

## SAMPLING THEOREMS ASSOCIATED WITH DIFFERENTIAL OPERATORS WITH FINITE RANK PERTURBATIONS

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ABSTRACT. We derive a sampling theorem associated with first order self-adjoint eigenvalue problem with a finite rank perturbation. The class of the sampled integral transforms is of finite Fourier type where the kernel has an additional perturbation.

### 1. Introduction

In the papers of Campbell [11], Haddad et al. [18] and Everitt-Poulkou [16] the classical sampling theorem of Whittaker-Kotel'nikov-Shannon (WKS) has been connected to first order self-adjoint eigenvalue problems. The WKS sampling theorem gives us the ability to reconstruct elements of the Paley-Wiener space from their values at the integers. This space is called the space of band-limited functions in physical and engineering terminology. The Paley-Wiener space  $B_\pi^2$  is defined to be the set of all  $L^2(\mathbb{R})$ -functions whose Fourier transforms vanish outside  $[-\pi, \pi]$ . Thus  $f \in B_\pi^2$  if there exists a unique  $L^2(-\pi, \pi)$ -function  $g(\cdot)$  for which

$$(1.1) \quad f(t) = \int_{-\pi}^{\pi} g(x) e^{-ixt} dx.$$

Equivalently, [7, 22],  $B_\pi^2$  coincides with the class of all  $L^2(\mathbb{R})$ -entire functions with exponential type  $\pi$ . Also it is known that  $B_\pi^2 \subset L^2(\mathbb{R})$  is a reproducing kernel Hilbert space with the reproducing kernel

$$(1.2) \quad K(t, s) = \frac{\sin \pi(t - s)}{\pi(t - s)},$$

cf. e.g. [19]. The classical sampling theorem of Whittaker-Kotel'nikov-Shannon (WKS), states that if  $f \in B_\pi^2$ , then  $f$  can be recovered from its values at the

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integers via the sampling series

$$(1.3) \quad f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(t-k)}{\pi(t-k)}, \quad t \in \mathbb{C}.$$

The convergence of series (1.3) is uniform on  $\mathbb{R}$  and on compact subsets of  $\mathbb{C}$  and it is absolute on  $\mathbb{C}$ . Moreover series (1.3) converges in the  $L^2(\mathbb{R})$ -norm. See [9, 20, 23, 26]. The proof of (1.3) could be derived using several ways, see e.g. [19, 27].

In the works of Campbell [11], Haddad et al. [18] and Everitt-Poulkou [16] mentioned above the authors indicated that WKS sampling theorem could be established in connection with the first order eigenvalue problem

$$(1.4) \quad \ell(y) := iy' = ty, \quad y(\pi) = y(-\pi) \quad -\pi \leq x \leq \pi, \quad t \in \mathbb{C}.$$

The connection comes from the fact that the kernel  $\exp(-ixt)$  of (1.1) is a solution of the differential equation of (1.4) for all  $t \in \mathbb{C}$  and the sampling points in (1.3), i.e., the integers, are nothing but the eigenvalues of the problem (1.4). Another fact that plays a major role in the derivation of the WKS sampling theorem associated with (1.4) is that the set of eigenfunctions of (1.4), namely  $\{e^{-ikx}\}_{k=-\infty}^{\infty}$  is generated by a single function, namely the solution (kernel). In another direction, still related to (1.4), Haddad et al. [18] derived WKS sampling theorem using Green's function of (1.4).

In this article we study the sampling theorem associated with (1.4) when the differential operator contains a finite rank perturbation. As we see below several changes will occur in the spectral properties of the perturbed eigenvalue problem. These changes will affect the associated sampling results. The first change will be in the class of transforms where the results hold. It will not be in general a Paley-Wiener space, but a perturbed one where the kernel  $\exp(-ixt)$  will be replaced by  $\exp(-ixt) + \kappa(x, t)$ , see the examples below for concrete examples. On the other hand, unlike (1.4), the eigenvalues will not be always simple. Therefore we will derive two sampling results, one when all eigenfunctions are generated by one single function and the other by using Green's function, employing a technique of [1, 2, 6]. In the next two sections we study a spectral analysis of the perturbed problem. We use and extend the techniques of Catchpole [12] and Stakgold [25] as well as the theory of finite rank operators as in [17]. Then we derive two sampling theorems as well as illustrative examples with comparisons in the last two sections.

It is worthwhile to mention that the authors have treated the rank-one perturbation situation in [5]. However, because of one dimensionality, there was no need for a deep treatment based on the theory of finite rank operators as we did here. See also [24]. A treatment of the discrete situation could be also found in [4].

**2. Fundamental solutions and eigenvalues**

Let  $n \in \mathbb{Z}^+$  be fixed. Consider the boundary-value problem

$$(2.1) \quad \ell_r(y) := iy' + \sum_{k=1}^n r_k(x) \int_{-\pi}^{\pi} r_k(\tau) y(\tau) d\tau = ty, \quad -\pi \leq x \leq \pi, \quad t \in \mathbb{C},$$

$$(2.2) \quad V(y) := y(\pi) - y(-\pi) = 0.$$

Here  $r_k(\cdot)$ ,  $k = 1, \dots, n$ , are  $n$  linearly independent  $L^2(-\pi, \pi)$ -real-valued functions. Let us denote this problem by  $\Pi_n$ . The unperturbed problem (1.4) will be denoted by  $\Pi$ . We can see that Problem  $\Pi_n$  is self adjoint with real eigenvalues only. Now we try to find the general solution of equation (2.1). We use the technique established by Catchpole in [12], see also [25]. Write (2.1) as the linear equation

$$(2.3) \quad y' + ity = i \sum_{k=1}^n \rho_k r_k(x), \quad \rho_k := \langle y, r_k \rangle = \int_{-\pi}^{\pi} y(\tau) r_k(\tau) d\tau,$$

which has the solution

$$(2.4) \quad y = e^{-itx} \left( i \sum_{k=1}^n \rho_k \int_{-\pi}^x r_k(\tau) e^{it\tau} d\tau + c \right) = c e^{-itx} + \sum_{k=1}^n \rho_k P_k(x, t),$$

where  $c$  is an arbitrary constant, and

$$(2.5) \quad P_k(x, t) := i e^{-itx} \int_{-\pi}^x r_k(\tau) e^{it\tau} d\tau, \quad k = 1, \dots, n.$$

Note that  $P_k(\cdot, t)$  is the unique solution of the inhomogeneous initial value problem

$$(2.6) \quad iy' + r_k(x) = ty, \quad y(-\pi) = 0.$$

Therefore, a solution  $\phi$  of equation (2.1) satisfies the integral equation

$$(2.7) \quad \phi = c e^{-itx} + \sum_{k=1}^n \langle \phi, r_k \rangle P_k, \quad \langle u, v \rangle := \int_{-\pi}^{\pi} u(\tau) \overline{v(\tau)} d\tau.$$

Applying the results of [17, p. 66] concerning the inversion formula for finite rank operators we get the following lemma.

**Lemma 2.1.** *If for some  $t$*

$$(2.8) \quad C(t) := \det(a_{ij})_{1 \leq i, j \leq n} \neq 0, \quad a_{ij} := \delta_{ij} - \langle P_j, r_i \rangle,$$

*then the general solution of (2.1) has the form*

$$(2.9) \quad \phi_c(x, t) := c \varphi(x, t) - \frac{c}{C(t)} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \langle \varphi, r_1 \rangle \\ a_{21} & a_{22} & \dots & a_{2n} & \langle \varphi, r_2 \rangle \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \langle \varphi, r_n \rangle \\ P_1 & P_2 & \dots & P_n & 0 \end{pmatrix},$$

where  $\varphi(x, t) := e^{-itx}$  and  $c$  is an arbitrary constant.

We will denote the solution by  $\phi(x, t)$  when  $c = 1$ .

This solution can be directly obtained by multiplying (2.7) by  $r_j(\cdot)$  and integrating over  $[-\pi, \pi]$ , and then we get the nonhomogeneous system

$$(2.10) \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \langle \phi_c, r_1 \rangle \\ \langle \phi_c, r_2 \rangle \\ \vdots \\ \langle \phi_c, r_n \rangle \end{pmatrix} = \begin{pmatrix} \langle \varphi, r_1 \rangle \\ \langle \varphi, r_2 \rangle \\ \vdots \\ \langle \varphi, r_n \rangle \end{pmatrix},$$

which, under the conditions of the lemma, can be solved by Cramer’s rule leading to (2.9). Equation (2.10) is also useful for our investigations.

It is concluded from the previous lemma that, when  $C(t)$  does not vanish, the linear space of solutions of equation (2.1) is a one dimensional space as is the case of (1.4). Also a solution is uniquely determined by one initial condition. So in this case all eigenvalues when  $C(t) \neq 0$  are simple. The situation is different when  $C(t) = 0$ . We will seek the solution when  $C(t) = 0$ . First we have the following lemma.

**Lemma 2.2.** *Assume that  $C(t) = 0$ . Then*

$$(2.11) \quad R(x, t) = \sum_{k=1}^n \alpha_k P_k(x, t),$$

is a non-trivial solution of (2.1) if  $(\alpha_1, \dots, \alpha_n)$  is a non-trivial solution of the homogenous system

$$(2.12) \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

*Proof.* Obviously  $P_1, \dots, P_n$  are linearly independent on  $[-\pi, \pi]$ . We assign from (2.11) in (2.1), using that  $P_k(\cdot, t)$  is the solution of (2.6), to obtain for arbitrary  $\alpha_1, \dots, \alpha_n$ ,

$$(2.13) \quad \sum_{k=1}^n \left( \sum_{j=1}^n \alpha_j \langle P_j, r_k \rangle - \alpha_k \right) r_k(x) = 0 \quad \text{a.e.}$$

The linearly independence of  $r_k(\cdot)$ ,  $k = 1, \dots, n$  and equation (2.13) lead to the system

$$(2.14) \quad \sum_{j=1}^n \alpha_j \langle P_j, r_k \rangle - \alpha_k = 0, \quad k = 1, \dots, n,$$

which has a non-trivial solution for  $\alpha_1, \dots, \alpha_n$ , since  $\det(a_{ij}) = C(t) = 0$ .  $\square$

As we notice in system (2.10) when  $C(t) = 0$ , the system cannot be guaranteed to have a solution unless  $\langle \varphi, r_k \rangle = 0, k = 1, \dots, n$ . So we assume throughout the rest of this article that this condition occurs when  $C(t) = 0$ . Therefore  $\varphi$  is a solution of (2.1) and we have the following lemma.

**Lemma 2.3.** *Let  $\nu :=$  the rank of the matrix  $(a_{ij})_{1 \leq i, j \leq n}$ . Therefore any solution of (2.1) when  $C(t) = 0$  has the form*

$$(2.15) \quad \psi_c(x, t) := c\varphi(x, t) + \sum_{k=1}^{n-\nu} \gamma_k R_k(x, t),$$

where  $\{R_k(\cdot, t)\}_{k=1}^{n-\nu}$  are all linearly independent solutions of (2.1) of the form (2.11) and  $\{\gamma_k\}_{k=1}^{n-\nu}$  are arbitrary constants.

As we saw if  $C(t) = 0$ , equation (2.1) may have  $n + 1$  linearly independent solutions. Also the same initial condition determines infinitely many solutions.

Now we investigate the eigenvalues of the problem  $\Pi_n$ . From now on  $\phi(\cdot, t)$  denotes the function (2.9) with  $c = 1$ . If  $C(t) \neq 0$ , then  $t$  is an eigenvalue if and only if

$$(2.16) \quad \Delta(t) := V(\phi(\cdot, t)) = 0.$$

In this case if  $t^*$  satisfies (2.16), then  $t^*$  is a simple eigenvalue with the eigenfunction  $\phi(x, t^*)$ , i.e.,  $\phi(\cdot, t)$  will generate all eigenfunctions corresponding to zeros of  $\Delta(t)$ . Now we consider the real zeros of  $C(t)$ .

**Lemma 2.4.** *Let  $C(t) = 0$  for some real  $t$ . Then  $t$  is an eigenvalue of  $\Pi_n$  with multiplicity  $n - \nu$  if it is not an eigenvalue of  $\Pi$ . Otherwise it is of multiplicity  $n - \nu + 1$ .*

*Proof.* If  $C(t) = 0$  for some real  $t$ , we show that  $P_k(\cdot, t)$  satisfies (2.2). Since

$$\langle \varphi_k, r \rangle = \int_{-\pi}^{\pi} e^{-it\tau} r_k(\tau) d\tau = 0,$$

then taking the complex conjugate and using that  $r_k$  and  $t$  are real, we get

$$\int_{-\pi}^{\pi} e^{it\tau} r_k(\tau) d\tau = 0, \quad k = 1, \dots, n.$$

Hence,  $P_k(\pi, t) = 0$  and since  $P_k(-\pi, t) = 0$ , then  $V(R_k) = 0$ , where

$$\{R_k(x, t)\}_{k=1}^{n-\nu}$$

are defined in Lemma 2.3. Thus  $R_k(x, t)$  satisfies the boundary condition (2.2) in addition to equation (2.1). Hence  $R_k(x, t)$  is an eigenfunction of  $\Pi_n$  and the first part follows. The eigenvalue  $t$  will have another eigenfunction, namely

$$\psi(x, t) = c\varphi(x, t) + \sum_{k=1}^{n-\nu} \gamma_k R_k(x, t),$$

if

$$(2.17) \quad 0 = V(\psi) = cV(\varphi) + \sum_{k=1}^{n-\nu} \gamma_k V(R_k) = cV(\varphi).$$

The above equation will have a non-trivial solution if and only if  $t$  is an eigenvalue of  $\Pi$  which cannot have more than one linearly independent solution, since  $\Pi$  has only simple eigenvalues. This completes the proof.  $\square$

As is seen the multiple eigenvalues of  $\Pi_n$  are only the real zeros of  $C(t)$ . Fortunately, as we will see in the next lemma, the number of eigenvalues with multiplicity more than one is finite.

**Lemma 2.5.** *The function  $C(t)$  has at most a finite number of real zeros.*

*Proof.* From Lemma 2.4 above, any real zero of  $C(t)$  is an eigenvalue. We assume that  $t$  is real. The function  $C(t)$  cannot have zeros with finite limit points and therefore the only possible limit points for the zeros of  $C(t)$  are  $\pm\infty$ . By definition of  $P_k(\cdot, t)$ , we have

$$\langle P_k, r_j \rangle = i \int_{-\pi}^{\pi} e^{-itx} r_j(x) \int_{-\pi}^x r_k(\tau) e^{it\tau} d\tau dx.$$

Since for  $x \in [-\pi, \pi]$ ,  $e^{-itx} r_j(x)$ , is bounded on  $\mathbb{R}$  as a function of  $t$ , then by Riemann-Lebesgue's Lemma, [8, p. 167], we obtain for  $x \in [-\pi, \pi]$ ,

$$(2.18) \quad \lim_{t \rightarrow \pm\infty} f(x, t) := \lim_{t \rightarrow \pm\infty} e^{-itx} r_j(x) \int_{-\pi}^x r_k(\tau) e^{it\tau} d\tau = 0.$$

Therefore, for any sequence  $t_k$  of real numbers with  $\lim_{k \rightarrow \infty} t_k = \pm\infty$ , we obtain

$$\lim_{k \rightarrow \infty} f(x, t_k) = 0, \quad x \in [-\pi, \pi].$$

Also

$$|f(x, t)| \leq |r_j(x)| \int_{-\pi}^x |r_k(\tau)| d\tau := g(x).$$

The function  $g(x) \in L^1(-\pi, \pi)$ . Hence, from Lebesgue's dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \int_{-\pi}^{\pi} f(x, t_k) dx = 0$$

for all real sequences  $t_k$  with  $t_k \rightarrow \pm\infty$ . Thus

$$\lim_{t \rightarrow \pm\infty} \int_{-\pi}^{\pi} f(x, t) dx = 0.$$

Then  $C(t)$ , cannot have large zeros since  $\lim_{t \rightarrow \pm\infty} C(t) = 1$ . Therefore,  $C(t)$  has only a finite number of real zeros.  $\square$

Finally we discuss the multiplicity of the eigenvalues in the following lemma.

**Lemma 2.6.** *The zeros of  $\Delta(t)$  are simple zeros. Let  $t^*$  be a real zero of  $C(t)$  with algebraic multiplicity  $s$  and  $\mu :=$  the multiplicity of the eigenvalue  $t^*$ . If  $t^*$  is not an eigenvalue of  $\Pi$ , then  $\mu \leq s$ , otherwise,  $\mu \leq s + 1$ .*

*Proof.* For differentiable functions  $y, z$  we can derive the following Lagrange’s identity

$$(2.19) \quad \langle \ell_r y, z \rangle = i [y(x)\bar{z}(x)]_{-\pi}^{\pi} - \langle y, \ell_r z \rangle.$$

Let  $t$  and  $s$  be different complex numbers. Applying (2.19) with  $y = \phi(\cdot, t)$  and  $z = \phi(\cdot, s)$ , we obtain

$$(2.20) \quad (t - s) \int_{-\pi}^{\pi} \phi(x, t)\bar{\phi}(x, s) dx = i \left( \phi(\pi, t)\bar{\phi}(\pi, s) - \phi(-\pi, t)\bar{\phi}(-\pi, s) \right).$$

Replace  $s$  by  $t_k$  for some  $k$ , and use  $\phi(\pi, t_k) = \phi(-\pi, t_k)$ , then for,  $t \neq t_k$ ,

$$(2.21) \quad \int_{-\pi}^{\pi} \phi(x, t)\phi(x, t_k) dx = i\bar{\phi}(\pi, t_k) \frac{\phi(\pi, t) - \phi(-\pi, t)}{t - t_k} = i\bar{\phi}(\pi, t_k) \frac{\Delta(t)}{t - t_k}.$$

Taking the limit in (2.21) as  $t$  approaches  $t_k$ , we have

$$(2.22) \quad \|\phi(x, t_k)\|^2 := \int_{-\pi}^{\pi} |\phi(x, t)|^2 dx = i\bar{\phi}(\pi, t_k)\Delta'(t_k).$$

Equation (2.22) shows that the eigenvalues are all simple zeros of  $\Delta(t)$ , since for all  $n$ ,  $\phi(-\pi, t_k) = \phi(\pi, t_k) \neq 0$ , and the left hand side also does not vanish because it is the norm of an eigenfunction.

The same discussion of [21, p. 15] can be applied for the real zeros of  $C(t)$  to get the last statement.  $\square$

### 3. Green’s function and eigenfunctions expansion

Green’s function of  $\Pi_n$  could be derived using the technique of Stakgold [25]. It appears while seeking solutions of the inhomogeneous boundary-value problem. Indeed, let  $f(\cdot)$  be continuous on  $[-\pi, \pi]$  and let  $g(x, \xi, t)$  be Green’s function of the problem

$$(3.1) \quad iy' - ty = f(x), \quad V(y) = 0, \quad t \in \mathbb{C}.$$

This means that any solution of (3.1) is

$$(3.2) \quad y = \int_{-\pi}^{\pi} g(x, \xi, t)f(\xi) d\xi,$$

where

$$(3.3) \quad g(x, \xi, t) = \frac{i}{1 - e^{2it\pi}} \begin{cases} e^{it(\xi-x+2\pi)}, & \xi \leq x, \\ e^{it(\xi-x)}, & x \leq \xi, \end{cases} \\ = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \frac{e^{-ikx} e^{ik\xi}}{k - t}.$$

We seek a solution of

$$(3.4) \quad iy' - ty + \sum_{k=1}^n r_k(x) \int_{-\pi}^{\pi} r_k(\tau) y(\tau) d\tau = f(x), \quad V(y) = 0,$$

in terms of Green's function  $g(x, \xi, t)$ . The following lemma is needed for the construction of Green's function.

**Lemma 3.1.** *Let  $t \in \mathbb{C}$ ,  $t \neq k$ ,  $k \in \mathbb{Z}$  and*

$$(3.5) \quad (A_t r_k)(x) := \int_{-\pi}^{\pi} g(x, \xi, t) r_k(\xi) d\xi, \quad b_{ij} = \delta_{ij} + \langle A_t r_j, r_i \rangle.$$

*If  $t$  is not an eigenvalue of  $\Pi_n$ , then*

$$(3.6) \quad B := \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix} \neq 0.$$

*Proof.* As we show in (3.1)–(3.2) the function  $(A_t r_k)(x)$  uniquely solves

$$(3.7) \quad iy' - ty = r_k(x), \quad V(y) = 0.$$

Assume that (3.6) is not true. Let  $\alpha_k$  be a non zero solution of the system

$$(3.8) \quad \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

We show that  $R_A(x, t)$  is an eigenfunction of (2.1)–(2.2), where  $R_A(x, t) = \sum_{k=1}^n \alpha_k (A_t r_k)(x)$ . Since each  $(A_t r_k)(x)$  satisfies (2.2), it remains to indicate that  $R_A(x, t)$  satisfies (2.1). Equation (3.8) leads to

$$\sum_{k=1}^n \left( \sum_{j=1}^n \alpha_j \langle A_t r_j, r_k \rangle + \alpha_k \right) r_k(x) = 0.$$

Equivalently,

$$\sum_{j=1}^n \alpha_j \left( t A_t r_j(x) + r_j(x) \right) + \sum_{k=1}^n \sum_{j=1}^n \alpha_j \langle A_t r_j, r_k \rangle r_k(x) = t \sum_{j=1}^n \alpha_j A_t r_j(x).$$

Since  $(A_t r_k)(x)$  satisfies (3.7), the last equation become

$$\ell_r(R_A(x, t)) = t R_A(x, t),$$

which is a contradiction. □



**Theorem 3.2.** *Let  $t$  be not an eigenvalue of  $\Pi_n$  and  $f(\cdot)$  be an  $L^2(-\pi, \pi)$ -function. Then the inhomogeneous boundary-value problem (3.4) has the unique solution*

$$y(x) = \int_{-\pi}^{\pi} G(x, \xi, t) f(\xi) d\xi,$$

where  $G(x, \xi, t)$  is Green's function of (3.4), which is given uniquely by (3.9)

$$G(x, \xi, t) := g(x, \xi, t) + \frac{1}{B} \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} & (A_t^* r_1)(\xi) \\ b_{21} & b_{22} & \dots & b_{2n} & (A_t^* r_2)(\xi) \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} & (A_t^* r_n)(\xi) \\ (A_t r_1)(x) & (A_t r_2)(x) & \dots & (A_t r_n)(x) & 0 \end{vmatrix},$$

where  $(A_t^* r_n)(\xi) := \int_{-\pi}^{\pi} g(x, \xi, t) r_n(x) dx$ .

*Proof.* Let  $t \in \mathbb{C}$  be neither an eigenvalue of  $\Pi$  nor an eigenvalue of  $\Pi_n$ . Since  $g(x, \xi, t)$  is Green's function of (3.1), then the solution of the problem (3.4) is given by

$$(3.10) \quad y(x) = (A_t f)(x) - \sum_{k=1}^n \langle y, r_k \rangle (A_t r_k)(x).$$

Again a direct application of [17, p. 66] will leads to

$$y(x) = (A_t f)(x) + \frac{1}{B} \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} & \langle A_t f, r_1 \rangle \\ b_{21} & b_{22} & \dots & b_{2n} & \langle A_t f, r_2 \rangle \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} & \langle A_t f, r_n \rangle \\ (A_t r_1)(x) & (A_t r_2)(x) & \dots & (A_t r_n)(x) & 0 \end{vmatrix}.$$

Using  $(A_t f)(x) = \int_{-\pi}^{\pi} g(x, \xi, t) f(\xi) d\xi$ , we obtain

$$\begin{aligned} \langle A_t f, r_k \rangle &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(x, \xi, t) f(\xi) r_k(x) d\xi dx \\ &= \int_{-\pi}^{\pi} f(\xi) \left( \int_{-\pi}^{\pi} g(x, \xi, t) r_k(x) dx \right) d\xi \\ &= \int_{-\pi}^{\pi} f(\xi) (A_t^* r_k)(\xi) d\xi, \end{aligned}$$

we get

$$(3.11) \quad y(x) = \int_{-\pi}^{\pi} G(x, \xi, t) f(\xi) d\xi.$$

We indicate that  $G(x, \xi, t)$  is defined and analytic at  $t = k$ ,  $k \in \mathbb{Z}$  if  $k$  is an eigenvalue of  $\Pi$  and not an eigenvalue of  $\Pi_n$ . Indeed, from (3.3) we can find a

neighborhood of  $t = k \in \mathbb{Z}$ ,  $D_k$ , such that

$$g(x, \xi, t) = \frac{e^{-ikx} e^{ik\xi}}{2\pi(k-t)} + g_0(x, \xi, t), \quad t \in D_k^* = D_k - \{k\},$$

where  $g_0(x, \xi, t) := g_0$  is regular in  $D_k$ . Let  $t \in D_k^*$ , and

$$g_l := g_l(x, t) = \int_{-\pi}^{\pi} g_0 r_l(\xi) d\xi, \quad g_l^* := \int_{-\pi}^{\pi} g_0 r_l(x) dx,$$

$$g_{jl} := g_{jl}(t) = \delta_{jl} + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_0 r_l(\xi) r_j(x) d\xi dx, \quad \Omega_l = \int_{-\pi}^{\pi} e^{-ikx} r_l(x) dx.$$

Thus

$$(3.12) \quad (A_t r_l)(x) = g_l - \frac{e^{-ikx} \overline{\Omega}_l}{2\pi(t-k)}, \quad (A_t^* r_l)(\xi) = g_l^* - \frac{e^{ik\xi} \Omega_l}{2\pi(t-k)}, \quad b_{jl} = g_{jl} - \frac{\overline{\Omega}_l \Omega_j}{2\pi(t-k)}.$$

Substituting in (3.6) we obtain  $B = \frac{B_0(t)}{(2\pi(t-k))^n}$ , where

$$B_0(t) = \begin{vmatrix} 2\pi(t-k)g_{11} - |\Omega_1|^2 & 2\pi(t-k)g_{12} - \Omega_1 \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{1n} - \Omega_1 \overline{\Omega}_n \\ 2\pi(t-k)g_{21} - \Omega_2 \overline{\Omega}_1 & 2\pi(t-k)g_{22} - |\Omega_2|^2 & \cdots & 2\pi(t-k)g_{2n} - \Omega_2 \overline{\Omega}_n \\ \vdots & \vdots & \ddots & \vdots \\ 2\pi(t-k)g_{n1} - \Omega_n \overline{\Omega}_1 & 2\pi(t-k)g_{n2} - \Omega_n \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{nn} - |\Omega_n|^2 \end{vmatrix}.$$

Also the  $(n+1) \times (n+1)$  determinant in (3.9) will have the form  $\frac{A_0(x, \xi, t)}{(2\pi(t-k))^{n+1}}$ ,

where

$$A_0(x, \xi, t) = \begin{vmatrix} 2\pi(t-k)g_{11} - |\Omega_1|^2 & 2\pi(t-k)g_{12} - \Omega_1 \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{1n} - \Omega_1 \overline{\Omega}_n & 2\pi(t-k)g_1^* - e^{ik\xi} \Omega_1 \\ 2\pi(t-k)g_{21} - \Omega_2 \overline{\Omega}_1 & 2\pi(t-k)g_{22} - |\Omega_2|^2 & \cdots & 2\pi(t-k)g_{2n} - \Omega_2 \overline{\Omega}_n & 2\pi(t-k)g_2^* - e^{ik\xi} \Omega_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\pi(t-k)g_{n1} - \Omega_n \overline{\Omega}_1 & 2\pi(t-k)g_{n2} - \Omega_n \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{nn} - |\Omega_n|^2 & 2\pi(t-k)g_n^* - e^{ik\xi} \Omega_n \\ 2\pi(t-k)g_1 - e^{-ikx} \overline{\Omega}_1 & 2\pi(t-k)g_2 - e^{-ikx} \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_n - e^{-ikx} \overline{\Omega}_n & 0 \end{vmatrix}.$$

Substituting in (3.9) we obtain for  $t \in D_k^*$ ,

$$(3.13) \quad G(x, \xi, t) = g_0 + \frac{e^{-ikx} e^{ik\xi}}{2\pi(k-t)} + \frac{A_0(x, \xi, t)}{2\pi(t-k)B_0(t)}$$

$$= g_0 + \frac{C_0(x, \xi, t)}{2\pi(t-k)B_0(t)}, \quad C_0 := A_0(x, \xi, t) - e^{-ikx} e^{ik\xi} B_0(t).$$

Notice that from Lemma 3.1 above  $B_0 \neq 0$  if  $t \in \mathbb{C} - \mathbb{Z}$  is not an eigenvalue of  $\Pi_n$ . Using the properties of determinants, we can see that

$$C_0(x, \xi, t) = \begin{vmatrix} 2\pi(t-k)g_{11} - |\Omega_1|^2 & 2\pi(t-k)g_{12} - \Omega_1 \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{1n} - \Omega_1 \overline{\Omega}_n & 2\pi(t-k)g_1^* - e^{ik\xi} \Omega_1 \\ 2\pi(t-k)g_{21} - \Omega_2 \overline{\Omega}_1 & 2\pi(t-k)g_{22} - |\Omega_2|^2 & \cdots & 2\pi(t-k)g_{2n} - \Omega_2 \overline{\Omega}_n & 2\pi(t-k)g_2^* - e^{ik\xi} \Omega_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\pi(t-k)g_{n1} - \Omega_n \overline{\Omega}_1 & 2\pi(t-k)g_{n2} - \Omega_n \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_{nn} - |\Omega_n|^2 & 2\pi(t-k)g_n^* - e^{ik\xi} \Omega_n \\ 2\pi(t-k)g_1 - e^{-ikx} \overline{\Omega}_1 & 2\pi(t-k)g_2 - e^{-ikx} \overline{\Omega}_2 & \cdots & 2\pi(t-k)g_n - e^{-ikx} \overline{\Omega}_n & -e^{-ikx} e^{ik\xi} \end{vmatrix}.$$

Assume that  $\Omega_1 \neq 0$ . Using elementary operations on rows of  $B_0(t)$  and  $C_0(x, \xi, t)$  we get

$$B_0(t) = (2\pi(t-k))^{n-1} B_1(t) \text{ and } C_0(x, \xi, t) = (2\pi(t-k))^n C_1(x, \xi, t), \quad t \in D_k^*,$$

where

$$B_1(t) = \begin{pmatrix} 2\pi(t-k)g_{11} - |\Omega_1|^2 & 2\pi(t-k)g_{12} - \Omega_1\bar{\Omega}_2 & \cdots & 2\pi(t-k)g_{1n} - \Omega_1\bar{\Omega}_n \\ g_{21} - g_{11}\frac{\Omega_2}{\Omega_1} & g_{22} - g_{12}\frac{\Omega_2}{\Omega_1} & \cdots & g_{2n} - g_{1n}\frac{\Omega_2}{\Omega_1} \\ \vdots & \vdots & & \vdots \\ g_{n1} - g_{11}\frac{\Omega_n}{\Omega_1} & g_{n2} - g_{12}\frac{\Omega_n}{\Omega_1} & \cdots & g_{nn} - g_{1n}\frac{\Omega_n}{\Omega_1} \end{pmatrix},$$

and

$$C_1(x, \xi, t) = \begin{pmatrix} 2\pi(t-k)g_{11} - |\Omega_1|^2 & 2\pi(t-k)g_{12} - \Omega_1\bar{\Omega}_2 & \cdots & 2\pi(t-k)g_{1n} - \Omega_1\bar{\Omega}_n & 2\pi(t-k)g_1^* - e^{ik\xi}\Omega_1 \\ g_{21} - g_{11}\frac{\Omega_2}{\Omega_1} & g_{22} - g_{12}\frac{\Omega_2}{\Omega_1} & \cdots & g_{2n} - g_{1n}\frac{\Omega_2}{\Omega_1} & g_2^* - g_1^*\frac{\Omega_2}{\Omega_1} \\ \vdots & \vdots & & \vdots & \vdots \\ g_{n1} - g_{11}\frac{\Omega_n}{\Omega_1} & g_{n2} - g_{12}\frac{\Omega_n}{\Omega_1} & \cdots & g_{nn} - g_{1n}\frac{\Omega_n}{\Omega_1} & g_n^* - g_1^*\frac{\Omega_n}{\Omega_1} \\ g_1 - g_{11}\frac{e^{-ikx}}{\Omega_1} & g_2 - g_{12}\frac{e^{-ikx}}{\Omega_1} & \cdots & g_n - g_{1n}\frac{e^{-ikx}}{\Omega_1} & -g_1^*\frac{e^{-ikx}}{\Omega_1} \end{pmatrix}.$$

Substituting in (3.13), we get

$$(3.14) \quad G(x, \xi, t) = g_0(x, \xi, t) + \frac{C_1(x, \xi, t)}{B_1(t)}, \quad t \in D_k^*.$$

Therefore  $G(x, \xi, t)$  can be defined at  $t = k$  to be

$$(3.15) \quad G(x, \xi, k) := \lim_{t \rightarrow k} G(x, \xi, t) = g_0(x, \xi, k) + \frac{C_1(x, \xi, k)}{B_1(k)}.$$

If  $\Omega_1 = 0$  and  $\Omega_s \neq 0, 2 \leq s \leq n$ , we can make the above elementary operations after replacing the  $s^{th}$  row in  $B_0(t)$  and  $C_0(x, \xi, t)$  by the first row. If  $\Omega_j = 0$  for all  $j$ , then  $e^{-ikx}$  will be an eigenfunction of  $\Pi_n$  corresponding to  $t = k$ , contradicting the assumption.  $\square$

**Lemma 3.3.** (1) *The problem  $\Pi_n$  has infinitely many real eigenvalues with no finite limit point and the eigenfunctions corresponding to different eigenvalues are orthogonal. The set of eigenfunctions is an orthogonal basis of  $L^2(-\pi, \pi)$ .*

(2) *For a fixed  $t \in \mathbb{C}$ ,  $G(x, \xi, t)$  has the eigenfunction expansion*

$$(3.16) \quad G(x, \xi, t) = \sum_{k \in \mathbb{Z}} \frac{\phi_k(x)\bar{\phi}_k(\xi)}{t_k - t}, \quad t \neq t_k,$$

where  $\{t_k\}_{k \in \mathbb{Z}}$  are the eigenvalues and  $\{\phi_k(\cdot)\}_{k \in \mathbb{Z}}$  is a corresponding complete orthonormal set of eigenfunctions of  $\Pi_n$  and the convergence is in the  $L^2((-\pi, \pi) \times (-\pi, \pi))$ -norm.

*Proof.* (1) Assume first that zero is not an eigenvalue of  $\Pi_n$ . Set

$$G(x, \xi) := G(x, \xi, 0),$$

hence  $G(x, \xi)$  is the Green's function of  $\Pi_n$ , i.e., any solution of

$$(3.17) \quad iy + \sum_{k=1}^n r_k \langle y, r_k \rangle = f, \quad V(y) = 0,$$

is

$$(3.18) \quad y(x) = \int_{-\pi}^{\pi} G(x, \xi) f(\xi) d\xi.$$

Replacing  $f$  in (3.17) and (3.18) by  $ty$ , establishes the equivalence between  $\Pi_n$  and the Fredholm integral equation

$$(3.19) \quad y(x) = t \int_{-\pi}^{\pi} G(x, \xi) y(\xi) d\xi.$$

From (3.9) we notice that  $G(x, \xi)$  is symmetric. To prove the first statement it suffices to show that  $G(x, \xi)$  is closed. This can be easily proved using the method of Stakgold [25]. The orthogonality holds for eigenfunctions corresponding to different eigenvalues. As for those belong to the same eigenvalue, we use the Gram-Schmidt procedure.

If zero is an eigenvalue of  $\Pi_n$ , we replace the eigenvalue parameter  $t$  by  $t - c$ , where  $c$  is a constant different from all eigenvalues of  $\Pi_n$ . Hence the new problem has the same eigenfunctions but zero is not an eigenvalue.

(2) Since  $G(x, \xi)$  has the  $L^2$ -convergent expansion, cf. [13],

$$(3.20) \quad G(x, \xi) = \sum_{k \in \mathbb{Z}} \frac{\phi_k(x)\bar{\phi}_k(\xi)}{t_k},$$

and for  $t \in \mathbb{C}$ ,  $t \neq t_k$ ,  $\{t_k - t\}_{k \in \mathbb{Z}}$  are the eigenvalues of (3.4) with the eigenfunctions  $\{\phi_k(\cdot)\}_{k \in \mathbb{Z}}$  and  $G(x, \xi, t)$  is Green's function of (3.4), then

$$(3.21) \quad G(x, \xi, t) = \sum_{k \in \mathbb{Z}} \frac{\phi_k(x)\bar{\phi}_k(\xi)}{t_k - t}, \quad t \neq t_k,$$

where the convergence is in the  $L^2((-\pi, \pi) \times (-\pi, \pi))$ -norm. □

Finally we discuss the asymptotic behavior of the eigenvalues. We have

$$(3.22) \quad \begin{aligned} |\langle \varphi, r_k \rangle| &= \left| \int_{-\pi}^{\pi} e^{-ixt} r_k(x) dx \right| \leq \|r(\cdot)\|_{L^1(-\pi, \pi)}, \\ |P_k(x, t)| &= \left| i e^{-ixt} \int_{-\pi}^x e^{-i\tau t} r_k(\tau) d\tau \right| \leq \|r_k(\cdot)\|_{L^1(-\pi, \pi)}, \\ |\langle P_k, r_j \rangle| &\leq \|r_k(\cdot)\|_{L^1(-\pi, \pi)} \|r_j(\cdot)\|_{L^1(-\pi, \pi)}. \end{aligned}$$

Since  $\lim_{t \rightarrow \pm\infty} C(t) = 1$ , there exists a positive number  $L$  such that  $|C(t)| > 1/2$ ,  $|t| > L$  ( $t$  is real). From (2.9), (2.16) and (3.22) we get

$$(3.23) \quad \Delta(t) = -2i \sin \pi t + o(1) \quad \text{as } t \rightarrow \pm\infty.$$

The  $o$ -term tends to zero as  $t \rightarrow \pm\infty$ . Since (3.23) has infinite number of zeros, the zeros of  $\Delta(t)$  satisfies

$$(3.24) \quad \sin \pi t = o(1), \quad \text{as } t \rightarrow \pm\infty.$$

Therefore the eigenvalues for large  $|t|$  has the asymptote

$$(3.25) \quad t_n \sim n \quad \text{as } n \rightarrow \pm\infty.$$

In the way from (2.9) we can get

$$(3.26) \quad |\phi(x, t) - \varphi(x, t)| = o(1).$$

Therefore,

$$(3.27) \quad \phi(x, t) \sim \varphi(x, t) \quad \text{as } t \rightarrow \pm\infty.$$

#### 4. Perturbed sampling theorems

According to the spectral analysis of the previous sections, the sampling analysis associated with  $\Pi_n$  will be divided into two cases, i.e., when  $C(t)$  has real zeros or not. Assume that  $\{t_k\}_{k \in \mathbb{Z}}$  denotes the set of all eigenvalues of  $\Pi_n$ , which are all simple. Although when  $C(t)$  has no real zeros the function  $\phi$  in (2.9) will generate all the eigenfunctions,  $\phi$  may be not defined for all  $t \in \mathbb{C}$ , because  $C(t)$  may have complex zeros. So in this case we assume that the eigenvalue parameter  $t$  is real.

**Theorem 4.1.** *Assume that  $C(t) \neq 0$  for all  $t \in \mathbb{R}$ ,  $g(\cdot) \in L^2(-\pi, \pi)$  and*

$$(4.1) \quad f(t) = \int_{-\pi}^{\pi} g(x) \phi(x, t) dx, \quad t \in \mathbb{R}.$$

*Then  $f(t)$  is an entire function which can be reconstructed via the sampling series*

$$(4.2) \quad f(t) = \sum_{k \in \mathbb{Z}} f(t_k) \frac{\Delta(t)}{(t - t_k) \Delta'(t_k)}.$$

*The sampling series (4.2) converges absolutely and uniformly on  $\mathbb{R}$ .*

*Proof.* Since  $\{\phi(\cdot, t_k)\}_{k \in \mathbb{Z}}$  is a complete orthogonal set of  $L^2(-\pi, \pi)$ , then applying Parseval's relation to (4.1) leads to

$$(4.3) \quad f(t) = \sum_{k \in \mathbb{Z}} \frac{\langle g(\cdot), \phi(\cdot, t_k) \rangle \langle \phi(\cdot, t), \phi(\cdot, t_k) \rangle}{\|\phi(\cdot, t_k)\|^2} = \sum_{k \in \mathbb{Z}} \frac{f(t_k) \langle \phi(\cdot, t), \phi(\cdot, t_k) \rangle}{\|\phi(\cdot, t_k)\|^2}.$$

Combining (2.21)–(4.3) we obtain (4.2) with a pointwise convergence on  $\mathbb{R}$ . It remains to prove the uniform convergence of (4.2). Let

$$\hat{\phi}(\cdot, t_k) = \phi(\cdot, t_k) / \|\phi(\cdot, t_k)\|.$$

We use (4.3) and the Cauchy-Schwartz' inequality to obtain

$$\left| f(t) - \sum_{|k| \leq N} \langle g, \hat{\phi}_k \rangle \langle \phi, \hat{\phi}_k \rangle \right| \leq \left[ \sum_{|k| > N} |\langle g, \hat{\phi}_k \rangle|^2 \right]^{\frac{1}{2}} \left[ \sum_{|k| > N} |\langle \phi, \hat{\phi}_k \rangle|^2 \right]^{\frac{1}{2}}.$$

But, in view of Bessel's inequality

$$\sum_{|k| > N} |\langle g, \hat{\phi}_k \rangle|^2 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

Again from Bessel's inequality, we have

$$\sum_{|k| > N} |\langle \phi, \hat{\phi}_k \rangle|^2 \leq \|\phi(x, t)\|^2.$$

We will show that  $\|\phi(x, t)\|^2$  is bounded on  $\mathbb{R}$ . From (2.5) and (2.9) it is enough to show that  $1/C(t)$  is bounded. Since  $\lim_{t \rightarrow \pm\infty} C(t) = 1$ , there exists a positive number  $L$  such that  $|C(t)| > 1/2, |t| > L$ . Since  $|C(t)|$  is continuous on  $[-L, L]$ , it assumes its minimum  $m$  on this interval and clearly  $m > 0$ . Hence  $1/|C(t)| \geq M, M = \max\{1/m, 2\}$  for all  $t \in \mathbb{R}$ .  $\square$

In the following we study the sampling problem associated with  $\Pi_n$  when  $C(t)$  has real zeros. In this case we have multiple eigenvalues. Therefore, the sampling problem may be treated by extending the ideas of [3, 14, 15] to the case of multiple eigenvalues with multiplicity  $\geq 2$ , or by the use of Green’s function. Here we use the last technique. We introduce a sampling theorem associated with Green’s function of problem  $\Pi_n$  above.

Since an eigenvalue  $t_k$  may have more than one eigenfunction, then expansion (3.16) may have the form

$$(4.4) \quad G(x, \xi, t) = \sum_{k \in \mathbb{Z}} \sum_{\nu=1}^{\nu_k} \frac{\phi_{k,\nu}(x) \overline{\phi_{k,\nu}(\xi)}}{t_k - t}, \quad t \neq t_k,$$

where  $\nu_k$  is the multiplicity of  $t_k$ , i.e.,  $1 \leq \nu_k \leq k - \nu + 1$ . Let  $\xi_0 \in [-\pi, \pi]$  such that  $\phi_k(\xi_0) \neq 0$  for all  $k$ . Such an  $\xi_0$  exists since an eigenfunction may vanish only on a subset of measure zero of  $[-\pi, \pi]$ . Define the function  $G_0(x, t) \in L^2(-\pi, \pi)$  to be

$$(4.5) \quad G_0(x, t) := G(x, \xi_0, t).$$

Since  $\{\phi_k(\cdot)\}_{k \in \mathbb{Z}}$  is a complete orthonormal set of  $L^2(-\pi, \pi)$ , (4.5) can be viewed as the Fourier expansion of  $G_0(x, t)$  with the Fourier coefficients  $\frac{\phi_k(\xi_0)}{t_k - t}, t \neq t_k$ . Also,  $G_0(x, t)$  is a meromorphic function with simple poles  $t_k$ . The residue at each pole  $t_k$  is

$$(4.6) \quad r_k = \sum_{\nu=1}^{\nu_k} \phi_{k,\nu}(x) \overline{\phi_{k,\nu}(\xi_0)}.$$

Define for  $t \in \mathbb{C}$  the following product

$$(4.7) \quad \omega(t) = \begin{cases} t \prod_{n=-\infty, n \neq 0}^{\infty} \left(1 - \frac{t}{t_n}\right) \exp(t/t_n), & \text{if } t_0 = 0 \text{ is an eigenvalue,} \\ \prod_{n=-\infty}^{\infty} \left(1 - \frac{t}{t_n}\right) \exp(t/t_n), & \text{if zero is not an eigenvalue,} \end{cases}$$

which is convergent since the eigenvalues  $\{t_k\}_{k \in \mathbb{Z}}$  have the asymptotes (3.25). We may omit the factor  $\exp(t/t_n)$  if the product is convergent without it. Define the function  $\Phi(\cdot, t)$  to be

$$(4.8) \quad \Phi(x, t) := \omega(t) G_0(x, t), \quad t \in \mathbb{C}.$$

It is an entire function of  $t$  for each fixed  $x$ . The second sampling theorem of this paper is the following. It gives another perturbed WKS sampling theorem, which is a perturbed form of the WKS’s theorem as derived by Haddad et al. [18].

**Theorem 4.2.** *Let  $g \in L^2(-\pi, \pi)$  and*

$$(4.9) \quad F(t) = \int_{-\pi}^{\pi} \bar{g}(x)\Phi(x, t) dx, \quad t \in \mathbb{C}.$$

*Then  $F(t)$  is an entire function that admits the sampling representation*

$$(4.10) \quad F(t) = \sum_{k \in \mathbb{Z}} F(t_k) \frac{\omega(t)}{(t - t_k)\omega'(t_k)}.$$

*The sampling series (4.10) is absolutely uniformly convergent on any compact subset of  $\mathbb{C}$  and uniformly on  $\mathbb{R}$ .*

*Proof.* Since both  $g$  and  $\Phi$  are  $L^2$ -functions and  $\{\phi_k(\cdot)\}_{k \in \mathbb{Z}}$  is a complete orthonormal set in  $L^2(-\pi, \pi)$ , then

$$(4.11) \quad g(x) = \sum_{k \in \mathbb{Z}} \langle g, \phi_k \rangle \phi_k(x), \quad \Phi(x, t) = \sum_{k \in \mathbb{Z}} \langle \Phi, \phi_k \rangle \phi_k(x)$$

are the Fourier series of  $g$  and  $\Phi$ , respectively. Here  $\langle g, \phi_k \rangle$  and  $\langle \Phi, \phi_k \rangle$  are the Fourier coefficients. Using Parseval's identity, we get

$$(4.12) \quad F(t) = \sum_{k \in \mathbb{Z}} \overline{\langle g, \phi_k \rangle} \langle \Phi, \phi_k \rangle.$$

In the view of (4.4) above, equation (4.12) can be rewritten in the form

$$(4.13) \quad F(t) = \sum_{k \in \mathbb{Z}} \sum_{\nu=1}^{\nu_k} \overline{\langle g, \phi_{k,\nu} \rangle} \langle \Phi, \phi_{k,\nu} \rangle.$$

From the definition of  $\Phi$ , we obtain

$$(4.14) \quad \langle \Phi, \phi_{k,\nu} \rangle = \frac{\omega(t)}{t_k - t} \bar{\phi}_{k,\nu}(\xi_0).$$

Since

$$(4.15) \quad F(t) = \omega(t) \int_{-\pi}^{\pi} \bar{g}(x)H_0(x, t)dx$$

and  $H_0(x, t)$  has simple poles at the eigenvalues with the residues (4.6), then

$$(4.16) \quad \begin{aligned} F(t_k) &= \lim_{t \rightarrow t_k} \frac{\omega(t)}{t - t_k} \int_{-\pi}^{\pi} (t - t_k)\bar{g}(x)H_0(x, t) dx \\ &= -\omega'(t_k) \sum_{k=1}^{\nu_k} \bar{\phi}_{k,\nu}(\xi_0) \int_{-\pi}^{\pi} \bar{g}(x)\phi_{k,\nu}(x) dx \\ &= -\omega'(t_k) \sum_{k=1}^{\nu_k} \bar{\phi}_{k,\nu}(\xi_0) \overline{\langle g, \phi_{k,\nu} \rangle}. \end{aligned}$$

Substituting from (4.14) and (4.16) in (4.13), one gets (4.10).

The proof of uniform and absolute convergence on  $\mathbb{R}$  can be established with a slight modification of that of the previous theorem. As for uniform

and absolute convergence on  $\mathbb{C}$  we use the same arguments of the proof of the previous theorem and the identity of [13, p. 50] that guarantees the boundedness of  $\|\Phi(\cdot, t)\|$  on compact subsets of  $\mathbb{C}$ .  $\square$

### 5. Examples

**Example 5.1.** Consider the perturbed eigenvalue problem

$$(5.1) \quad iy' + \sum_{k=1}^2 r_k(x) \int_{-\pi}^{\pi} r_k(\tau) y(\tau) d\tau = ty, \quad y(\pi) - y(-\pi) = 0, \quad -\pi \leq x \leq \pi,$$

where

$$(5.2) \quad r_1(x) = \begin{cases} 1, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad r_2(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ 1, & 0 < x \leq \pi. \end{cases}$$

In the previous notations we have

$$(5.3) \quad P_1(x, t) = \begin{cases} \frac{1 - e^{-it(x+\pi)}}{t}, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad P_2(x, t) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \frac{1 - e^{-itx}}{t}, & 0 < x \leq \pi. \end{cases}$$

The function  $C(t)$  will be

$$(5.4) \quad C(t) = \frac{e^{-2i\pi t} (e^{i\pi t} [(\pi - t)t + i] - i)^2}{t^4},$$

which does not vanish for any  $t \in \mathbb{R}$ . Hence, cf. (2.9),

$$(5.5) \quad \phi(x, t) = e^{-itx} - \frac{(-1 + e^{i\pi t})t}{ie^{i\pi t}((\pi - t)t + i) + 1} \left( e^{i\pi t} P_1(x, t) + P_2(x, t) \right),$$

is the solution for any  $t \in \mathbb{R}$ , and the eigenvalues are the zeros of

$$(5.6) \quad \Delta(t) = (1 - e^{i\pi t}) \left( \frac{(\pi - t)t}{e^{i\pi t}((\pi - t)t + i) - i} + 1 \right).$$

Some of the eigenvalues can be computed explicitly and the other cannot be computed concretely. The eigenvalues are  $t_{2k} = 2k$ ,  $k \in \mathbb{Z}^* := \mathbb{Z} - \{0\}$ , with corresponding eigenfunctions  $\phi(x, 2k) = e^{-2ikx}$ , and the other eigenvalues  $t_{2k+1}$  are the zeros of

$$(5.7) \quad \tan \frac{\pi t}{2} = (\pi - t)t, \quad t \neq 0.$$

Note that the zeros of (5.7) have the asymptotes  $t_{2k+1} \sim 2k + 1$  as we expect from (3.25). Following Theorem 4.1 above, the transform

$$(5.8) \quad f(t) = \int_{-\pi}^{\pi} g(x) \left( e^{-itx} - \frac{(-1 + e^{i\pi t})t}{ie^{i\pi t}((\pi - t)t + i) + 1} \left( e^{i\pi t} P_1(x, t) + P_2(x, t) \right) \right) dx, \quad t \in \mathbb{R},$$



has the sampling form

$$(5.9) \quad f(t) = \sum_{k=0}^{\infty} f(t_{2k+1}) \frac{\Delta(t)}{(t - t_{2k+1})\Delta'(t_{2k+1})} + \sum_{k \in \mathbb{Z}^*} f(2k) \frac{\Delta(t)}{2i\pi(2k - t)}.$$

Now we illustrate the figure of perturbed transform (5.8) and the unperturbed one when  $g(x) = 1$ . We have

$$(5.10) \quad f_u(t) = \int_{-\pi}^{\pi} e^{-itx} dx = \frac{2 \sin t\pi}{t},$$

(5.11)

$$f_p(t) = \int_{-\pi}^{\pi} \left( e^{-itx} - \frac{(-1 + e^{i\pi t})t}{ie^{i\pi t}((\pi - t)t + i) + 1} \left( e^{i\pi t} P_1(x, t) + P_2(x, t) \right) \right) dx \\ = \frac{2 \sin t\pi}{t} + \frac{(e^{2i\pi t} - 1)(i(e^{-i\pi t} - 1) - \pi t)}{t(i e^{i\pi t}((\pi - t)t + i) + 1)}.$$

Figures below illustrate  $\Re f_p(t)$ ,  $\Re f_u(t)$  and  $\Im f_p(t)$ ,  $\Im f_u(t)$ , respectively. Notice that  $\Im f_u(t) \equiv 0$  on  $\mathbb{R}$  and  $\Im f_p(t)$  is very small. Also  $\Re f_u(t)$  and  $\Re f_p(t)$  eventually merges.

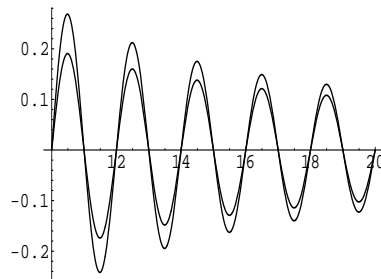


FIGURE 1.  $\Re f_p(t)$  and  $\Re f_u(t)$ , when  $10 < t < 20$ .

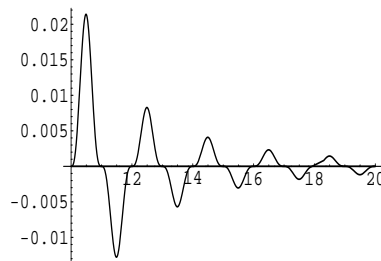


FIGURE 2.  $\Im f_p(t)$  and  $\Im f_u(t)$ , when  $10 < t < 20$ .

**Example 5.2.** Consider the problem (5.1) with

$$(5.12) \quad r_1(x) = \begin{cases} 1, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad r_2(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \sqrt{\frac{2}{\pi}}, & 0 < x \leq \pi. \end{cases}$$

After some computations we have

$$(5.13) \quad P_1(x, t) = \begin{cases} \frac{1-e^{-it(x+\pi)}}{t}, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad P_2(x, t) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \sqrt{\frac{2}{\pi}} \frac{1-e^{-itx}}{t}, & 0 < x \leq \pi. \end{cases}$$

The function  $C(t)$  will be

$$(5.14) \quad C(t) = \det(a_{ij})_{1 \leq i, j \leq 2} = \begin{vmatrix} 1 - \frac{i-ie^{-i\pi t} + \pi t}{t^2} & 0 \\ 0 & 1 - \frac{2(i(1-e^{-i\pi t}) + \pi t)}{\pi t^2} \end{vmatrix} \\ = -\frac{e^{-2i\pi t} (e^{i\pi t}((\pi - t)t + i) - i) (e^{i\pi t}(\pi(t - 2)t - 2i) + 2i)}{\pi t^4}.$$

The function  $C(t)$  has a unique real zero at  $t = 2$ . From (5.14) the rank of  $(a_{ij})_{1 \leq i, j \leq 2}$ , cf. (2.8) is one when  $t = 2$ , it is a simple eigenvalue as zero of  $C(t)$ . Obviously  $\langle \varphi(x, 2), r_1(x) \rangle = \langle \varphi(x, 2), r_2(x) \rangle = 0$ . Since  $t = 2$  is also an eigenvalue of the problem  $\Pi$ , then it is a double eigenvalue of the problem and the corresponding eigenfunctions are

$$P_2(x, 2) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \frac{1-e^{-2ix}}{\sqrt{2\pi}}, & 0 < x \leq \pi, \end{cases} \quad \varphi(x, 2) = e^{-2ix}.$$

For  $t \neq 2$ , the solution of the integro-differential equation is

$$(5.15) \quad \phi(x, t) = e^{-itx} + t(-1 + e^{i\pi t}) \\ \times \frac{e^{i\pi t} [-2 + e^{i\pi t}(i\pi(t-2)t + 2)] P_1(x, t) + \sqrt{2\pi} [-1 + e^{i\pi t}(1 - i(\pi - t)t)] P_2(x, t)}{(e^{i\pi t}((\pi - t)t + i) - i)(e^{i\pi t}(\pi(t - 2)t - 2i) + 2i)},$$

and the rest of the eigenvalues are the zeros of

$$(5.16) \quad \Delta(t) = (1 - e^{i\pi t}) \left( \frac{\pi(t - 2)t}{e^{i\pi t}(\pi(t - 2)t - 2i) + 2i} + 1 \right).$$

Also some of the eigenvalues are  $t_{2k} = 2k$ ,  $k \in \mathbb{Z}^* := \mathbb{Z} - \{0\}$ , with corresponding eigenfunctions  $\varphi(x, 2k) = e^{-2ikx}$ . The other eigenvalues  $t_{2k+1}$  are the zeros of

$$(5.17) \quad \tan \frac{\pi t}{2} = \pi \left(1 - \frac{t}{2}\right)t, \quad t \neq 0, 2.$$

Here we have

$$(5.18) \quad (A_t r_1)(x) = \frac{1}{t(1 + e^{it\pi})} \begin{cases} e^{-itx} - (1 + e^{it\pi}), & -\pi \leq x \leq 0, \\ -e^{it(\pi-x)}, & 0 < x \leq \pi, \end{cases}$$

$$(5.19) \quad (A_t r_2)(x) = \frac{\sqrt{\frac{2}{\pi}}}{t(1 + e^{it\pi})} \begin{cases} -e^{-itx}, & -\pi \leq x \leq 0, \\ e^{it(\pi-x)} - (1 + e^{it\pi}), & 0 < x \leq \pi. \end{cases}$$

Thus  $G(x, \xi, t)$  can be computed from (3.9). Hence if  $\xi_0$  is chosen in  $[-\pi, \pi]$  and  $\Phi(x, t)$  is as in (4.8) above, then the transform

$$F(t) = \int_0^1 \bar{g}(x)\Phi(x, t) dx, \quad t \in \mathbb{C}, \quad g(\cdot) \in L^2(-\pi, \pi),$$

admits the sampling representation

$$(5.20) \quad f(t) = \sum_{k \in \mathbb{Z}} f(t_{2k+1}) \frac{\omega(t)}{(t - t_{2k+1})\omega'(t_{2k+1})} + \sum_{k \in \mathbb{Z}^*} f(2k) \frac{\omega(t)}{2i\pi(2k - t)}.$$

**Example 5.3.** Consider the problem (5.1) with

$$(5.21) \quad r_1(x) = \begin{cases} \sqrt{\frac{2}{\pi}}, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad r_2(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \sqrt{\frac{2}{\pi}}, & 0 < x \leq \pi. \end{cases}$$

We have the following

$$(5.22) \quad P_1(x, t) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1 - e^{-it(x+\pi)}}{t}, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \quad P_2(x, t) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \sqrt{\frac{2}{\pi}} \frac{1 - e^{-itx}}{t}, & 0 < x \leq \pi. \end{cases}$$

The function  $C(t)$  will be

$$(5.23) \quad C(t) = \det(a_{ij})_{1 \leq i, j \leq 2} = \begin{vmatrix} 1 - \frac{2(i(1 - e^{-i\pi t}) + \pi t)}{\pi t^2} & 0 \\ 0 & 1 - \frac{2(i(1 - e^{-i\pi t}) + \pi t)}{\pi t^2} \end{vmatrix} \\ = \frac{e^{-2i\pi t} (e^{i\pi t} (\pi(t - 2)t - 2i) + 2i)^2}{\pi^2 t^4},$$

The function  $C(t)$  has a unique zero at  $t = 2$ . Since from (5.23) the rank of  $(a_{ij})_{1 \leq i, j \leq 2}$  is two when  $t = 2$ , it is a double eigenvalue as a zero of  $C(t)$ . Obviously  $\langle \varphi(x, 2), r_1(x) \rangle = \langle \varphi(x, 2), r_2(x) \rangle = 0$ . Since  $t = 2$  is also an eigenvalue of the problem  $\Pi$ , then it is a triple eigenvalue of the problem and the corresponding eigenfunctions are

$$(5.24) \quad P_1(x, 2) = \begin{cases} \frac{1 - e^{-2ix}}{\sqrt{2\pi}}, & -\pi \leq x \leq 0, \\ 0, & 0 < x \leq \pi, \end{cases} \\ P_2(x, 2) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ \frac{1 - e^{-2ix}}{\sqrt{2\pi}}, & 0 < x \leq \pi, \end{cases} \\ \varphi(x, 2) = e^{-2ix}.$$

If  $t \neq 2$  the solution of the integro-differential equation here is

$$(5.25) \quad \phi(x, t) = e^{-itx} - \frac{(-1 + e^{i\pi t})\sqrt{2\pi t}}{e^{i\pi t}(-i\pi(t-2)t-2) + 2} \left( e^{i\pi t} P_1(x, t) + P_2(x, t) \right),$$

with

$$(5.26) \quad \Delta(t) = (1 - e^{i\pi t}) \left( \frac{\pi(t-2)t}{e^{i\pi t}(\pi(t-2)t-2i) + 2i} + 1 \right).$$

Again some of the eigenvalues are  $t_{2k} = 2k$ ,  $k \in \mathbb{Z}^* := \mathbb{Z} - \{0\}$ , and the rest are the zeros of (5.17). Here we have

$$(5.27) \quad (A_t r_1)(x) = \frac{\sqrt{\frac{2}{\pi}}}{t(1 + e^{it\pi})} \begin{cases} e^{-itx} - (1 + e^{it\pi}), & -\pi \leq x \leq 0, \\ -e^{it(\pi-x)}, & 0 < x \leq \pi, \end{cases}$$

and

$$(5.28) \quad (A_t r_2)(x) = \frac{\sqrt{\frac{2}{\pi}}}{t(1 + e^{it\pi})} \begin{cases} -e^{-itx}, & -\pi \leq x \leq 0, \\ e^{it(\pi-x)} - (1 + e^{it\pi}), & 0 < x \leq \pi. \end{cases}$$

Also  $G(x, \xi, t)$  can be computed from (3.9). If  $\xi_0 \in [-\pi, \pi]$  and  $\Phi(x, t)$  is as in (4.8) above, then the transform

$$F(t) = \int_0^1 \bar{g}(x)\Phi(x, t) dx, \quad t \in \mathbb{C}, \quad g(\cdot) \in L^2(-\pi, \pi),$$

admits the sampling representation

$$(5.29) \quad f(t) = \sum_{k \in \mathbb{Z}} f(t_{2k+1}) \frac{\omega(t)}{(t - t_{2k+1})\omega'(t_{2k+1})} + \sum_{k \in \mathbb{Z}^*} f(2k) \frac{\omega(t)}{2i\pi(2k - t)}.$$

**Example 5.4.** Here we consider a problem of rank one, which is

$$(5.30) \quad iy' + \frac{1}{2\pi} \int_{-\pi}^{\pi} y(\tau) d\tau = ty, \quad y(\pi) - y(-\pi) = 0.$$

In this case we have,

$$(5.31) \quad P(x, t) = \frac{1}{\sqrt{2\pi}} \left( \frac{1 - e^{-it(x+\pi)}}{t} \right), \quad C(t) = 1 + \frac{i}{2\pi t^2} (2it\pi + e^{-2it\pi} - 1),$$

$$(5.32) \quad \phi(x, t) = e^{-itx} + \frac{\sin t\pi(1 - e^{-it(x+\pi)})}{\pi t^2 C(t)}, \quad \Delta(t) = -2i \sin t\pi \left( 1 + \frac{i(1 - e^{-2it\pi})}{2\pi t^2 C(t)} \right).$$

One sees that the eigenvalues are the zeros of  $\Delta(t)$ , which are  $t_n = n$ ,  $n \in \mathbb{Z}^*$  with the corresponding eigenfunctions  $\{e^{-in x}\}_{n \in \mathbb{Z}^*}$  and  $t = 1$ , the only real zero of  $C(t)$ , with corresponding eigenfunctions  $\phi(x, 1) = e^{-ix}$  and  $P(x, 1) = \frac{1 - e^{-i(x+\pi)}}{\sqrt{2\pi}}$ . This means that  $t = 1$  is a double eigenvalue of the problem. Also

$$(5.33) \quad \omega(t) = \prod_{n=-\infty, n \neq 0}^{\infty} \left( 1 - \frac{t}{t_n} \right) = \frac{\sin \pi t}{\pi t}, \quad \omega'(t_n) = \frac{(-1)^n}{n}.$$

The transform defined as in Theorem 4.2 above has the sampling representation

$$(5.34) \quad F(t) = \sum_{n \in \mathbb{Z}^*} F(n) \frac{n \sin \pi(t-n)}{\pi t(t-n)} \quad t \in \mathbb{C}.$$

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