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A TRANSLATION THEOREM FOR THE GENERALIZED FOURIER–FEYNMAN TRANSFORM ASSOCIATED WITH GAUSSIAN PROCESS ON FUNCTION SPACE

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Abstract. In this paper we define a generalized analytic Fourier–Feynman transform associated with Gaussian process on the function space $C_{a,b}[0,T]$. We establish the existence of the generalized analytic Fourier–Feynman transform for certain bounded functionals on $C_{a,b}[0,T]$. We then proceed to establish a translation theorem for the generalized transform associated with Gaussian process.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space. The concept of the analytic Fourier–Feynman transform on $C_0[0,T]$, initiated by Brue [3], has been developed in the literature. This transform and its properties are similar in many respects to the ordinary Fourier function transform. For an elementary introduction to the analytic Fourier–Feynman transform, see [29] and the references cited therein. Various kinds of the study for the analytic Fourier–Feynman transform and related topics were developed on abstract Wiener space [1, 2, 11, 12, 13, 25], space of abstract Wiener space valued continuous functions on compact interval in $\mathbb{R}$ [8, 9, 10, 17, 18, 19], and the analogue of Wiener space [20, 28].

Let $D = [0,T]$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. A generalized Brownian motion process (GBMP) on $\Omega \times D$ is a Gaussian process $Y \equiv \{Y_t\}_{t\in D}$

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such that $Y_0 = 0$ almost everywhere, and for any $0 \le s < t \le T$,

$$Y_t - Y_s \sim N\big(a(t) - a(s),\, b(t) - b(s)\big),$$

where $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2, $a(t)$ is a continuous real-valued function on $[0, T]$, and $b(t)$ is a monotonically increasing continuous real-valued function on $[0, T]$. Thus, the GBMP Y is determined by the functions $a(t)$ and $b(t)$. For more details, see [31, 32]. Note that when $a(t) \equiv 0$ and $b(t) = t$, the GBMP is a standard Brownian motion (Wiener process).

In [14, 16], the authors defined the generalized analytic Feynman integral and the generalized analytic Fourier–Feynman transform (GFFT) on the function space $C_{a,b}[0, T]$, and studied their properties and related topics. The function space $C_{a,b}[0, T]$, induced by a GBMP, was introduced by Yeh in [31], and was used extensively in [14, 15, 16, 21, 23]. There have also been several recent attempts to construct financial mathematical theories using this process [22, 24, 26].

In this paper, using the Gaussian processes $\mathcal{Z}_k$ defined on the function space $C_{a,b}[0, T]$ (see Section 4 below), we define a GFFT associated with the process $\mathcal{Z}_k$(the $\mathcal{Z}_k$-GFFT). We then establish the existence of the $\mathcal{Z}_k$-GFFT for certain bounded functionals on $C_{a,b}[0, T]$. We also proceed to establish a translation theorem for the generalized transform.

The steps contained in establishing the results involving $\mathcal{Z}_k$-GFFTs are quite complicated, because the GBMP and the Gaussian process $\mathcal{Z}_k$ used in this paper are subject to drifts and are non-stationary in time. However, by choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the Wiener space $C_0[0, T]$, and so the expected results on $C_0[0, T]$ are immediate corollaries of the results in this paper.

2. Preliminaries

In this section, we briefly list some of the preliminaries from [14, 16, 21] that we will need to establish our results in the next sections.

Let $a(t)$ be an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$ and $a'(t) \in L^2[0, T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$. The GBMP Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$. By [32, Theorem 14.2], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel σ-algebra of $C_{a,b}[0, T]$. We then complete this function space to obtain $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$ where $\mathcal{W}(C_{a,b}[0, T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0, T]$.

We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0,T] \times [0,T]$ is also the GBMP determined by $a(t)$ and $b(t)$. For more detailed studies about this function space $C_{a,b}[0,T]$, see [14, 15, 16, 21, 31].

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\,\cdot\,)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$. If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] := \left\{ v : \int_0^T v^2(s)db(s) < +\infty \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\},$$

where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} := \int_0^T u(t)v(t)dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue-Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$.

Let

$$C'_{a,b}[0,T] := \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) := z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} := \int_0^T Dw_1(t)Dw_2(t)db(t)$$

is a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by equation (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)db(s), \quad t \in [0,T].$$

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty \tag{2.2}$$

from which it follows that

$$\int_0^T |Da(t)|^2 d[b(t)+|a|(t)] = \int_0^T \left|\frac{a'(t)}{b'(t)}\right|^2 d[b(t)+|a|(t)]$$
$$< M\|a'\|_{L^2[0,T]} + M^2 \int_0^T |a'(t)|^2 d|a|(t) < +\infty,$$

where $M = \sup_{t\in[0,T]}(1/b'(t))$. Thus, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w,a)_{C'_{a,b}} := \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)da(t).$$

For each $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we let

$$(w,x)^\sim := \int_0^T Dw(t)dx(t).$$

This integral is called the Paley-Wiener-Zygmund (PWZ) stochastic integral, see [21]. Our definition of the PWZ stochastic integral is different than the definition given in [14, 16, 23]. But we will emphasize that the following fundamental facts are still true:

(i) The PWZ stochastic integral $(w,x)^\sim$ is defined for s-a.e. $x \in C_{a,b}[0,T]$.

(ii) It follows from the definition of the PWZ stochastic integral and from Parseval's equality that if $w \in C'_{a,b}[0,T]$ and $x \in C'_{a,b}[0,T]$, then $(w,x)^\sim$ exists and we have $(w,x)^\sim = (w,x)_{C'_{a,b}}$.

(iii) If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^\sim$ equals the Riemann-Stieltjes integral $\int_0^T z(t)dx(t)$ for μ-a.e. $x \in C_{a,b}[0,T]$.

(iv) The PWZ stochastic integral has the expected linearity properties. That is, for any real number c, $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we have

$$(w,cx)^\sim = c(w,x)^\sim = (cw,x)^\sim.$$

(v) For each $w \in C'_{a,b}[0,T]$, $(w,x)^\sim$ is a Gaussian random variable with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. For all $w_1, w_2 \in C'_{a,b}[0,T]$, we have

$$\int_{C_{a,b}[0,T]} (w_1,x)^\sim (w_2,x)^\sim d\mu(x) = (w_1,w_2)_{C'_{a,b}} + (w_1,a)_{C'_{a,b}}(w_2,a)_{C'_{a,b}}.$$

Thus, if $\{w_1, \dots, w_n\}$ is an orthogonal set in $C'_{a,b}[0,T]$, then the Gaussian random variables $(w_j, x)^\sim$'s are independent.

From the assertion (v) above, we obtain the very important integration formula on the function space $C_{a,b}[0,T]$. Let $\{w_1, \dots, w_n\}$ be an orthogonal set of functions in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$, and let $f : \mathbb{R}^n \to \mathbb{C}$ be a Lebesgue measurable function. Then

$$\begin{aligned} &\int_{C_{a,b}[0,T]} f\big((w_1,x)^\sim, \dots, (w_n,x)^\sim\big) d\mu(x) \\ &= \Big(\prod_{j=1}^n 2\pi \|w_j\|^2_{C'_{a,b}}\Big)^{-n/2} \int_{\mathbb{R}^n} f(u_1,\dots,u_n) \\ &\quad \times \exp\Big\{ -\sum_{j=1}^n \frac{[u_j - (w_j,a)_{C'_{a,b}}]^2}{2\|w_j\|^2_{C'_{a,b}}} \Big\} du_1 \cdots du_n \end{aligned} \tag{2.3}$$

in the sense that if either side of equation (2.3) exists, both sides exist and equality holds.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively. Furthermore, for each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of λ, i.e., $\lambda^{1/2}$ is always chosen to have nonnegative real part, so that $\lambda^{-1/2} = (\lambda^{1/2})^{-1}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$. Then we have the following: for $\lambda \in \mathbb{C}$ with $\lambda = \alpha + i\beta$,

$$\lambda^{-1/2} \equiv (\lambda^{1/2})^{-1} = \sqrt{\tfrac{\sqrt{\alpha^2+\beta^2}+\alpha}{2(\alpha^2+\beta^2)}} - i\,\mathrm{sign}(\beta)\sqrt{\tfrac{\sqrt{\alpha^2+\beta^2}-\alpha}{2(\alpha^2+\beta^2)}}, \tag{2.4}$$

where $\mathrm{sign}(\beta) = 1$ if $\beta \ge 0$ and $\mathrm{sign}(\beta) = -1$ if $\beta < 0$.

The following integration formula is used several times in this paper:

$$\int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\} du = \sqrt{\frac{\pi}{\alpha}} \exp\Big\{\frac{\beta^2}{4\alpha}\Big\} \tag{2.5}$$

for complex numbers α and β with $\mathrm{Re}(\alpha) > 0$.

3. Gaussian process and the commutative algebra $(C^*_{a,b}[0,T], \odot)$

For each $t \in [0,T]$, let $\chi_{[0,t]}$ denote the characteristic function of the interval $[0,t]$ and for $k \in C'_{a,b}[0,T]$ with $Dk = h$ and with $\|k\|^2_{C'_{a,b}} = \int_0^T h^2(t)db(t) > 0$, let $\mathcal{Z}_k(x,t)$ be the PWZ stochastic integral

$$\mathcal{Z}_k(x,t) := (D^{-1}(h\chi_{[0,t]}), x)^\sim. \tag{3.1}$$

Let $\gamma_k(t) := \int_0^t h(u)da(u)$ and let $\beta_k(t) := \int_0^t h^2(u)db(u)$. Then the stochastic process $\mathcal{Z}_k : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ is a Gaussian process with mean function

$$\int_{C_{a,b}[0,T]} \mathcal{Z}_k(x,t) d\mu(x) = \int_0^t h(u)da(u) = \gamma_k(t)$$

and covariance function

$$\int_{C_{a,b}[0,T]} \big(\mathcal{Z}_k(s)-\gamma_k(s)\big)\big(\mathcal{Z}_k(t)-\gamma_k(t)\big)d\mu(x)$$
$$= \int_0^{\min\{s,t\}} h^2(u)db(u) = \beta_k(\min\{s,t\}).$$

In addition, by [32, Theorem 21.1], $\mathcal{Z}_k(\cdot,t)$ is stochastically continuous in t on $[0,T]$. If $h = Dk$ is of bounded variation on $[0,T]$, then, for all $x \in C_{a,b}[0,T]$, $\mathcal{Z}_k(x,t)$ is continuous in t. Of course if $k(t) \equiv b(t)$, then $\mathcal{Z}_b(x,t) = x(t)$. Furthermore, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the Gaussian process (3.1) with $k(t) \equiv t$ is an ordinary Wiener process.

Let $C^*_{a,b}[0,T]$ be the set of functions k in $C'_{a,b}[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, let the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$ be defined by

$$w \odot k := D^{-1}(DwDk), \ \text{ i.e., } D(w\odot k) = DwDk,$$

where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. Then we observe the following algebraic structures:

- $C'_{a,b}[0,T] \times C^*_{a,b}[0,T] \ni (w,k) \mapsto w \odot k \in C'_{a,b}[0,T]$.
- For every $w \in C'_{a,b}[0,T]$ and every $k_1, k_2 \in C^*_{a,b}[0,T]$,
$$(w\odot k_1)\odot k_2 = w \odot (k_1 \odot k_2)$$
and
$$w\odot (k_1 + k_2) = w\odot k_1 + w \odot k_2.$$
- For every $w_1, w_2 \in C'_{a,b}[0,T]$ and every $k \in C^*_{a,b}[0,T]$,
$$(w_1 + w_2)\odot k = w_1 \odot k + w_2 \odot k.$$
- For every $w_1, w_2 \in C'_{a,b}[0,T]$ and every $k \in C^*_{a,b}[0,T]$,
$$(w_1, w_2\odot k)_{C'_{a,b}} = (w_1 \odot k, w_2)_{C'_{a,b}}.$$

We also observe that for $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$,

$$\begin{aligned}\|w\odot k\|_{C'_{a,b}} &= (w\odot k, w\odot k)^{1/2}_{C'_{a,b}}\\ &= \bigg[\int_0^T \{Dw(t)\}^2\{Dk(t)\}^2 db(t)\bigg]^{1/2}\\ &\le \|Dk\|_\infty \bigg[\int_0^T \{Dw(t)\}^2 db(t)\bigg]^{1/2}\\ &= \|Dk\|_\infty \|w\|_{C'_{a,b}},\end{aligned} \tag{3.2}$$

where $\|\cdot\|_\infty$ denotes the essential supremum norm.

Remark 3.1. $(C^*_{a,b}[0,T],\odot)$ is a commutative algebra with the identity $b(\cdot)$.

For $w\in C'_{a,b}[0,T]$ and $k\in C^*_{a,b}[0,T]$, it follows that

$$(w,\mathcal{Z}_k(x,\cdot))^\sim=\int_0^T Dw(t)d\bigg(\int_0^t Dk(s)dx(s)\bigg)=\int_0^T Dw(t)Dk(t)dx(t)=(w\odot k,x)^\sim \tag{3.3}$$

for s-a.e $x\in C_{a,b}[0,T]$. Thus, throughout the remainder of this paper, we require k to be in $C^*_{a,b}[0,T]$ for each process $\mathcal{Z}_k$. This will ensure that the Lebesgue-Stieltjes integrals

$$\|w\odot k\|^2_{C'_{a,b}}=\int_0^T (Dw(t))^2(Dk(t))^2db(t),$$

and

$$(w\odot k,a)_{C'_{a,b}}=\int_0^T Dw(t)Dk(t)Da(t)db(t)=\int_0^T Dw(t)Dk(t)da(t)$$

will exist for all $w\in C'_{a,b}[0,T]$ and $k\in C^*_{a,b}[0,T]$.

4. Generalized analytic Fourier–Feynman transform associated with Gaussian process

We define the $\mathcal{Z}_k$-function space integral (namely, the function space integral associated with the Gaussian process $\mathcal{Z}_k$) for functionals F on $C_{a,b}[0,T]$ as the formula

$$I_k[F]\equiv I_{k,x}[F(\mathcal{Z}_k(x,\cdot))]:=\int_{C_{a,b}[0,T]}F(\mathcal{Z}_k(x,\cdot))d\mu(x)$$

whenever the integral exists.

Definition 4.1. Let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that

$$J_F(\mathcal{Z}_k;\lambda):=I_{k,x}[F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))]$$

exists and is finite for all $\lambda>0$. Let Λ be a domain in $\mathbb{C}_+$ such that $(0,+\infty)\cap\Lambda$ is an open interval of positive real numbers. If there exists a function $J^*_F(\mathcal{Z}_k;\lambda)$ analytic on Λ such that $J^*_F(\mathcal{Z}_k;\lambda)=J_F(\mathcal{Z}_k;\lambda)$ for all $\lambda\in(0,+\infty)\cap\Lambda$, then $J^*_F(\mathcal{Z}_k;\lambda)$ is defined to be the analytic $\mathcal{Z}_k$-function space integral (namely, the analytic function space integral associated with the Gaussian process $\mathcal{Z}_k$) of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda\in\Lambda$ we write

$$I_k^{\mathrm{an}_\lambda}[F]\equiv I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))]\equiv\int^{\mathrm{an}_\lambda}_{C_{a,b}[0,T]}F\big(\mathcal{Z}_k(x,\cdot)\big)d\mu(x):=J^*_F(\mathcal{Z}_k;\lambda). \tag{4.1}$$

Let q be a nonzero real number and let Γ_q be a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that $(0,+\infty)\cap\Gamma_q$ is an open interval of positive real numbers.

Let F be a measurable functional whose analytic $\mathcal{Z}_k$-function space integral exists for all λ in $\mathrm{Int}(\Gamma_q)$, the interior of Γ_q in $\widetilde{\mathbb{C}}_+$. If the following limit exists, we call it the generalized analytic $\mathcal{Z}_k$-Feynman integral (namely, the generalized analytic Feynman integral associated with the process $\mathcal{Z}_k$) of F with parameter q and we write

$$I_k^{\mathrm{anf}_q}[F] \equiv I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] := \lim_{\lambda\to -iq} I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))], \tag{4.2}$$

where λ approaches $-iq$ through values in $\mathrm{Int}(\Gamma_q)$.

Next we state the definition of the $\mathcal{Z}_k$-GFFT on function space.

Definition 4.2. Let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a scale-invariant measurable functional on $C_{a,b}[0,T]$. Let q be a nonzero real number, and let Γ_q be a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that for all $\lambda \in \mathrm{Int}(\Gamma_q)$ and $y \in C_{a,b}[0,T]$, the following analytic $\mathcal{Z}_k$-function space integral

$$T_{\lambda,k}(F)(y) := I_{k,x}^{\mathrm{an}_\lambda}[F(y+\mathcal{Z}_k(x,\cdot))]$$

exists. For $p \in (1,2]$, we define the L_p analytic $\mathcal{Z}_k$-GFFT (namely, the GFFT associated with the process $\mathcal{Z}_k$), $T_{q,k}^{(p)}(F)$ of F, by the formula,

$$T_{q,k}^{(p)}(F)(y) := \underset{\substack{\lambda\to -iq \\ \lambda\in \mathrm{Int}(\Gamma_q)}}{\mathrm{l.i.m.}}\ T_{\lambda,k}(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in \mathrm{Int}(\Gamma_q)}} \int_{C_{a,b}[0,T]} \left|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\right|^{p'} d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $\mathcal{Z}_k$-GFFT, $T_{q,k}^{(1)}(F)$ of F, by the formula

$$T_{q,k}^{(1)}(F)(y) := \lim_{\substack{\lambda\to -iq \\ \lambda\in \mathrm{Int}(\Gamma_q)}} T_{\lambda,k}(F)(y) = I_{k,x}^{\mathrm{anf}_q}[F(y+\mathcal{Z}_k(x,\cdot))] \tag{4.3}$$

if it exists.

We note that for $1 \le p \le 2$, $T_{q,k}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q,k}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,k}^{(p)}(G)$ exists and $T_{q,k}^{(p)}(G) \approx T_{q,k}^{(p)}(F)$. Moreover, from equations (4.1), (4.2) and (4.3), we have

$$I_k^{\mathrm{anf}_q}[F] \equiv I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] = T_{q,k}^{(1)}(F)(0) \tag{4.4}$$

if both side exist.

Remark 4.3. Note that if $k \equiv b$ on $[0,T]$, then the generalized analytic $\mathcal{Z}_b$-Feynman integral, $I_b^{\mathrm{anf}_q}[F]$, and the L_p analytic $\mathcal{Z}_b$-GFFT, $T_{q,b}^{(p)}(F)$, agree with the previous definitions of the generalized analytic Feynman integral and the L_p analytic GFFT respectively [14, 16].

5. Bounded cylinder functionals

A functional F is called a cylinder functional on $C_{a,b}[0,T]$ if there exists a finite subset $\{w_1,\dots,w_m\}$ of $C'_{a,b}[0,T]$ such that

$$F(x)=\phi((w_1,x)^\sim,\dots,(w_m,x)^\sim),\quad x\in C_{a,b}[0,T], \tag{5.1}$$

where ϕ is a $\mathbb{C}$-valued Lebesgue measurable function on $\mathbb{R}^m$. It is easy to show that for given cylinder functional F of the form (5.1) there exists an orthogonal set $\{e_1,\dots,e_n\}$ of functions in $C'_{a,b}[0,T]\setminus\{0\}$ such that F is expressed as

$$F(x)=f((e_1,x)^\sim,\dots,(e_n,x)^\sim),\quad x\in C_{a,b}[0,T], \tag{5.2}$$

where f is a $\mathbb{C}$-valued Lebesgue measurable function on $\mathbb{R}^n$. Thus we lose no generality in assuming that every cylinder functional on $C_{a,b}[0,T]$ is of the form (5.2).

For $k\in C^*_{a,b}[0,T]$ with $\|k\|_{C'_{a,b}}>0$, let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) above and let F be given by equation (5.2). Then by equation (3.3),

$$\begin{aligned}F(\mathcal{Z}_k(x,\cdot))&=f((e_1,\mathcal{Z}_k(x,\cdot))^\sim,\dots,(e_n,\mathcal{Z}_k(x,\cdot))^\sim)\\&=f((e_1\odot k,x)^\sim,\dots,(e_n\odot k,x)^\sim).\end{aligned}$$

Even though the subset $\mathcal{A}=\{e_1,\dots,e_n\}$ of $C'_{a,b}[0,T]$ is orthogonal, the subset

$$\mathcal{A}\odot k\equiv\{e\odot k: e\in\mathcal{A}\}$$

of $C'_{a,b}[0,T]$ need not be orthogonal.

Given an orthogonal set $\mathcal{A}=\{e_1,\dots,e_n\}$ of functions in $C'_{a,b}[0,T]\setminus\{0\}$, let $\mathcal{O}^*(\mathcal{A})$ be the class of all nonzero elements $k\in C^*_{a,b}[0,T]$ such that $\mathcal{A}\odot k$ is orthogonal in $C'_{a,b}[0,T]$. Since $\dim C'_{a,b}[0,T]=\infty$, infinitely many elements k exist in $\mathcal{O}^*(\mathcal{A})$.

Example 5.1. For every $\rho\in\mathbb{R}\setminus\{0\}$, $\rho b(\cdot)$ is an element of $\mathcal{O}^*(\mathcal{A})$ for any orthogonal set $\mathcal{A}$ in $C'_{a,b}[0,T]$.

Example 5.2. Given any orthogonal set $\mathcal{A}=\{e_1,\dots,e_n\}$ of functions in $C'_{a,b}[0,T]$, each of whose elements is in $C^*_{a,b}[0,T]\setminus\{0\}$, let $L(S)$ be the subspace of $C'_{a,b}[0,T]$ which is spanned by $S=\{e_i\odot e_j:1\le i<j\le n\}$, and let $L(S)^\perp$ be the orthogonal complement of $L(S)$. Let

$$\mathcal{P}^*(\mathcal{A}):=\{k\in C^*_{a,b}[0,T]:k\odot k\in L(S)^\perp \text{ and } \|k\|_{C'_{a,b}}>0\}.$$

Since $\dim L(S)$ is finite, and $C^*_{a,b}[0,T]$ is dense in $C'_{a,b}[0,T]$, $\dim(L(S)^\perp\cap C^*_{a,b}[0,T])=\infty$ and so $\mathcal{P}^*(\mathcal{A})$ has infinitely many elements.

Let k be an element of $\mathcal{P}^*(\mathcal{A})$. It is easy to show that $\|e_j\odot k\|_{C'_{a,b}}>0$ for all $j\in\{1,\dots,n\}$. From the definition of the $\mathcal{P}^*(\mathcal{A})$, we see that for $i,j\in\{1,\dots,n\}$ with $i\neq j$,

$$\begin{aligned}(e_i\odot k,e_j\odot k)_{C'_{a,b}}&=\int_0^T De_i(t)De_j(t)(Dk)^2(t)db(t)\\&=(e_i\odot e_j,k\odot k)_{C'_{a,b}}=0.\end{aligned}$$

From these, we see that $\mathcal{A} \odot k$ is an orthogonal set in $C'_{a,b}[0,T]$ for any k in $\mathcal{P}^*(\mathcal{A})$, i.e., $\mathcal{P}^*(\mathcal{A}) \subset \mathcal{O}^*(\mathcal{A})$.

We clearly observe that for orthogonal sets $\mathcal{A}_1$ and $\mathcal{A}_2$ in $C'_{a,b}[0,T]$ with $\mathcal{A}_1 \subset \mathcal{A}_2$, $\mathcal{O}^*(\mathcal{A}_2) \subset \mathcal{O}^*(\mathcal{A}_1)$.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of $\mathbb{C}$-valued Borel measures on $\mathcal{B}(\mathbb{R}^n)$. It is well known that a $\mathbb{C}$-valued Borel measure ν necessarily has a finite total variation $\|\nu\|$, and $\mathcal{M}(\mathbb{R}^n)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

For $\nu \in \mathcal{M}(\mathbb{R}^n)$, the Fourier transform $\widehat{\nu}$ of ν is a $\mathbb{C}$-valued function defined on $\mathbb{R}^n$ by the formula

$$\widehat{\nu}(\vec{u}) := \int_{\mathbb{R}^n} \exp\bigg\{ i \sum_{j=1}^{n} u_j v_j \bigg\} d\nu(\vec{v}), \tag{5.3}$$

where $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$ are in $\mathbb{R}^n$.

Let $\mathcal{A} = \{e_1, \dots, e_n\}$ be an orthogonal set of functions in $C'_{a,b}[0,T] \setminus \{0\}$. Define the functional $F : C_{a,b}[0,T] \to \mathbb{C}$ by

$$F(x) = \widehat{\nu}((e_1, x)^{\sim}, \dots, (e_n, x)^{\sim}), \quad x \in C_{a,b}[0,T], \tag{5.4}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where $\widehat{\nu}$ is the Fourier transform of ν in $\mathcal{M}(\mathbb{R}^n)$. Then F is a bounded cylinder functional because $|\widehat{\nu}(\vec{u})| \le \|\nu\| < +\infty$.

Given an orthogonal subset $\mathcal{A} = \{e_1, \dots, e_n\}$ of $C'_{a,b}[0,T] \setminus \{0\}$, let $\widehat{\mathfrak{T}}_{\mathcal{A}}$ be the space of all functionals F on $C_{a,b}[0,T]$ having the form (5.4). Note that $F \in \widehat{\mathfrak{T}}_{\mathcal{A}}$ implies that F is scale-invariant measurable on $C_{a,b}[0,T]$. Throughout the rest of this paper, we fix the orthogonal set $\mathcal{A}$.

Lemma 5.3. *Let $\mathcal{A} = \{e_1, \dots, e_n\}$ be an orthogonal subset of $C'_{a,b}[0,T] \setminus \{0\}$. Then, for every $k \in \mathcal{O}^*(\mathcal{A})$ and all $\zeta \in \mathbb{C}_+$, the function space integral*

$$K \equiv I_{k,x}\bigg[\exp\bigg\{ i\zeta \sum_{j=1}^{n} (e_j, \mathcal{Z}_k(x,\cdot))^{\sim} v_j \bigg\} \bigg]$$

exists and is given by the formula

$$K = \exp\bigg\{ -\frac{\zeta^2}{2} \sum_{j=1}^{n} \|e_j \odot k\|^2_{C'_{a,b}} v_j^2 + i\zeta \sum_{j=1}^{n} (e_j \odot k, a)_{C'_{a,b}} v_j \bigg\}. \tag{5.5}$$

Proof. Using (3.3), (2.3), Fubini's theorem, and (2.5), it follows immediately that equation (5.5) holds for all $\zeta \in \mathbb{C}_+$. $\square$

For notational convenience we use the following notation throughout this paper:

(5.6)

$\mathcal{W}_{e_1,\dots,e_n}(\lambda, k; v_1, \dots, v_n)$

$$\equiv \mathcal{W}_{\vec{e}}(\lambda, k; \vec{v}) = \exp\bigg\{ -\frac{1}{2\lambda}\sum_{j=1}^{n} \|e_j \odot k\|^2_{C'_{a,b}} v_j^2 + i\lambda^{-1/2}\sum_{j=1}^{n}(e_j \odot k, a)_{C'_{a,b}} v_j \bigg\}$$

for an orthogonal subset $\mathcal{A} = \{e_1, \dots, e_n\}$ of $C'_{a,b}[0,T]\backslash\{0\}$, $k \in \mathcal{O}^*(\mathcal{A})$, $\lambda \in \widetilde{\mathbb{C}}_+$ and $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$.

In next theorem, we establish the existence of the analytic $\mathcal{Z}_k$-function space integral $T_{\lambda,k}(F)(y) = I^{\mathrm{an}_\lambda}_{k,x}[F(y + \mathcal{Z}_k(x,\cdot))]$ of the functionals F in $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

Theorem 5.4. *Let $F \in \widehat{\mathfrak{T}}_{\mathcal{A}}$ be given by equation* (5.4) *and let k be an element of $\mathcal{O}^*(\mathcal{A})$. Then for all $\lambda \in \mathbb{C}_+$, $T_{\lambda,k}(F)$ exists and is given by the formula*

$$T_{\lambda,k}(F)(y) = \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \bigg\}\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})\,d\nu(\vec{v}) \tag{5.7}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where $\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})$ is given by equation (5.6) *above.*

Proof. By (5.4), (5.3), Fubini's theorem, (5.5) with ζ replaced with $\lambda^{-1/2}$, and (5.6), we have that for all $\lambda > 0$ and s-a.e. $y \in C_{a,b}[0,T]$,

$$\begin{aligned} J_{F(y+\cdot)}(\mathcal{Z}_k;\lambda) &\equiv I_{k,x}[F(y + \lambda^{-1/2}\mathcal{Z}_k(x,\cdot))] \\ &= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \bigg\}\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})\,d\nu(\vec{v}). \end{aligned}$$

Now let

$$J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda) := \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \bigg\}\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})\,d\nu(\vec{v})$$

for $\lambda \in \mathbb{C}_+$. Then $J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda) = J_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)$ for all $\lambda > 0$. We will use the Morera theorem to show that $J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)$ is analytic on $\mathbb{C}_+$ as a function of λ. Let $\{\lambda_l\}_{l=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ such that $\lambda_l \to \lambda$. Then $\lambda_l^{-1/2} \to \lambda^{-1/2}$ and $\mathrm{Re}(\lambda_l) > 0$ for all $l \in \mathbb{N}$. Thus it follows that for each $l \in \mathbb{N}$,

$$\begin{aligned} &\bigg|\exp\bigg\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \bigg\}\mathcal{W}_{\vec{e}}(\lambda_l, k; \vec{v})\bigg| \\ &= \big|\mathcal{W}_{\vec{e}}(\lambda_l, k; \vec{v})\big| \\ &= \bigg|\exp\bigg\{ -\frac{1}{2\lambda_l}\sum_{j=1}^{n} \|e_j \odot k\|^2_{C'_{a,b}} v_j^2 + i\lambda_l^{-1/2}\sum_{j=1}^{n}(e_j \odot k, a)_{C'_{a,b}} v_j \bigg\}\bigg| \\ &= \exp\bigg\{ -\frac{\mathrm{Re}(\lambda_l)}{2|\lambda_l|^2}\sum_{j=1}^{n} \|e_j \odot k\|^2_{C'_{a,b}} v_j^2 - \mathrm{Im}(\lambda_l^{-1/2})\sum_{j=1}^{n}(e_j \odot k, a)_{C'_{a,b}} v_j \bigg\}\bigg| \\ &= \exp\bigg\{ -\frac{1}{2}\sum_{j=1}^{n}\bigg[\frac{\sqrt{\mathrm{Re}(\lambda_l)}\|e_j \odot k\|_{C'_{a,b}}}{|\lambda_l|}v_j + \frac{|\lambda_l|\mathrm{Im}(\lambda_l^{-1/2})(e_j \odot k, a)_{C'_{a,b}}}{\sqrt{\mathrm{Re}(\lambda_l)}\|e_j \odot k\|_{C'_{a,b}}}\bigg]^2 \end{aligned}$$

$$+\frac{1}{2}\sum_{j=1}^{n}\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2(e_j\odot k,a)^2_{C'_{a,b}}}{\mathrm{Re}(\lambda_l)\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}$$

$$\leq \exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}.$$

Since $\nu\in\mathcal{M}(\mathbb{R}^n)$, we see that

$$\bigg|\int_{\mathbb{R}^n}\exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}d\nu(\vec{v})\bigg|$$

$$\leq \int_{\mathbb{R}^n}\exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}d|\nu|(\vec{v})$$

$$=\exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}\|\nu\|<+\infty$$

for each $l\in\mathbb{N}$. Furthermore we have that

$$\lim_{l\to\infty}\int_{\mathbb{R}^n}\exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}d|\nu|(\vec{v})$$

$$=\lim_{l\to\infty}\exp\bigg\{\frac{|\lambda_l|^2(\mathrm{Im}(\lambda_l^{-1/2}))^2}{2\mathrm{Re}(\lambda_l)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}|\nu|(\mathbb{R}^n)$$

$$=\exp\bigg\{\frac{|\lambda|^2(\mathrm{Im}(\lambda^{-1/2}))^2}{2\mathrm{Re}(\lambda)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}|\nu|(\mathbb{R}^n)$$

$$=\int_{\mathbb{R}^n}\exp\bigg\{\frac{|\lambda|^2(\mathrm{Im}(\lambda^{-1/2}))^2}{2\mathrm{Re}(\lambda)}\sum_{j=1}^{n}\frac{(e_j\odot k,a)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}d|\nu|(\vec{v}).$$

Thus, by Theorem 4.17 in [27, p. 92], $J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)$ is continuous on $\mathbb{C}_+$. Since

$$g(\lambda)\equiv\exp\bigg\{i\sum_{j=1}^{n}(e_j,y)^{\sim}v_j\bigg\}\mathcal{W}_{\vec{e}}(\lambda,k;\vec{v})$$

is analytic on $\mathbb{C}_+$, applying Fubini's theorem, we have

$$\int_{\triangle}J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)d\lambda=\int_{\mathbb{R}^n}\int_{\triangle}g(\lambda)d\lambda d\nu(\vec{v})=0$$

for all rectifiable simple closed curve $\triangle$ lying in $\mathbb{C}_+$. Thus by the Morera theorem, $J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)$ is analytic on $\mathbb{C}_+$. Therefore the analytic function space integral

$$J^*_{F(y+\cdot)}(\mathcal{Z}_k;\lambda)=I^{\mathrm{an}_\lambda}_{k,x}[F(y+\mathcal{Z}_k(x,\cdot))]\equiv T_{\lambda,k}(F)(y)$$

exists on $\mathbb{C}_+$ and is given by equation (5.7) for all $\lambda \in \mathbb{C}_+$. $\square$

6. $\mathcal{Z}_k$-generalized Fourier–Feynman transforms of bounded cylinder functionals

The following observation will be very useful in the development of our results for the $\mathcal{Z}_k$-GFFT of functionals F in $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

If $a(t) \equiv 0$ on $[0, T]$, then for all functionals F given by equation (5.4), the L_1 analytic $\mathcal{Z}_k$-GFFT $T_{q,k}^{(1)}(F)$ will always exist for all real $q \neq 0$ and be given by the formula

$$\begin{aligned}T_{q,k}^{(1)}(F)(y) &= \int_{\mathbb{R}^n} \exp\bigg\{ i \sum_{j=1}^n (e_j, y)^\sim v_j \bigg\} \mathcal{W}_{\vec{e}}(-iq, k; \vec{v}) d\nu(\vec{v}) \\ &= \int_{\mathbb{R}^n} \exp\bigg\{ i \sum_{j=1}^n (e_j, y)^\sim v_j - \frac{i}{2q} \sum_{j=1}^n \|e_j \odot k\|_{C'_{a,b}}^2 v_j^2 \bigg\} d\nu(\vec{v}).\end{aligned}$$

However for $a(t)$ as in Section 2, and proceeding formally using equations (5.4) and (5.7), we see that $T_{q,k}^{(1)}(F)(y)$ will be given by equation (6.5) below if it exists. But the integral on the right-hand side of (6.5) might not exist if the real part of

$$\mathrm{Log}\mathcal{W}_{\vec{e}}(-iq, k; \vec{v}) = \bigg\{ -\frac{i}{2q} \sum_{j=1}^n \|e_j \odot k\|_{C'_{a,b}}^2 v_j^2 + i(-iq)^{-1/2} \sum_{j=1}^n (e_j \odot k, a)_{C'_{a,b}} v_j \bigg\}$$

is positive. However, by the Cauchy-Schwartz inequality and (3.2),

$$\big|\mathcal{W}_{\vec{e}}(-iq, k; \vec{v})\big| \leq \exp\bigg\{ \frac{\|a\|_{C'_{a,b}} \|Dk\|_\infty}{\sqrt{|2q|}} \sum_{j=1}^n \|e_j\|_{C'_{a,b}} |v_j| \bigg\},$$

and so the L_1 analytic $\mathcal{Z}_k$-GFFT $T_q^{(1)}(F)$ of F will certainly exist provided the associated measure ν of F satisfies the condition

$$\int_{\mathbb{R}^n} \exp\bigg\{ \frac{\|a\|_{C'_{a,b}} \|Dk\|_\infty}{\sqrt{|2q|}} \sum_{j=1}^n \|e_j\|_{C'_{a,b}} |v_j| \bigg\} d|\nu|(\vec{v}) < +\infty. \tag{6.1}$$

Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the classical Wiener space $C_0[0, T]$ and $(e_j \odot k, a)_{C'_{a,b}} = 0$ for all $j = 1, \ldots, n$. Hence for all $\lambda \in \widetilde{\mathbb{C}}_+$,

$$\begin{aligned}\big|\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})\big| &= \bigg| \exp\bigg\{ -\frac{1}{2\lambda} \sum_{j=1}^n \|e_j \odot k\|_{C'_{a,b}}^2 v_j^2 \bigg\} \bigg| \\ &= \exp\bigg\{ -\frac{\mathrm{Re}(\lambda)}{2|\lambda|^2} \sum_{j=1}^n \|e_j \odot k\|_{C'_{a,b}}^2 v_j^2 \bigg\} \leq 1.\end{aligned}$$

Given a positive real number q_0, let

$$\Gamma_{q_0} = \{\lambda \in \widetilde{\mathbb{C}}_+ : |\mathrm{Im}(\lambda^{-1/2})| < (2q_0)^{-1/2}\} \tag{6.2}$$

and let $\Upsilon_{q_0} = \{\lambda \in \widetilde{\mathbb{C}}_+ : |\lambda| > q_0\}$. Then we can observe the following:

(i) The set Γ_{q_0} is an unbounded open set in $\widetilde{\mathbb{C}}_+$, the topological subspace of $\mathbb{C}$.

(ii) For any real q with $|q| > q_0$, $-iq$ is an element of Γ_{q_0}. In fact, we have the equality $(-iq)^{-1/2} = 1/\sqrt{2|q|} + i\mathrm{sign}(q)/\sqrt{2|q|}$ by equation (2.4).

(iii) For any real q with $|q| > q_0$, Γ_{q_0} is a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ so that $(0, +\infty) \subset \Gamma_{q_0}$. More precisely, we observe $-iq \in \Upsilon_{q_0} \subset \Gamma_{q_0}$.

(iv) For all $\lambda \in \Gamma_{q_0}$, we have the inequality

$$\big|\mathcal{W}_{\vec{e}}(\lambda, k; \vec{v})\big| \le \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{|2q|}} \sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\}. \tag{6.3}$$

Given a positive real q_0 and an element $k \in \mathcal{O}^*(\mathcal{A})$, we define a subclass $\widehat{\mathfrak{T}}_{\mathcal{A}}^{q_0,k}$ of $\widehat{\mathfrak{T}}_{\mathcal{A}}$ by $F \in \widehat{\mathfrak{T}}_{\mathcal{A}}^{q_0,k}$ if and only if the associated measure ν of F by (5.3) satisfies the condition (6.1) with q replaced with q_0.

We will emphasize the fact that $\cap_{q>0}\widehat{\mathfrak{T}}_{\mathcal{A}}^{q,k}$ is not empty.

Given $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ and $\vec{\sigma^2} = (\sigma_1^2, \ldots, \sigma_n^2) \in \mathbb{R}^n$ with $\sigma_j^2 > 0$, $j = 1, \ldots, n$, let $\nu_{\vec{m},\vec{\sigma^2}}$ be the Gaussian measure defined by

$$\nu_{\vec{m},\vec{\sigma^2}}(B) = \bigg(\prod_{j=1}^n 2\pi\sigma_j^2\bigg)^{-\frac{1}{2}} \int_B \exp\bigg\{-\sum_{j=1}^n \frac{(v_j - m_j)^2}{2\sigma_j^2}\bigg\} d\vec{v},\ B \in \mathcal{B}(\mathbb{R}^n). \tag{6.4}$$

Then $\nu_{\vec{m},\vec{\sigma^2}} \in \mathcal{M}(\mathbb{R}^n)$ and

$$\widehat{\nu_{\vec{m},\vec{\sigma^2}}}(\vec{u}) = \exp\bigg\{-\frac{1}{2}\sum_{j=1}^n \sigma_j^2 u_j^2 + i\sum_{j=1}^n m_j u_j\bigg\}.$$

Using equation (6.4), Fubini's theorem and equation (2.5), we see that for any nonzero real q,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{|2q|}} \sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\} d|\nu_{\vec{m},\vec{\sigma^2}}|(\vec{v}) \\
&= \prod_{j=1}^n \bigg[(2\pi\sigma_j^2)^{-1/2}\exp\bigg\{-\frac{m_j^2}{2\sigma_j^2}\bigg\} \\
&\qquad \times \int_{-\infty}^0 \exp\bigg\{-\frac{v_j^2}{2\sigma_j^2} + \bigg(\frac{m_j}{\sigma_j^2} - \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty\|e_j\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_j\bigg\} dv_j \\
&\qquad + (2\pi\sigma_j^2)^{-1/2}\exp\bigg\{-\frac{m_j^2}{2\sigma_j^2}\bigg\}
\end{aligned}$$

$$\times \int_0^{+\infty} \exp\bigg\{ -\frac{v_j^2}{2\sigma_j^2} + \bigg(\frac{m_j}{\sigma_j^2} + \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty \|e_j\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_j\bigg\}dv_j\bigg]$$

$$< \prod_{j=1}^n \bigg[(2\pi\sigma_j^2)^{-1/2}\exp\bigg\{-\frac{m_j^2}{2\sigma_j^2}\bigg\}$$

$$\times \int_{-\infty}^{+\infty} \exp\bigg\{ -\frac{v_j^2}{2\sigma_j^2} + \bigg(\frac{m_j}{\sigma_j^2} - \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty \|e_j\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_j\bigg\}dv_j$$

$$+ (2\pi\sigma_j^2)^{-1/2}\exp\bigg\{-\frac{m_j^2}{2\sigma_j^2}\bigg\}$$

$$\times \int_{-\infty}^{+\infty} \exp\bigg\{ -\frac{v_j^2}{2\sigma_j^2} + \bigg(\frac{m_j}{\sigma_j^2} + \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty \|e_j\|_{C'_{a,b}}}{\sqrt{|2q|}}\bigg)v_j\bigg\}dv_j\bigg]$$

$$< +\infty.$$

Theorem 6.1. *Given* $q_0 > 0$ *and* $k \in \mathcal{O}^*(\mathcal{A})$*, let* F *be an element of* $\widehat{\mathfrak{T}}_{\mathcal{A}}^{q_0,k}$*. Then for all real* q *with* $|q| > q_0$*, the* L_1 *analytic* $\mathcal{Z}_k$*-GFFT of* F*,* $T_{q,k}^{(1)}(F)$ *exists and is given by the formula*

$$T_{q,k}^{(1)}(F)(y) = \int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n (e_j, y)^\sim v_j\bigg\}\mathcal{W}_{\vec{e}}(-iq, k; \vec{v})d\nu(\vec{v}) \tag{6.5}$$

for s-a.e. $y \in C_{a,b}[0,T]$*, where* $\mathcal{W}_{\vec{e}}(-iq, k; \vec{v})$ *is given by equation* (5.6).

Proof. Let Γ_{q_0} be given by equation (6.2). It was shown in the proof of Theorem 5.4 that $T_{\lambda,k}(F)(y)$ is an analytic function of λ throughout $\mathbb{C}_+$. Thus, $T_{q,k}^{(1)}(F)(y)$ is analytic on the domain Γ_{q_0}.

Let $\{\lambda_l\}_{l=1}^\infty$ be any sequence in $\mathbb{C}_+$ which converges to $-iq$ through $\mathbb{C}_+$. Then, clearly, $\mathcal{W}_{\vec{e}}(\lambda_l, k; \vec{v})$ converges to $\mathcal{W}_{\vec{e}}(-iq, k; \vec{v})$. By Theorem 5.4, we know that the integral

$$T_{\lambda_l,k}(F)(y) = \int_{\mathbb{R}^n} \exp\bigg\{i\sum_{j=1}^n (e_j, y)^\sim v_j\bigg\}\mathcal{W}_{\vec{e}}(\lambda_l, k; \vec{v})d\nu(\vec{v})$$

exists for all $l \in \mathbb{N}$. Since $|\mathrm{Arg}(\lambda_l^{-1/2})| < \pi/4$ for every $l \in \mathbb{N}$ and $\lambda_l^{-1/2} = \mathrm{Re}(\lambda_l^{-1/2}) + i\mathrm{Im}(\lambda_l^{-1/2}) \to (-iq)^{-1/2} = 1/\sqrt{|2q|} + i\mathrm{sign}(q)/\sqrt{|2q|}$, we see that $\mathrm{Re}(\lambda_l^{-1/2}) > |\mathrm{Im}(\lambda_l^{-1/2})|$ for every $l \in \mathbb{N}$, and so there exists a sufficiently large $L \in \mathbb{N}$ such that $|\mathrm{Im}(\lambda_l^{-1/2})| < 1/\sqrt{|2q_0|}$, i.e., $\lambda_l \in \Gamma_{q_0}$ for every $l \geq L$. Thus for each $l \geq L$,

$$|\mathcal{W}_{\vec{e}}(\lambda_l, k; \vec{v})|$$

$$= \bigg|\exp\bigg\{-\frac{1}{2}\Big([\mathrm{Re}(\lambda_l^{-1/2})]^2 - [\mathrm{Im}(\lambda_l^{-1/2})]^2$$

$$+ i\mathrm{Re}(\lambda_l^{-1/2})\mathrm{Im}(\lambda_l^{-1/2})\Big)\sum_{j=1}^n \|e_j \odot k\|_{C'_{a,b}}^2 v_j^2$$

$$+ i\Big(\mathrm{Re}(\lambda_l^{-1/2}) + i\mathrm{Im}(\lambda_l^{-1/2})\Big)\sum_{j=1}^n (e_j \odot k, a)_{C'_{a,b}} v_j\Big\}\Big|$$

$$\le \exp\bigg\{ -\mathrm{Im}(\lambda_l^{-1/2})\sum_{j=1}^n (e_j \odot k, a)_{C'_{a,b}} v_j\bigg\}$$

$$\le \exp\bigg\{ \big|\mathrm{Im}(\lambda_l^{-1/2})\big|\sum_{j=1}^n \big|(e_j \odot k, a)_{C'_{a,b}} v_j\big|\bigg\}$$

$$< \exp\bigg\{ \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{|2q|}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\}$$

and so, by condition (6.1) with q replaced with q_0,

$$\begin{aligned}\big|T_{\lambda_l,k}(F)(y)\big| &\le \int_{\mathbb{R}^n}\big|\mathcal{W}_{\vec{e}}(\lambda_l,k;\vec{v})\big|d|\nu|(\vec{v})\\ &< \int_{\mathbb{R}^n}\exp\bigg\{ \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{|2q|}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\}d|\nu|(\vec{v})\\ &< \int_{\mathbb{R}^n}\exp\bigg\{ \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\}d|\nu|(\vec{v}) < +\infty.\end{aligned}$$

Also, by condition (6.1) with q replaced with q_0, we have

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^n}\exp\bigg\{ i\sum_{j=1}^n (e_j,y)^\sim v_j\bigg\}\mathcal{W}_{\vec{e}}(-iq,k;\vec{v})d\nu(\vec{v})\bigg|\\ &\le \int_{\mathbb{R}^n}\big|\mathcal{W}_{\vec{e}}(-iq,k;\vec{v})\big|d|\nu|(\vec{v})\\ &< \int_{\mathbb{R}^n}\exp\bigg\{ \frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\}d|\nu|(\vec{v}) < +\infty.\end{aligned}$$

Therefore the equation (6.5) follows from (4.3), (5.7) and the dominated convergence theorem. □

The following corollary follows from equations (4.4) and (6.5).

Corollary 6.2. *Let q_0, k, and F be as in Theorem 6.1. Then for all real q with $|q| > q_0$, the generalized analytic $\mathcal{Z}_k$-Feynman integral of F, $I_k^{\mathrm{anf}_q}[F]$ exists and is given by the formula*

$$I_k^{\mathrm{anf}_q}[F] = \int_{\mathbb{R}^n}\mathcal{W}_{\vec{e}}(-iq,k;\vec{v})d\nu(\vec{v}),$$

where $\mathcal{W}_{\vec{e}}(-iq,k;\vec{v})$ is given by equation (5.6).

Theorem 6.3. *Let q_0, k, and F be as in Theorem 6.1. Then for all $p \in (1,2]$ and all real q with $|q| > q_0$, the L_p analytic $\mathcal{Z}_k$-GFFT of F, $T_{q,k}^{(p)}(F)$ exists and is given by the right hand side of equation* (6.5).

Proof. It was shown in the proof of Theorem 5.4 that $T_{\lambda,k}(F)(y)$ given by equation (5.7) is an analytic function of λ throughout $\mathbb{C}_+$. In view of the definition of the L_p analytic $\mathcal{Z}_k$-GFFT, it suffices to show that for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma_{q_0}}} \int_{C_{a,b}[0,T]} \big|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\big|^{p'} d\mu(y) = 0.$$

Fixing $p \in (1,2]$ and using inequalities (6.3) and (6.1) with q replaced with q_0 respectively, we obtain that for all $\rho > 0$ and all $\lambda \in \Gamma_{q_0}$,

$$\begin{aligned}
&\big|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\big|^{p'}\\
&\le \bigg(\int_{\mathbb{R}^n} \bigg|\exp\bigg\{i\rho\sum_{j=1}^n (e_j,y)^\sim\bigg\}\bigg| \Big\{\big|\mathcal{W}_{\vec{e}}(\lambda,k;\vec{v})\big| + \big|\mathcal{W}_{\vec{e}}(-iq,k;\vec{v})\big|\Big\} d\nu(\vec{v})\bigg)^{p'}\\
&\le \bigg(2\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\bigg\} d|\nu|(\vec{v})\bigg)^{p'} < +\infty.
\end{aligned}$$

Hence by the dominated convergence theorem, we see that for all $p \in (1,2]$ and all $\rho > 0$,

$$\begin{aligned}
&\lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma_{q_0}}} \int_{C_{a,b}[0,T]} \big|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\big|^{p'} d\mu(y)\\
&= \int_{C_{a,b}[0,T]} \bigg|\int_{\mathbb{R}^n} \exp\bigg\{i\rho\sum_{j=1}^n (e_j,y)^\sim\bigg\}\\
&\qquad \times \lim_{\substack{\lambda\to -iq\\ \lambda\in\Gamma_{q_0}}} \Big\{\mathcal{W}_{\vec{e}}(\lambda,k;\vec{v}) - \mathcal{W}_{\vec{e}}(-iq,k;\vec{v})\Big\} d\nu(\vec{v})\bigg|^{p'} d\mu(y)\\
&= 0
\end{aligned}$$

and the theorem is established. □

Remark 6.4. Let q_0, k, and F be as in Theorem 6.1. For a real number q with $|q| > q_0$, define a set function $\nu_{q,k} : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\nu_{q,k}(B) := \int_B \mathcal{W}_{\vec{e}}(-iq,k;w)d\nu(w), \qquad B \in \mathcal{B}(\mathbb{R}^n),$$

where ν and F are related by equation (5.4). Then it is obvious that $\nu_{q,k}$ belongs to $\mathcal{M}(\mathbb{R}^n)$. In this case, by Theorems 6.1 and 6.3, and equation (6.5), we see that for all $p \in [1,2]$, the L_p analytic $\mathcal{Z}_k$-GFFT of F, $T_{q,k}^{(p)}(F)$, can be expressed as

$$T_{q,k}^{(p)}(F)(y) = \int_{\mathbb{R}^n} \exp\left\{ i\sum_{j=1}^{n}(e_j, y)^{\sim} v_j \right\} d\nu_{q,k}(\vec{v})$$

for s-a.e. $y \in C_{a,b}[0,T]$. Hence $T_{q,k}^{(p)}(F)$ belongs to $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

7. Translation theorem

It is well known that there is no quasi-invariant measure on infinite dimensional linear spaces (see for instance [30]). Thus, there is no quasi-invariant probability measure on the function space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$. Based on such circumstance, numerous constructions and applications of the translation theorem (Cameron-Martin theorem) for integrals on infinite-dimensional spaces have been studied in various research fields in Mathematics and Physics. The most of the results in the literature are concentrated on Wiener space.

Cameron-Martin translation theorem on classical Wiener space was introduced in [4, 5]. On the other hand, Cameron and Storvick [6, 7] presented a translation theorem for the analytic Feynman integral of functionals on the Wiener space $C_0[0,T]$ and Chang and Chung [15] derived a translation theorem for function space integral of functionals on $C_{a,b}[0,T]$. In this section, we will present a $\mathcal{Z}_k$-GFFT version of the translation theorem for functionals in $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

Given $q_0 > 0$ and $k \in \mathcal{O}^*(\mathcal{A})$, let F be an element of $\widehat{\mathfrak{T}}_{\mathcal{A}}^{q_0,k}$, and for $\theta \in C'_{a,b}[0,T]$ and $q \in \mathbb{R} \setminus \{0\}$, let

$$F^{q\theta}(x) := F(x)\exp\{-iq(\theta, x)^{\sim}\}. \tag{7.1}$$

Also, given the orthogonal set $\mathcal{A} = \{e_1, \ldots, e_n\}$ and $\theta \in C'_{a,b}[0,T]$, let

$$g_j = e_j/\|e_j\|_{C'_{a,b}}, \qquad j = 1, \ldots, n,$$

$$c_j^{\theta} = \begin{cases} (\theta, g_j)_{C'_{a,b}}, & j = 1, \ldots, n \\ \sqrt{\|\theta\|_{C'_{a,b}}^2 - \sum_{j=1}^{n}(\theta, g_j)_{C'_{a,b}}^2}, & j = n+1, \end{cases}$$

$$g_{n+1} \equiv g_{n+1}(\theta) = \frac{1}{c_{n+1}^{\theta}}\left[\theta - \sum_{j=1}^{n} c_j^{\theta} g_j\right], \quad \text{if } c_{n+1}^{\theta} \neq 0,$$

and $e_{n+1} = c_{n+1}^{\theta} g_{n+1}$. Then $\mathcal{A} \cup \{e_{n+1}\} = \{e_1, \ldots, e_n, e_{n+1}\}$ is an orthogonal set in $C'_{a,b}[0,T]$ and we obtain

$$\theta = \sum_{j=1}^{n+1} c_j^{\theta} g_j = \sum_{j=1}^{n+1} \frac{c_j^{\theta}}{\|e_j\|_{C'_{a,b}}} e_j.$$

For the complex measure ν associated with F by (5.4), let $\nu_{q,\vec{e},\theta}^t$ be the translation measure of ν defined by

$$\nu_{q,\vec{e},\theta}^t(B) := \nu\Big(B + \big(qc_1^{\theta}/\|e_1\|_{C'_{a,b}}, \ldots, qc_n^{\theta}/\|e_n\|_{C'_{a,b}}\big)\Big)$$

for $B \in \mathcal{B}(\mathbb{R}^n)$, and let δ_{-q} be the Dirac measure concentrated at $-q$ in $\mathbb{R}$. Then it follows that

$$
\begin{aligned}
&F^{q\theta}(x)\\
&= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j,x)^{\sim} v_j - iq\bigg[\sum_{j=1}^{n+1} c_j^{\theta}(g_j,x)^{\sim}\bigg]\bigg\} d\nu(\vec{v})\\
&= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j,x)^{\sim}\bigg[v_j - \frac{qc_j^{\theta}}{\|e_j\|_{C'_{a,b}}}\bigg] - iqc_{n+1}^{\theta}(g_{n+1},x)^{\sim}\bigg\} d\nu(\vec{v})\\
&= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j,x)^{\sim} r_j - iq(e_{n+1},x)^{\sim}\bigg\} d\nu_{q,\vec{e},\theta}^{t}(\vec{r})\\
&= \int_{\mathbb{R}^n} \exp\bigg\{ i\sum_{j=1}^{n}(e_j,x)^{\sim} r_j\bigg\}\bigg[\int_{\mathbb{R}} \exp\big\{i(e_{n+1},x)^{\sim} r_{n+1}\big\} d\delta_{-q}(r_{n+1})\bigg] d\nu_{q,\vec{e},\theta}^{t}(\vec{r})\\
&= \int_{\mathbb{R}^{n+1}} \exp\bigg\{ i\sum_{j=1}^{n}(e_j,x)^{\sim} r_j + i(e_{n+1},x)^{\sim} r_{n+1}\bigg\} d(\nu_{q,\vec{e},\theta}^{t}\times\delta_{-q})(\vec{r})\\
&= (\nu_{q,\vec{e},\theta}^{t}\,\widehat{\times\,\delta_{-q}})\big((e_1,x)^{\sim},\dots,(e_n,x)^{\sim},(e_{n+1},x)^{\sim}\big).
\end{aligned}
$$

One can easily see that $\nu_{q,\vec{e},\theta}^{t}\times\delta_{-q}$ is an element of $\mathcal{M}(\mathbb{R}^{n+1})$. Thus $(\nu_{q,\vec{e},\theta}^{t}\,\widehat{\times\,\delta_{-q}})$ belongs to $\widehat{\mathcal{M}}(\mathbb{R}^{n+1})$, the space of Fourier transforms of measures from $\mathcal{M}(\mathbb{R}^{n+1})$, and so the functional $F^{q\theta}$ given by (7.1) is an element of $\widehat{\mathfrak{T}}_{\mathcal{A}\cup\{e_{n+1}\}}$. Furthermore, for any real q with $|q|>q_0$, $F^{q\theta}$ is an element of $\widehat{\mathfrak{T}}_{\mathcal{A}\cup\{e_{n+1}\}}^{q_0,k}$, because

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^{n+1}\|e_j\|_{C'_{a,b}}|r_j|\bigg\} d\big|\nu_{q,\vec{e},\theta}^{t}\times\delta_{-q}\big|(\vec{r})\\
&= \int_{\mathbb{R}} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\|e_{n+1}\|_{C'_{a,b}}|r_{n+1}|\bigg\} d\delta_{-q}(r_{n+1})\\
&\quad\times\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^{n}\|e_j\|_{C'_{a,b}}|r_j|\bigg\} d\big|\nu_{q,\vec{e},\theta}^{t}\big|(\vec{r})\\
&= \exp\bigg\{\frac{|q|\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\|e_{n+1}\|_{C'_{a,b}}\bigg\}\\
&\quad\times\int_{\mathbb{R}^n} \exp\bigg\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^{n}\|e_j\|_{C'_{a,b}}\bigg|v_j - \frac{qc_j^{\theta}}{\|e_j\|_{C'_{a,b}}}\bigg|\bigg\} d|\nu|(\vec{r})\\
&\le \exp\bigg\{\frac{|q|\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^{n+1}|c_j^{\theta}|\bigg\}
\end{aligned}
$$

$$\times \int_{\mathbb{R}^n} \exp\left\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}} \sum_{j=1}^{n} \|e_j\|_{C'_{a,b}}|v_j|\right\} d|\nu|(\vec{r})$$
$$< +\infty.$$

If k is an element of $\mathcal{O}^*(\mathcal{A}\cup\{e_{n+1}\})$, one can evaluate the L_p analytic $\mathcal{Z}_k$-GFFT, $T^{(p)}_{q,k}(F^{q\theta})$, of the functional $F^{q\theta}$ given by (7.1) in view of Theorems 6.1 and 6.3. But we cannot ensure that $k\in\mathcal{O}^*(\mathcal{A})$ implies $k\in\mathcal{O}^*(\mathcal{A}\cup\{e_{n+1}\})$, because $\mathcal{O}^*(\mathcal{A}\cup\{e_{n+1}\})\subset\mathcal{O}^*(\mathcal{A})$.

The functional $F^{q\theta}$ given by (7.1) plays an important role in our translation theorems for the analytic $\mathcal{Z}_k$-Feynman integral and the analytic $\mathcal{Z}_k$-GFFT of functionals in $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

Theorem 7.1. *Let q_0, k, and F be as in Theorem 6.1. Let $Dk = h$, and given $\theta\in C'_{a,b}[0,T]$ with $D\theta=\varphi$, let $x_0\in C'_{a,b}[0,T]$ be given by*

$$x_0(t) := \int_0^t h(s)\varphi(s)db(s) = (k\odot\theta)(t). \tag{7.2}$$

Then for all real q with $|q|>q_0$, the generalized analytic $\mathcal{Z}_k$-Feynman integral $I_k^{\mathrm{anf}_q}[F^{q\theta}]$ exists. Furthermore, we have the following equality:

$$\begin{aligned} &I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot)+\mathcal{Z}_k(x_0,\cdot))]\\ &\equiv T^{(1)}_{q,k}(F)(\mathcal{Z}_k(x_0,\cdot))\\ &= \exp\left\{\frac{iq}{2}\|\theta\odot k\|^2_{C'_{a,b}} - (-iq)^{1/2}(\theta\odot k, a)_{C'_{a,b}}\right\} I_{k,x}^{\mathrm{anf}_q}[F^{q\theta}(\mathcal{Z}_k(x,\cdot))], \end{aligned} \tag{7.3}$$

where $F^{q\theta}$ is given by equation (7.1) above.

The following lemma will be very useful in the proof of Theorem 7.1. By Parseval's relation, one can obtain equations (7.4), (7.5) and (7.6) below.

Lemma 7.2. *Given an orthogonal set $\mathcal{A}=\{e_1,\dots,e_n\}$ in $C'_{a,b}[0,T]$, $k\in\mathcal{O}^*(\mathcal{A})$, and $\theta\in C'_{a,b}[0,T]$, let*

$$e^{\theta\odot k}_{n+1} = c^{\theta\odot k}_{n+1} g^{\theta\odot k}_{n+1},$$

where

$$c^{\theta\odot k}_{n+1} = \sqrt{\|\theta\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(\theta\odot k, g^k_j)^2_{C'_{a,b}}},$$

$$g^{\theta\odot k}_{n+1} = \frac{1}{c^{\theta\odot k}_{n+1}}\left[\theta\odot k - \sum_{j=1}^{n}(\theta\odot k, g^k_j)_{C'_{a,b}} g^k_j\right],$$

and where $g^k_j = e_j\odot k/\|e_j\odot k\|_{C'_{a,b}}$ for $j=1,\dots,n$. Then

$$\{e_1\odot k,\dots,e_n\odot k, e^{\theta\odot k}_{n+1}\}$$

is an orthogonal set. In this case, one can see that

$$\theta\odot k=\sum_{j=1}^{n}\frac{(\theta\odot k,e_j\odot k)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}e_j\odot k+e_{n+1}^{\theta\odot k}, \tag{7.4}$$

$$\|\theta\odot k\|^2_{C'_{a,b}}=\sum_{j=1}^{n}\frac{(\theta\odot k,e_j\odot k)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}+\|e_{n+1}^{\theta\odot k}\|^2_{C'_{a,b}}, \tag{7.5}$$

and

$$(\theta\odot k,a)_{C'_{a,b}}=\sum_{j=1}^{n}\frac{(\theta\odot k,e_j\odot k)_{C'_{a,b}}(\theta\odot k,a)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}+(e_{n+1}^{\theta\odot k},a)_{C'_{a,b}}. \tag{7.6}$$

Proof of Theorem 7.1. Using (7.1), (5.4), (3.3), Fubini's theorem, (7.4), (2.3), (2.5), (5.6), (7.5), and (7.6), it follows that for $\lambda>0$,

(7.7)

$$\begin{aligned}
&J_{F^{q\theta}}(\mathcal{Z}_k;\lambda)\\
:=&\int_{C_{a,b}[0,T]}F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))\exp\{-iq\lambda^{-1/2}(\theta,\mathcal{Z}_k(x,\cdot))^{\sim}\}d\mu(x)\\
=&\int_{C_{a,b}[0,T]}\bigg[\int_{\mathbb{R}^n}\exp\bigg\{i\lambda^{-1/2}\sum_{j=1}^{n}(e_j\odot k,x)^{\sim}v_j\\
&\qquad-iq\lambda^{-1/2}\bigg(\sum_{j=1}^{n}\frac{(\theta\odot k,e_j\odot k)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}e_j\odot k+e_{n+1}^{\theta\odot k},x\bigg)^{\sim}\bigg\}d\nu(\vec{v})\bigg]d\mu(x)\\
=&\int_{\mathbb{R}^n}\bigg[\int_{C_{a,b}[0,T]}\exp\bigg\{i\lambda^{-1/2}\sum_{j=1}^{n}(e_j\odot k,x)^{\sim}v_j\\
&\qquad\qquad-iq\lambda^{-1/2}\sum_{j=1}^{n}\frac{(\theta\odot k,e_j\odot k)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}(e_j\odot k,x)^{\sim}\\
&\qquad\qquad-iq\lambda^{-1/2}(e_{n+1}^{\theta\odot k},x)^{\sim}\bigg\}d\mu(x)\bigg]d\nu(\vec{v})\\
=&\int_{\mathbb{R}^n}\bigg[\Big(2\pi\|e_{n+1}^{\theta\odot k}\|^2_{C'_{a,b}}\Big)^{-1/2}\int_{\mathbb{R}}\exp\bigg\{-iq\lambda^{-1/2}u_0\\
&\qquad\qquad-\frac{\big[u_0-(e_{n+1}^{\theta\odot k},a)_{C'_{a,b}}\big]^2}{2\|e_{n+1}^{\theta\odot k}\|^2_{C'_{a,b}}}\bigg\}du_0\bigg]\\
&\times\bigg[\prod_{j=1}^{n}\Big(2\pi\|e_j\odot k\|^2_{C'_{a,b}}\Big)^{-1/2}\int_{\mathbb{R}}\exp\bigg\{i\lambda^{-1/2}v_ju_j
\end{aligned}$$

$$- iq\lambda^{-1/2}\frac{(\theta\odot k, e_j\odot k)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}u_j - \frac{\big[u_j-(e_j\odot k, a)_{C'_{a,b}}\big]^2}{2\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}du_j\bigg]d\nu(\vec{v})$$

$$= \exp\bigg\{\frac{\|e^{\theta\odot k}_{n+1}\|^2_{C'_{a,b}}}{2}\bigg[-iq\lambda^{-1/2}+\frac{(e^{\theta\odot k}_{n+1},a)_{C'_{a,b}}}{\|e^{\theta\odot k}_{n+1}\|^2_{C'_{a,b}}}\bigg]^2-\frac{(e^{\theta\odot k}_{n+1},a)^2_{C'_{a,b}}}{2\|e^{\theta\odot k}_{n+1}\|^2_{C'_{a,b}}}\bigg\}$$

$$\times\int_{\mathbb{R}^n}\bigg[\prod_{j=1}^n\exp\bigg\{\frac{\|e_j\odot k\|^2_{C'_{a,b}}}{2}\bigg[i\lambda^{-1/2}v_j - iq\lambda^{-1/2}\frac{(\theta\odot k, e_j\odot k)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}$$

$$+\frac{(\theta\odot k, a)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg]^2-\frac{(\theta\odot k,a)^2_{C'_{a,b}}}{2\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}\bigg]d\nu(\vec{v})$$

$$= \exp\bigg\{\frac{(-iq)^2}{2\lambda}\|e^{\theta\odot k}_{n+1}\|^2_{C'_{a,b}}+(-iq)\lambda^{-1/2}(e^{\theta\odot k}_{n+1},a)_{C'_{a,b}}\bigg\}$$

$$\times\int_{\mathbb{R}^n}\exp\bigg\{\frac{i(-iq)}{\lambda}\sum_{j=1}^n(\theta\odot k, e_j\odot k)_{C'_{a,b}}v_j$$

$$-\frac{1}{2\lambda}\sum_{j=1}^n\|e_j\odot k\|^2_{C'_{a,b}}v_j^2 + i\lambda^{-1/2}\sum_{j=1}^n(e_j\odot k, a)_{C'_{a,b}}v_j$$

$$+\frac{(-iq)^2}{2\lambda}\sum_{j=1}^n\frac{(\theta\odot k, e_j\odot k)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}$$

$$+(-iq)\lambda^{-1/2}\sum_{j=1}^n\frac{(\theta\odot k, e_j\odot k)_{C'_{a,b}}(\theta\odot k, a)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}\bigg\}d\nu(\vec{v})$$

$$= \exp\bigg\{\frac{(-iq)^2}{2\lambda}\bigg[\sum_{j=1}^n\frac{(\theta\odot k, e_j\odot k)^2_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}+\|e^{\theta\odot k}_{n+1}\|^2_{C'_{a,b}}\bigg]$$

$$+(-iq)\lambda^{-1/2}\bigg[\sum_{j=1}^n\frac{(\theta\odot k, e_j\odot k)_{C'_{a,b}}(\theta\odot k, a)_{C'_{a,b}}}{\|e_j\odot k\|^2_{C'_{a,b}}}+(e^{\theta\odot k}_{n+1},a)_{C'_{a,b}}\bigg]\bigg\}$$

$$\times\int_{\mathbb{R}^n}\exp\bigg\{\frac{i(-iq)}{\lambda}\sum_{j=1}^n(\theta\odot k, e_j\odot k)_{C'_{a,b}}v_j\bigg\}\mathcal{W}_{\vec{e}}(\lambda,k;\vec{v})d\nu(\vec{v})$$

$$= \exp\bigg\{\frac{(-iq)^2}{2\lambda}\|\theta\odot k\|^2_{C'_{a,b}}+(-iq)\lambda^{-1/2}(\theta\odot k,a)_{C'_{a,b}}\bigg\}$$

$$\times\int_{\mathbb{R}^n}\exp\bigg\{\frac{i(-iq)}{\lambda}\sum_{j=1}^n(\theta\odot k, e_j\odot k)_{C'_{a,b}}v_j\bigg\}\mathcal{W}_{\vec{e}}(\lambda,k;\vec{v})d\nu(\vec{v}).$$

Next, using the techniques similar to those used in the proof of Theorem 5.4, one can obtain the analytic $\mathcal{Z}_k$-function space integral $J^*_{F^{q\theta}}(\mathcal{Z}_k;\lambda)\equiv I_k^{\mathrm{an}_\lambda}[F^{q\theta}]$, as a function of λ on $\mathbb{C}_+$, such that $J^*_{F^{q\theta}}(\mathcal{Z}_k;\lambda)=J_{F^{q\theta}}(\mathcal{Z}_k;\lambda)$ for all $\lambda>0$.

Let Γ_{q_0} be the domain in $\widetilde{\mathbb{C}}_+$ given by (6.2) and let $\Delta > 1$ be given. Since $-iq \in \Gamma_{q_0}$ for any real number q with $|q| > q_0$, if $\lambda \to -iq$ in $\widetilde{\mathbb{C}}_+$ there exists a real number $\delta > 0$ such that for all $\lambda \in N_\delta(-iq) \cap \Gamma_{q_0}$ (the set $N_\delta(-iq)$ indicates the open neighborhood of $-iq$ with radius δ in $\mathbb{C}$),

$$\left| \exp\left\{ \frac{i(-iq)}{\lambda} \sum_{j=1}^{n} (\theta \odot k, e_j \odot k)_{C'_{a,b}} v_j \right\} \right| < \Delta. \tag{7.8}$$

Hence, in the last expression of (7.7), we observe that for all $\lambda \in N_\delta(-iq) \cap \Gamma_{q_0}$,

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \exp\left\{ \frac{i(-iq)}{\lambda} \sum_{j=1}^{n} (\theta \odot k, e_j \odot k)_{C'_{a,b}} v_j \right\} \mathcal{W}_{\vec{e}}(\lambda, k; \vec{v}) d\nu(\vec{v}) \bigg| \\
&\le \int_{\mathbb{R}^n} \left| \exp\left\{ \frac{i(-iq)}{\lambda} \sum_{j=1}^{n} (\theta \odot k, e_j \odot k)_{C'_{a,b}} v_j \right\} \right| \big| \mathcal{W}_{\vec{e}}(\lambda, k; \vec{v}) \big| d|\nu|(\vec{v}) \\
&\le \Delta \int_{\mathbb{R}^n} \exp\left\{ \frac{\|a\|_{C'_{a,b}} \|Dk\|_\infty}{\sqrt{2q_0}} \sum_{j=1}^{n} \|e_j\|_{C'_{a,b}} |v_j| \right\} d|\nu|(\vec{v}) \\
&< +\infty
\end{aligned}$$

by the inequalities (7.8) and (6.3) above. Thus, by the dominated convergence theorem, (7.2), (3.3) and (6.5), it follows that for real q with $|q| > q_0$,

$$\begin{aligned}
&I_k^{\mathrm{anf}_q}[F^{q\theta}] = \lim_{\substack{\lambda \to -iq \\ \lambda \in \Gamma_{q_0}}} I_k^{\mathrm{an}_\lambda}[F^{q\theta}] \\
&= \lim_{\substack{\lambda \to -iq \\ \lambda \in \Gamma_{q_0}}} \exp\left\{ \frac{(-iq)^2}{2\lambda} \|\theta \odot k\|^2_{C'_{a,b}} + (-iq)\lambda^{-1/2} (\theta \odot k, a)_{C'_{a,b}} \right\} \\
&\qquad \times \int_{\mathbb{R}^n} \exp\left\{ \frac{i(-iq)}{\lambda} \sum_{j=1}^{n} (\theta \odot k, e_j \odot k)_{C'_{a,b}} v_j \right\} \mathcal{W}_{\vec{e}}(\lambda, k; \vec{v}) d\nu(\vec{v}) \\
&= \exp\left\{ -\frac{iq}{2} \|\theta \odot k\|^2_{C'_{a,b}} + (-iq)^{1/2} (\theta \odot k, a)_{C'_{a,b}} \right\} \\
&\qquad \times \int_{\mathbb{R}^n} \exp\left\{ i \sum_{j=1}^{n} (e_j \odot k, x_0)_{C'_{a,b}} v_j \right\} \mathcal{W}_{\vec{e}}(-iq, k; \vec{v}) d\nu(\vec{v}) \\
&= \exp\left\{ -\frac{iq}{2} \|\theta \odot k\|^2_{C'_{a,b}} + (-iq)^{1/2} (\theta \odot k, a)_{C'_{a,b}} \right\} T^{(1)}_{q,k}(F)(\mathcal{Z}_k(x_0, \cdot)).
\end{aligned} \tag{7.9}$$

Thus, the generalized analytic $\mathcal{Z}_k$-Feynman integral $I_k^{\mathrm{anf}_q}[F^{q\theta}]$ exists. Equations (4.3) with y replaced with $\mathcal{Z}_k(x_0, \cdot)$ and (7.9) yield the equation (7.3). □

Theorem 7.3. *Let q_0, k, F, θ, and x_0 be as in Theorem 7.1. Then for all real q with $|q| > q_0$ and s-a.e. $y \in C_{a,b}[0,T]$,*

(7.10)
$$\begin{aligned} &T^{(1)}_{q,k}(F)(y+\mathcal{Z}_k(x_0,\cdot)) \\ &= \exp\left\{\frac{iq}{2}\|\theta\odot k\|^2_{C'_{a,b}} - (-iq)^{1/2}(\theta\odot k, a)_{C'_{a,b}} + iq(\theta,y)^{\sim}\right\}T^{(1)}_{q,k}(F^{q\theta})(y), \end{aligned}$$

where $F^{q\theta}$ is given by equation (7.1) above.

Proof. By Theorem 6.1, the analytic $\mathcal{Z}_k$-GFFT on the left hand side of equation (7.10) exists.

Given $y \in C_{a,b}[0,T]$, let

$$G_y(x) = F(y+x). \tag{7.11}$$

Clearly, for the complex measure ν associated with F by (5.4), the set function $\nu_y : \mathcal{B}(\mathbb{R}^n) \to \mathbb{C}$ given by $\nu_y(B) = \int_B \exp\{i\sum_{j=1}^n (e_j,y)^{\sim} v_j\}d\nu(\vec{v})$ is an element of $\mathcal{M}(\mathbb{R}^n)$. From this, we see that

$$G_y(x) = \int_{\mathbb{R}^n} \exp\left\{i\sum_{j=1}^n (e_j,x)^{\sim} v_j\right\}d\nu_y(\vec{v}) \tag{7.12}$$

for s-a.e. $x \in C_{a,b}[0,T]$, and that G_y belongs to $\widehat{\mathfrak{T}}_{\mathcal{A}}$. We also observe that given $y \in C_{a,b}[0,T]$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\left\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\right\}d|\nu_y|(\vec{v}) \\ &= \int_{\mathbb{R}^n} \exp\left\{\frac{\|a\|_{C'_{a,b}}\|Dk\|_\infty}{\sqrt{2q_0}}\sum_{j=1}^n \|e_j\|_{C'_{a,b}}|v_j|\right\}d|\nu|(\vec{v}) < +\infty, \end{aligned}$$

and that G_y is an element of $\widehat{\mathfrak{T}}^{q_0,k}_{\mathcal{A}}$.

Next let

$$G^{q\theta}_y(x) = G_y(x)\exp\{-iq(\theta,x)^{\sim}\}. \tag{7.13}$$

Then, using (4.3) with F replaced with $F^{q\theta}$, (7.1), (7.12) and (7.13), we obtain the equation

$$\begin{aligned} T^{(1)}_{q,k}(F^{q\theta})(y) &= I^{\mathrm{anf}_q}_{k,x}[F^{q\theta}(y+\mathcal{Z}_k(x,\cdot))] \\ &= \exp\{-iq(\theta,y)^{\sim}\}I^{\mathrm{anf}_q}_{k,x}[G^{q\theta}_y(\mathcal{Z}_k(x,\cdot))]. \end{aligned} \tag{7.14}$$

Since G_y is an element of $\widehat{\mathfrak{T}}^{q_0,k}_{\mathcal{A}}$, applying Theorem 7.1 with G_y instead of F, we guarantee the existence of the analytic $\mathcal{Z}_k$-GFFT $T^{(1)}_{q,k}(F^{q\theta})$ of $F^{q\theta}$.

Now, we need only to verify the equality in equation (7.10). But, applying equations (4.3), (7.11), (7.3) with F and $F^{q\theta}$ replaced with G_y and $G_y^{q\theta}$ respectively, and (7.14), it follows that for real q with $|q| > q_0$ and s-a.e. $y \in C_{a,b}[0,T]$,

$$\begin{aligned}
&T_{q,k}^{(1)}(F)(y + \mathcal{Z}_k(x_0,\cdot)) \\
&= I_{k,x}^{\mathrm{anf}_q}[G_y(\mathcal{Z}_k(x,\cdot) + \mathcal{Z}_k(x_0,\cdot))] \\
&= \exp\left\{\frac{iq}{2}\|\theta \odot k\|_{C'_{a,b}} - (-iq)^{1/2}(\theta \odot k, a)_{C'_{a,b}}\right\} I_{k,x}^{\mathrm{anf}_q}[G_y^{q\theta}(\mathcal{Z}_k(x,\cdot))] \\
&= \exp\left\{\frac{iq}{2}\|\theta \odot k\|_{C'_{a,b}} - (-iq)^{1/2}(\theta \odot k, a)_{C'_{a,b}} + iq(\theta, y)^\sim\right\} \\
&\quad \times I_{k,x}^{\mathrm{anf}_q}[\exp\{-iq(\theta, y)^\sim\} G_y^{q\theta}(\mathcal{Z}_k(x,\cdot))] \\
&= \exp\left\{\frac{iq}{2}\|\theta \odot k\|_{C'_{a,b}} - (-iq)^{1/2}(\theta \odot k, a)_{C'_{a,b}} + iq(\theta, y)^\sim\right\} T_{q,k}^{(1)}(F^{q\theta})(y)
\end{aligned}$$

as desired. □

Remark 7.4. In view of Theorem 6.3, it also follows that for all $p \in (1, 2]$ and for s-a.e. $y \in C_{a,b}[0,T]$,

$$\begin{aligned}
&T_{q,k}^{(p)}(F)(y + \mathcal{Z}_k(x_0,\cdot)) \\
&= T_{q,k}^{(1)}(F)(y + \mathcal{Z}_k(x_0,\cdot)) \\
&= \exp\left\{\frac{iq}{2}\|\theta \odot k\|^2_{C'_{a,b}} - (-iq)^{1/2}(\theta \odot k, a)_{C'_{a,b}} + iq(\theta, y)^\sim\right\} T_{q,k}^{(p)}(F^{q\theta})(y).
\end{aligned}$$

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