_2)$ for regular invertible ideals $\phi(I)$, $\phi(J_1)$, $\phi(J_2)$ of $\phi(R)$. So $\phi(I)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))(\phi(J_2)/Nil(\phi(R)))$ for invertible ideals $\phi(I)/Nil(\phi(R))$, $\phi(J_1)/Nil(\phi(R))$, $\phi(J_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4]. Since $(\phi(I))^k = \phi(I_1)\phi(I_2)$ for regular invertible ideals $\phi(I_1)$ and $\phi(I_2)$ of $\phi(R)$ and for some $k \geq 1$ with $\phi(I_1) \supseteq (\phi(J_1))^k$ and $\phi(I_2) \supseteq (\phi(J_2))^k$, then $(\phi(I)/Nil(\phi(R)))^k = (\phi(I_1)/Nil(\phi(R)))(\phi(I_2)/Nil(\phi(R)))$ for invertible ideals $\phi(I_1)/Nil(\phi(R))$, $\phi(I_2)/Nil(\phi(R))$ of $\phi(R)/Nil(\phi(R))$ by [14, Lemma 2.4] and for some $k \geq 1$, with $\phi(I_1)/Nil(\phi(R)) \supseteq (\phi(J_1)/Nil(\phi(R)))^k$ and $\phi(I_2)/Nil(\phi(R)) \supseteq (\phi(J_2)/Nil(\phi(R)))^k$. Hence, $\phi(R)/Nil(\phi(R))$ is a AQS domain and so by [5, Lemma 2.5], $R/Nil(R)$ is an AQS domain. Therefore, by Theorem 2.31, R is a ϕ-AQS ring. □

Corollary 2.33. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-AQS ring*;
(2) *$\phi(R)$ is a AQS ring*;
(3) *$R/Nil(R)$ is a AQS domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a AQS domain.*

Theorem 2.34. *Let $R \in \mathcal{H}$. If R is a ϕ-AQS ring, then R is an AQS ring.*

Proof. Suppose that R is a ϕ-AQS ring, then by Theorem 2.31, $R/Nil(R)$ is an AQS domain. Let I, J_1, J_2 be regular invertible ideals of R and $I \supseteq J_1J_2$. Then, by [14, Lemma 2.4], $I/Nil(R), J_1/Nil(R), J_2/Nil(R)$ are nonzero invertible ideals of $R/Nil(R)$ and $I/Nil(R) \supseteq J_1/Nil(R)J_2/Nil(R)$. So, $(I/Nil(R))^k = I_1/Nil(R)I_2/Nil(R)$ for an integer $k \geq 1$ and for some nonzero invertible ideals $I_1/Nil(R), I_2/Nil(R)$ of $R/Nil(R)$ with $I_i/Nil(R) \supseteq (J_i/Nil(R))^k$ for $i = 1, 2$. Therefore $I^k = I_1I_2$ for an integer $k \geq 1$ and some regular invertible ideals I_1, I_2 of R by [14, Lemma 2.4], with $I_i \supseteq J_i^k$ for $i = 1, 2$. Hence R is AQS ring. □

Theorem 2.35. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-AQS ring if and only if R is an AQS ring.*

Proof. Suppose that R is a AQS ring. Then $\phi(R) = R$ is an AQS ring. Hence, by Theorem 2.32, R is a ϕ-AQS ring. The converse is clear by Theorem 2.34. □

Corollary 2.36. *Let $R \in \mathcal{H}$. If R is a ϕ-AS ring, then R is a ϕ-AQS ring.*

Proof. Let R be a ϕ-AS ring, then $R/Nil(R)$ is an AS domain. So, by [1, Proposition 2.3], $R/Nil(R)$ is an AQS domain. Therefore, by Theorem 2.31, R is a ϕ-AQS ring. □

Corollary 2.37. *Let $R \in \mathcal{H}$. If R is a ϕ-quasi-Schreier ring, then R is a ϕ-AQS ring.*

Proof. Let R be a ϕ-quasi-Schreier ring, then by Theorem 2.13 $R/Nil(R)$ is a quasi-Schreier domain. So $R/Nil(R)$ is a AQS domain. Therefore, by Theorem 2.31, R is a ϕ-AQS ring. □

3. ϕ-GCD ring

Recall that a ring R is called a GCD ring if every two regular elements of R have a greatest common divisor.

Definition 3.1. A ring R is called a ϕ-GCD ring if every two nonnil elements of R have a greatest common divisor.

Theorem 3.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-GCD ring if and only if $\phi(R)$ is a GCD ring.*

Proof. Let R be a ϕ-GCD ring and $\phi(x), \phi(y)$ two regular elements of $\phi(R)$, i.e., $\phi(x), \phi(y) \in \phi(R) \setminus Z(\phi(R)) = \phi(R) \setminus Nil(\phi(R))$. Then $x, y \in R \setminus Nil(R)$. So, there is $d \in R \setminus Nil(R)$ such that $d \mid x$ and $d \mid y$ and if $c \in R \setminus Nil(R)$ with $c \mid x$ and $c \mid y$, then $c \mid d$. Therefore, $\phi(d) \in \phi(R) \setminus Nil(\phi(R)) = \phi(R) \setminus Z(\phi(R))$ such that $\phi(d) \mid \phi(x)$ and $\phi(d) \mid \phi(y)$ and if $\phi(c) \in \phi(R) \setminus Nil(\phi(R)) = \phi(R) \setminus Z(\phi(R))$ with $\phi(c) \mid \phi(x)$ and $\phi(c) \mid \phi(y)$, then $\phi(c) \mid \phi(d)$. So, $\phi(R)$ is a GCD ring. Conversely, let $\phi(R)$ be a GCD ring and $x, y \in R \setminus Nil(R)$. Then $\phi(x), \phi(y) \in \phi(R) \setminus Nil(\phi(R)) = \phi(R) \setminus Z(\phi(R))$. So, there is $\phi(d) \in \phi(R) \setminus Z(\phi(R)) = \phi(R) \setminus Nil(\phi(R))$ such that $\phi(d) \mid \phi(x), \phi(y)$ and if $\phi(c) \in \phi(R) \setminus Z(\phi(R)) = \phi(R) \setminus Nil(\phi(R))$ with $\phi(c) \mid \phi(x), \phi(y)$, then $\phi(c) \mid \phi(d)$. Thus, there is $d \in R \setminus Nil(R)$ such that $d \mid x$ and $d \mid y$ and if $c \in R \setminus Nil(R)$ with $c \mid x$ and $c \mid y$, then $c \mid d$. Therefore R is a ϕ-GCD ring. □

Theorem 3.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-GCD ring if and only if $R/Nil(R)$ is a GCD domain.*

Proof. Suppose that R is a ϕ-GCD ring and $x + Nil(R), y + Nil(R)$ be two nonzero elements of $R/Nil(R)$. Then x, y are two nonnil elements of R and so x, y have greatest common divisor. Hence $x + Nil(R), y + Nil(R)$ have a greatest common divisor. Therefore $R/Nil(R)$ is a GCD domain. Conversely, let $R/Nil(R)$ is a GCD domain and x, y be two nonnil elements of R. Then $x + Nil(R), y + Nil(R)$ are two nonzero elements of $R/Nil(R)$. Thus $x + Nil(R), y + Nil(R)$ have a greatest common divisor. So x, y have a greatest common divisor. Therefore R is a ϕ-GCD ring. □

Corollary 3.4. *Let $R \in \mathcal{H}$. The following are equivalent:*
(1) *R is a ϕ-GCD ring;*
(2) *$\phi(R)$ is a GCD ring;*
(3) *$R/Nil(R)$ is a GCD domain;*
(4) *$\phi(R)/Nil(\phi(R))$ is a GCD domain.*

Theorem 3.5. *Let $R \in \mathcal{H}$. If R is a ϕ-GCD ring, then R is a GCD ring.*

Proof. Suppose that R is a ϕ-GCD ring, then $R/Nil(R)$ is a GCD domain. Let a, b be two regular elements of R. Then $a + Nil(R), b + Nil(R)$ are two elements of $R/Nil(R)$. So $a+Nil(R), b+Nil(R)$ have greatest common divisor. So, a, b have greatest common divisors. Therefore R is a GCD ring. □

Theorem 3.6. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-GCD ring if and only if R is a GCD ring.*

Proof. Suppose that R is a GCD ring. Then $\phi(R) = R$ is a GCD ring. Hence, by Theorem 3.2, R is a ϕ-GCD ring. The converse is clear by Theorem 3.5. □

Corollary 3.7. *Let $R \in \mathcal{H}$. If R is a ϕ-GCD ring, then R is a ϕ-Schreier ring.*

Proof. Let R be a ϕ-GCD ring. So $R/Nil(R)$ is a GCD domain and thus, by [15, Theorem 2.4], $R/Nil(R)$ is a Schreier domain. Therefore R is a ϕ-Schreier ring. □

Definition 3.8. A ϕ-generalized GCD ring (ϕ-$GGCD$ ring) is a ring in which every intersection of two ϕ-invertible ideals is a ϕ-invertible ideal.

Theorem 3.9. *Let $R \in \mathcal{H}$. Then R is a ϕ-$GGCD$ ring if and only if $R/Nil(R)$ is a $GGCD$ domain.*

Proof. Suppose that R is a ϕ-$GGCD$ ring and let $I/Nil(R), J/Nil(R)$ be two nonzero invertible ideals of $R/Nil(R)$. So, by Lemma 2.12, I, J are two ϕ-invertible ideals of R. Therefore, $I \cap J$ is a ϕ-invertible ideal of R. Thus, by Lemma 2.12, $I \cap J/Nil(R) = I/Nil(R) \cap J/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Hence $R/Nil(R)$ is a $GGCD$ domain. Conversely, suppose that $R/Nil(R)$ is a $GGCD$ domain. let I, J be two ϕ-invertible ideals of R. Then, by Lemma 2.12, $I/Nil(R), J/Nil(R)$ are two nonzero invertible ideals of $R/Nil(R)$. So $I \cap J/Nil(R) = I/Nil(R) \cap J/Nil(R)$ is an invertible ideal of $R/Nil(R)$. Therefore, by Lemma 2.12, $I \cap J$ is a ϕ-invertible ideal of R. Hence R is a ϕ-$GGCD$ ring. □

Theorem 3.10. *Let $R \in \mathcal{H}$. Then R is a ϕ-$GGCD$ ring if and only if $\phi(R)$ is a $GGCD$ ring.*

Proof. Let R be a ϕ-$GGCD$ ring and $\phi(I), \phi(J)$ be two invertible ideals of $\phi(R)$. Then I, J are two ϕ-invertible ideals of R and so $I \cap J$ is a ϕ-invertible ideal of R. Hence $\phi(I) \cap \phi(J) = \phi(I \cap J)$ is an invertible ideal of $\phi(R)$. Therefore $\phi(R)$ is a $GGCD$ ring. Conversely, let $\phi(R)$ be a $GGCD$ ring and I, J be two ϕ-invertible ideals of R. Then $\phi(I), \phi(J)$ are two invertible ideals of $\phi(R)$ and so $\phi(I \cap J) = \phi(I) \cap \phi(J)$ is an invertible ideal of $\phi(R)$. Therefore $I \cap J$ is a ϕ-invertible ideal of R. Thus R is a ϕ-$GGCD$ ring. □

Corollary 3.11. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-$GGCD$ ring*;

(2) *$\phi(R)$ is a $GGCD$ ring*;

(3) *$R/Nil(R)$ is a $GGCD$ domain;*
(4) *$\phi(R)/Nil(\phi(R))$ is a $GGCD$ domain.*

Theorem 3.12. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-$GGCD$ ring if and only if R is a $GGCD$ ring.*

Proof. In this case $\phi(R) = R$. □

Corollary 3.13. *Let $R \in \mathcal{H}$. If R is a ϕ-GCD ring, then R is a ϕ-$GGCD$ ring.*

Corollary 3.14. *Let $R \in \mathcal{H}$. If R is a ϕ-Prüfer ring, then R is a ϕ-$GGCD$ ring.*

Proof. Let R be a ϕ-Prüfer ring. So, by [5, Theorem 2.6], $R/Nil(R)$ is a Prüfer domain. Then, by [4, Theorem 1], $R/Nil(R)$ is a $GGCD$ domain. Therefore, by Theorem 3.9, R is a ϕ-$GGCD$ ring. □

Corollary 3.15. *Let $R \in \mathcal{H}$. If R is a ϕ-$GGCD$ ring, then R is a ϕ-quasi-Schreier ring.*

Proof. Let R be a ϕ-$GGCD$ ring. So, by Theorem 3.9, $R/Nil(R)$ is a $GGCD$ domain. Then, by [18, Proposition 2.2], $R/Nil(R)$ is a quasi-Schreier domain. Therefore, by Theorem 2.13, R is a ϕ-quasi-Schreier ring. □

Definition 3.16. A ϕ-almost GCD ring (ϕ-$AGCD$ ring) is a ring in which for every two nonnil principal ideals I and J of R, there exists some $k \geq 1$ such that $I^k \cap J^k$ is a nonnil principal ideal of R.

Theorem 3.17. *Let $R \in \mathcal{H}$. Then R is a ϕ-$AGCD$ ring if and only if $R/Nil(R)$ is a $AGCD$ domain.*

Proof. Suppose that R is a ϕ-$AGCD$ ring and let $I/Nil(R), J/Nil(R)$ be two nonzero principal ideals of $R/Nil(R)$. Then, by [5, Lemma 3.1], I, J are two nonnil principal ideals of R. So, there exists some $k \geq 1$ such that $I^k \cap J^k$ is a nonnil principal ideal of R. Therefore, by [5, Lemma 3.1], $(I/Nil(R))^k \cap (J/Nil(R))^k = I^k \cap J^k/(Nil(R))^k$ is a nonzero principal ideal of $R/Nil(R)$. Hence, $R/Nil(R)$ is a $AGCD$ domain. Conversely, suppose that $R/Nil(R)$ is a $AGCD$ domain. Let I, J be two nonnil principal ideal of R. Then, by [5, Lemma 3.1], $I/Nil(R), J/Nil(R)$ be two nonzero principal ideals of $R/Nil(R)$. Therefore there exists some $k \geq 1$ such that $(I/Nil(R))^k \cap (J/Nil(R))^k = I^k \cap J^k/(Nil(R))^k$ is a nonzero principal ideal of $R/Nil(R)$. So, by [5, Lemma 3.1], $I^k \cap J^k$ is a nonnil principal ideal of R. Hence, R is a ϕ-$AGCD$ ring. □

Theorem 3.18. *Let $R \in \mathcal{H}$. Then R is a ϕ-$AGCD$ ring if and only if $\phi(R)$ is a $AGCD$ ring.*

Proof. Let R be a ϕ-$AGCD$ ring and $\phi(I), \phi(J)$ be two regular principal ideals of $\phi(R)$. Then I, J are two nonnil principal ideals of R and so there exists some $k \geq 1$ such that $I^k \cap J^k$ is a nonnil principal ideal of R. Hence $\phi(I)^k \cap \phi(J)^k =$

$\phi(I^k \cap J^k)$ is an regular principal ideal of $\phi(R)$. Therefore $\phi(R)$ is a $AGCD$ ring. Conversely, let $\phi(R)$ is a $AGCD$ ring and I, J be two nonnil principal ideals of R. Then $\phi(I), \phi(J)$ are two regular principal ideals of $\phi(R)$ and so there exists some $k \geq 1$ such that $\phi(I^k \cap J^k) = \phi(I)^k \cap \phi(J)^k$ is a regular principal ideal of $\phi(R)$. Therefore $I^k \cap J^k$ is a nonnil principal ideal of R. Thus R is a ϕ-$AGCD$ ring. □

Corollary 3.19. *Let $R \in \mathcal{H}$. The following are equivalent*:

(1) *R is a ϕ-$AGCD$ ring*;
(2) *$\phi(R)$ is a $AGCD$ ring*;
(3) *$R/Nil(R)$ is a $AGCD$ domain*;
(4) *$\phi(R)/Nil(\phi(R))$ is a $AGCD$ domain.*

Theorem 3.20. *Let $R \in \mathcal{H}$. If R is a ϕ-$AGCD$ ring, then R is an $AGCD$ ring.*

Proof. Suppose that R is a ϕ-$AGCD$ ring. Then, by Theorem 3.17, $R/Nil(R)$ is a $AGCD$ domain. Let I and J be two regular principal ideal of R. So, by [5, Lemma 3.1], $I/Nil(R)$ and $J/Nil(R)$ are two principal ideals of $R/Nil(R)$. So there exists $k \geq 1$ such that $(I/Nil(R))^k \cap (J/Nil(R))^k$ is a principal ideal of $R/Nil(R)$. Therefore $I^k \cap J^k$ is a principal ideal of R. Hence R is a $AGCD$ ring. □

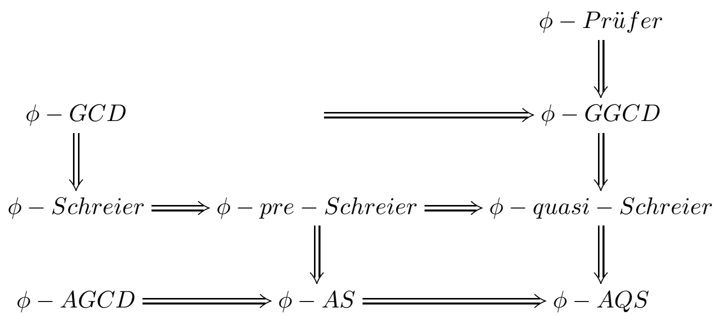
Theorem 3.21. *Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is a ϕ-$AGCD$ ring if and only if R is an $AGCD$ ring.*

Proof. Suppose that R is an $AGCD$ ring. Then $\phi(R) = R$ is an $AGCD$ ring. Hence, by Theorem 3.18, R is a ϕ-$AGCD$ ring. The converse is clear by Theorem 3.20. □

Corollary 3.22. *Let $R \in \mathcal{H}$. If R is a ϕ-$AGCD$ ring, then R is a ϕ-AS ring.*

Proof. Let R be a ϕ-$AGCD$ ring. So, by Theorem 3.17, $R/Nil(R)$ is an $AGCD$ domain. Thus, by [17, Proposition 2.2], $R/Nil(R)$ is an AS domain. Therefore, by Theorem 2.23, R is a ϕ-AS ring. □

Therefore, by Corollaries 2.18, 2.29, 2.36, 2.37, 3.7, 3.13, 3.14, 3.15 and 3.22, we have the following implications.



4. Examples

Our non-domain examples of ϕ-X- rings where $X =$ Schreier, quasi-Schreier, AS, AQS, GCD, GGCD, AGCD are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in [21]. We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R \text{ and } b \in B\}$, and let (r, b) and (s, c) be to elements of $R(+)B$. Define

(1) $(r, b) = (s, c)$ if $r = s$ and $b = c$.

(2) $(r, b) + (s, c) = (r + s, b + c)$.

(3) $(r, b)(s, c) = (rs, bs + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity.

Example 4.1. Let D be a X-domain with quotient field L where $X =$ Schreier, quasi-Schreier, AS, AQS, GCD, GGCD, AGCD. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-X-ring which is not a X-domain.

Proof. Since D is a domain, $\{0\}$ is a prime ideal of D and $Nil(D) = \{0\}$. Hence $Nil(R) = \{0\}(+)L$ is a prime ideal of R. $Nil(R)$ is a divided ideal, because let $(a, x) \in R \setminus Nil(R)$ and $(0, y) \in Nil(R)$. Since $(0, y) = (a, x)(0, y/a)$, $(0, y) \in (a, x)$. Hence $R \in \mathcal{H}$. Also, $R/Nil(R)$ is ring-isomorphic to D. Since D is a X-domain, so $R/Nil(R)$ is a X-domain and therefore R is a ϕ-X ring. But R is not a domain, because $(0, l_1)(0, l_2) = (0, 0)$ for each $l_1, l_2 \in L$. □

The following is an example of a ring $R \in \mathcal{H}$ which is a X-ring but not a ϕ-X-ring where $X =$Schreier, quasi-Schreier, AS, AQS, GCD, GGCD, AGCD.

Example 4.2. Let D be an integral domain with quotient field L which is not a X-domain where $X =$ Schreier, quasi-Schreier, AS, AQS, GCD, GGCD, AGCD. Set $R = D(+)(L/D)$. Then $R \in \mathcal{H}$ is a X-ring which is not a ϕ-X-ring.

Proof. By previous example, $Nil(R) = \{0\}(+)(L/D)$ is a divided prime ideal of R and thus $R \in \mathcal{H}$. Since every nonunit of R is zero divisor, we conclude that R is a X-ring. Since $R/Nil(R)$ is ring-isomorphic to D and D is not a X-domain, so R is not a ϕ-X-ring. □

It is clear that a ϕ-pre-Schreier ring is a ϕ-AS ring. In following example we show that the converse is not true.

Example 4.3. Let $D = \mathbb{Z}[\sqrt{-3}]$. So by [17], D is an AS domain such that is not a pre-Schreier domain. Let $R = D(+)L$ where L is the quotient field of D. Then $Nil(R) = \{0\}(+)L$ and $R \in \mathcal{H}$. Since $R/Nil(R)$ is ring-isomorphic to D, so $R/Nil(R)$ is an AS domain such that is not a pre-Schreier domain. Therefore, by Theorem 2.5 and Theorem 2.23, R is a ϕ-AS ring which is not a ϕ-pre-Schreier ring.

Example 4.4. Let $D = \mathbb{Z}[X^2, X^3]$. So, by [1, Proposition 4.8], D is an AQS domain which is neither AS domain nor quasi-Schreier domain. Set $R = D(+)L$ where L is the quotient field of D. Hence $Nil(R) = \{0\}(+)L$, $R \in \mathcal{H}$ and

$R/Nil(R)$ is ring-isomorphic to D. So $R/Nil(R)$ is an AQS domain which is neither AS domain nor quasi-Schreier domain. Therefore, by Theorem 2.31, Theorem 2.23 and Theorem2.13, R is a ϕ-AQS ring which is neither ϕ-AS-ring nor ϕ-quasi-Schreier ring.

Example 4.5. Let D be a Dedekind non-principal domain. Then by [18], D is a quasi-Schreier domain but is not a pre-Schreier domain. Let $R = D(+)L$ where L is the quotient field of D. Hence $Nil(R) = \{0\}(+)L$, $R \in \mathcal{H}$ and $R/Nil(R)$ is ring-isomorphic to D. So $R/Nil(R)$ is a quasi-Schreier domain such that is not a pre-Schreier domain. Hence, by Theorem 2.13 and Theorem 2.5, R is a ϕ-quasi-Schreier ring which is not a ϕ-pre-Schreier ring.

Example 4.6. Let $D = \mathbb{Q}[[X^2, X^3]]$. So, by [17, Remark 2.4], D is an AS domain which is not $AGCD$ domain. Set $R = D(+)L$ where L is the quotient field of D. By the same argument, R is a ϕ-AS ring which is not a ϕ-$AGCD$ ring.

Example 4.7. Let D be a Prüfer non Bezout domain. Then D is a $GGCD$ domain which is not a GCD domain. Let $R = D(+)L$ where L is the quotient field of D. Hence $Nil(R) = \{0\}(+)L$, $R \in \mathcal{H}$ and $R/Nil(R)$ is ring-isomorphic to D. Thus $R/Nil(R)$ is a $GGCD$ domain which is not a GCD domain. Therefore, by Theorem 3.9 and Theorem 3.3, R is a ϕ-$GGCD$ ring which is not a ϕ-GCD ring.

Example 4.8. Let A be a Prüfer domain which has two nonzero nonunits a, b such that $a \in \bigcap_n b^n A$. Consider the domain $D = A + XA[1/b][X]$. Then, by [7, Example 11], D is a quasi-Schreier domain which is not a $GGCD$ domain. Set $R = D(+)L$ where L is the quotient field of D. Hence $Nil(R) = \{0\}(+)L$, $R \in \mathcal{H}$ and $R/Nil(R)$ is ring-isomorphic to D. So $R/Nil(R)$ is a quasi-Schreier domain which is not a $GGCD$ domain. Therefore, by Theorem 2.13 and Theorem 3.9, R is a ϕ-quasi-Schreier ring but is not a ϕ-$GGCD$ ring.

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References

[1] Z. Ahmad and T. Dumtrescu, *Almost quasi Schreier domains*, Comm. Algebra **41** (2013), no. 5, 1685–1696.

[2] M. M. Ali and D. J. Smith, *Generalized GCD rings*, Beiträge Algebra Geom. **42** (2001), no. 1, 219–233.

[3] D. D. Anderson, *π-domains divisorial ideals and overrings*, Glasgow Math. J. **19** (1978), no. 1, 199–203.

[4] D. D. Anderson and D. F. Anderson, *Generalized GCD domains*, Comment. Math. St. Univ. Pauli **28** (1979), 215–221.

[5] D. F. Anderson and A. Badawi, *On ϕ-Prüfer rings and ϕ-Bezout rings*, Houston J. Math. **2** (2004), no. 2, 331–343.

[6] ______, *On ϕ-Dedekind rings and ϕ-Krull rings*, Houston J. Math. **4** (2005), no. 4, 1007–1022.
[7] D. D. Anderson, T. Dumiterscu, and M. Zafrullah, *Quasi Schreier domains II*, Comm. Algebra **35** (2007), no. 7, 2096–2104.
[8] A. Badawi, *On ϕ-pseudo-valuation rings*, Lecture Notes Pure Appl. Math., vol. **205**, 101–110, Marcel Dekker, New York/Basel, 1999.
[9] ______, *On divided commutative rings*, Comm. Algebra **27** (1999), no. 3, 1465–1474.
[10] ______, *On ϕ-pseudo- valuation rings II*, Houston J. Math. **26** (2000), no. 3, 473–480.
[11] ______, *On ϕ-chained rings and ϕ-pseudo-valuation rings*, Houston J. Math. **27** (2001), no. 4, 725–736.
[12] ______, *On divided rings and ϕ-pseudo-valuation rings*, Int. J. Commutative Rings **1** (2002), 51–60.
[13] ______, *On nonnil-Noetherian rings*, Comm. Algebra **31** (2003), no. 4, 1669–1677.
[14] A. Badawi and Thomas G. Lucas, *On ϕ-Mori rings*, Houston J. Math. **32** (2006), no. 1, 1–32.
[15] P. M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc. **64** (1968), 251–264.
[16] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. **67** (1976), no. 2, 353–363.
[17] T. Dumtrescu and W. Khalid, *Almost-Schreier domains*, Comm. Algebra **38** (2010), 2981–2991.
[18] T. Dumtrescu and R. Moldovan, *Quasi-Schreier domains*, Math. Reports. **5** (2003), 121–126.
[19] R. Gilmer, *Multiplication ideal theory*, Queen's, Kingston, 1999.
[20] M. Griffin, *Prüfer rings with zerodivisors*, J. Reine Angew. Math. **240** (1970), 55–67.
[21] J. A. Huckaba, *Commutative rings with zero divisors*, New York, Dekker, 1988.
[22] S. McAdam and D. E. Rush, *Schreier rings*, Bull. London Math. Soc. **10** (1978), no. 1, 77–80.
[23] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), no. 1-3, 29–62.
[24] ______, *On a property of pre-Schreier domains*, Comm. Algebra **15** (1987), no. 9, 1895–1920.

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