

Existence and Stability Results on Nonlinear Delay Integro-Differential Equations with Random Impulses

ARUMUGAM VINODKUMAR*

Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham University, Coimbatore-641 112, Tamil Nadu, India
e-mail: vinod026@gmail.com, a_vinodkumar@cb.amrita.edu

MUTHUSAMY GOWRISANKAR

Department of Mathematics, Annapoorana Engineering College, Salem-636308, Tamil Nadu, India
e-mail: tamilangowri@gmail.com

PRATHIBAN MOHANKUMAR

Department of Mathematics, A. C. T College of Engineering and Technology, Kancheepuram- 603 107, Tamil Nadu, India
e-mail: profmohankumar@yahoo.co.in

ABSTRACT. In this paper, the existence, uniqueness, stability via continuous dependence and Ulam stabilities of nonlinear integro-differential equations with random impulses are studied under sufficient condition. The results are obtained by using Leray-Schauder alternative fixed point theorem and Banach contraction principle.

1. Introduction

Mathematical modelling of real-life problems in many engineering and scientific disciplines usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics. For details, see [1, 2, 9, 32] and the references therein.

Impulsive differential equations are well known to model problems from many areas of science and engineering. There has been much research activity concerning

* Corresponding Author.

Received July 9, 2014; revised October 29, 2014; accepted February 21, 2015.

2010 Mathematics Subject Classification: 45J05, 34D20, 34A12, 60H99.

Key words and phrases: Delay integro-differential equations, random impulses, Leray-Schauder alternative fixed point theorem, contraction Principle.

the theory of impulsive differential equations see [13, 15]. The impulses may exist at deterministic or random points. There are a lot of papers which investigate the properties of deterministic impulses see [3, 10, 13, 15] and the references therein.

When the impulses exist at random, the solutions of the equation behave as a stochastic process. It is quite different from deterministic impulsive differential equations and stochastic differential equations. Iwankiewicz et al [12], investigated dynamic response of non-linear systems to poisson distributed random impulses. Tatsuyuki et al [17] presented a mathematical model of random impulse to depict drift motion of granules in charya cells due to myosin-actin interaction. In [16], Sanz-Serna et al first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. Wu and Meng first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by the Liapunov's direct method in [27]. In [28], Wu and Duan discussed oscillation, stability and boundedness of solutions to the model by comparing the solutions of this system with the corresponding non-impulsive differential system. In [29], Wu, Guo and Lin discussed the existence and uniqueness in mean square of solutions to certain random impulsive differential systems employing Cauchy-Schwarz inequality, Lipschitz condition and techniques in stochastic analysis. In [30], Wu, Guo and Zhou first brought forward random impulsive functional differential equations and considered p -moment stability of solutions to these models using Liapunov's function coupled with Razumikhin technique. Then, Wu, Guo and Zhai considered almost sure stability of solutions to random impulsive functional differential equations by Liapunov's function coupled with Razumikhin technique in [31]. In [4], the author studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [5] through Banach fixed point method for the system of differential equations with random impulsive effect. The author [6], studied the existence results for the random impulsive neutral functional differential equations with delays. In [20, 21], the author studied existence results of random impulsive differential inclusions with delays via fixed point theory. In [33], the authors generalized the distribution of random impulses with the Erlang distribution. Using the Erlang distribution in [19, 8], the authors studied qualitative behavior of the random impulsive semilinear differential equations and neutral functional differential equations.

The stabilities like continuous dependence, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, exponential stability and asymptotic stability have attracted the attention of many mathematicians (see [7, 11, 18, 22, 23, 26, 24, 25, 14] and the references therein). In [22], the authors have given the Ulam's type stability and data dependence for fractional differential equations (FDEs). JinRong Wang et al. [23] studied stability of FDEs using fixed point theorem in a generalized complete metric space. In [24], JinRong Wang et al. studied Ulam's stability for the nonlinear impulsive FDEs.

Motivated by the above mentioned works, the main purpose of this paper is

to study of random impulsive delay integro differential equations. We relaxed the Lipschitz condition on the impulsive term and under our assumption it is enough to be bounded. We extend the results of Hyers-Ulam stability and Hyers- Ulam-Rassias stability to fill the gab in delay integro differential equation. We utilize the technique developed [13, 15, 29, 14].

The paper will be organized as follows: In section 2, we recall briefly the notations, definitions and preliminary facts which are used throughout this paper. In section 3, we investigate the existence of solutions of nonlinear delay integro-differential equations with random impulses by using Leray - Schauder alternative fixed point theory. An interesting feature of this method is that this yields simultaneously the existence and maximal interval of existence and further we investigated the existence and uniqueness of solutions of random impulsive nonlinear delay integro-differential equations by relaxing the linear growth condition. In section 4, we study the stability through continuous dependence on initial conditions of random impulsive nonlinear delay integro-differential equations. The Hyers- Ulam stability and Hyers Ulam-Rassias stability of the solutions of nonlinear delay integro-differential differential systems is investigated in section 5.

2. Preliminaries

Let \mathbb{R}^n be the n -dimensional Euclidean space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \stackrel{def.}{=} (0, d_k)$ for $k = 1, 2, \dots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_k follow Erlang distribution, where $k = 1, 2, \dots$ and let τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \dots$. For the sake of simplicity, we denote $\mathbb{R}_\tau = [\tau, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$.

We consider nonlinear delay integro-differential equation with random impulses of the form

$$(2.1) \quad \begin{cases} x'(t) &= \int_0^t F(t, s, x(\sigma(s))) ds, t \neq \xi_k, \quad t \geq \tau, \\ x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \dots, \\ x_{t_0} &= \varphi \end{cases}$$

where the functional $F : \Delta \times \mathcal{C} \rightarrow \mathbb{R}^n$, $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$ is the set of piecewise continuous functions mapping $[-r, 0]$ in to \mathbb{R}^n with some given $r > 0$; $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \dots$. Here $t_0 \in R_\tau$ is an arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \dots < \lim_{k \rightarrow \infty} \xi_k = \infty$; $b_k : D_k \rightarrow R^{n \times n}$ is a matrix-valued function for each $k = 1, 2, \dots$; $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{t-r \leq s \leq t} |x(s)|$ for each t satisfying $\tau \leq t \leq T$ $\|\cdot\|$ is any given norm in X , here Δ denotes the set $\{(t, s) : 0 \leq s \leq t < \infty\}$.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote \mathcal{F}_t the σ -algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_p = L_p(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ denote the Banach space

of all \mathcal{F}_t - measurable p^{th} integrable random variables with values in \mathbb{R}^n .

Assume that $T > t_0$ is any fixed time to be determined later and let \mathcal{B} denote the Banach space $\mathcal{B}([t_0 - r, T], L_p)$, the family of all \mathcal{F}_t -measurable, \mathbb{C} -valued random variables ψ with the norm

$$\|\psi\|_{\mathcal{B}} = \left(\sup_{t_0 \leq t \leq T} E \|\psi\|_t^p \right)^{1/p}.$$

Let $L_p^0(\Omega, \mathcal{B})$ denote the family of all \mathcal{F}_0 - measurable, \mathcal{B} - valued random variable φ .

Definition 2.1. A map $F(t, s, x) : \Delta \times \mathcal{C} \rightarrow X$, for all $t \in [\tau, T]$, $F(t, \cdot, \cdot)$ satisfies \mathcal{L}^p -Caratheodory, if

- (i) $s \rightarrow F(t, s, x)$ is measurable for each $x \in \mathcal{C}$;
- (ii) $x \rightarrow F(t, s, x)$ is continuous for almost all $t \in [\tau, T]$;
- (iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1([\tau, T], \mathbb{R}^+)$ such that $\sup_{E\|x\|^p \leq m} E \|F(t, s, x)\|^p \leq \alpha_m(t)$, for $t \in [\tau, T]$, a.e.

Definition 2.2. For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}, t_0 - r \leq t \leq T\}$ is called a solutions to equation (2.1) in $(\Omega, P, \{\mathcal{F}_t\})$, if

- (i) $x(t) \in \mathbb{R}^n$ is \mathcal{F}_t -adapted for $t \geq t_0$;
- (ii) $x(t_0 + s) = \varphi(s) \in L_2^0(\Omega, F)$ when $s \in [-r, 0]$

$$(2.2) \quad \begin{aligned} x(t) = & \sum_{k=0}^{+\infty} \left(\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right. \\ & \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right) I_{[\xi_k, \xi_{k+1})}(t), t \in [t_0, T] \end{aligned}$$

, where $\prod_{j=m}^n (\cdot) = 1$ as $m > n$, $\prod_{j=i}^k b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i)$, and $I_A(\cdot)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Our existence theorem is based on the following theorem, which is a version of the topological transversality theorem.

Theorem 2.1. Let B be a convex subset of a Banach space E and assume that $0 \in B$. Let $F : B \rightarrow B$ be a completely continuous operator and let

$U(F) = \{x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$;
then either $U(F)$ is unbounded or F has a fixed point.

3. Existence Results

In this section, we prove the existence theorem by using the following hypothesis.
(H_1): The function $F : [t_0, T] \times [t_0, T] \times \mathcal{C} \rightarrow \mathfrak{R}^n$ is continuous, $F(t, s, 0) = 0$, and it satisfies the Lipschitz continuous with respect to x , ie.,

$$E\|F(t, s, x_1) - F(t, s, x_2)\|^p \leq L(t, s, E\|x_1\|^p, E\|x_2\|^p)E\|x_1 - x_2\|_s^p, \\ (t, s) \in \Delta, x_1, x_2 \in \mathfrak{R}^n,$$

where $L : [t_0, T] \times [t_0, T] \times \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$ and is monotonically nondecreasing with respect to the second and third arguments.

(H_2): There exists a continuous function $p : [t_0, T] \times [t_0, T] \rightarrow (0, \infty)$ such that

$$E\|F(t, s, x)\|^p \leq p(t, s)H(E\|x\|_s^p), \quad (t, s) \in \Delta, x \in \mathfrak{R}^n,$$

where $H : \mathfrak{R}^+ \rightarrow (0, \infty)$ is a continuous nondecreasing function.

(H_3): $\sigma : [t_0, T] \rightarrow [t_0, T]$, is a continuous functions such that $\sigma(t) \leq t$.

(H_4): $E\left\{\max_{i,k} \prod_{j=i}^k \|b_j(\tau_j)\|\right\}$ is uniformly bounded that there is $c > 0$ such that

$$E\left\{\max_{i,k} \prod_{j=i}^k \|b_j(\tau_j)\|\right\} \leq C \text{ for all } \tau_j \in D_j, \quad j = 1, 2, \dots.$$

Theorem 3.1. *If the hypothesis (H_2) – (H_4) hold, then system (2.1) has a solution $x(t)$, defined on $[t_0, T]$ provided that the following inequality is satisfied*

$$(3.1) \quad M_1 \int_{t_0}^T p(s, s) ds < \int_{c_1}^{\infty} \frac{ds}{H(s)},$$

where $M_1 = 2^{p-1} \max\{1, C^p\}(T - t_0)^2$, $c_1 = 2^{p-1} C^p E\|\varphi\|^p$ and $C^p \geq \frac{1}{2^{p-1}}$.

Proof. Let T be an arbitrary number $t_0 < T < +\infty$ satisfying (3.1). We transform the problem (2.1) into a fixed point problem. We consider the operator $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ defined by

$$\Phi x(t) = \begin{cases} \varphi(t - t_0), & t \in [t_0 - r, t_0], \\ \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right. \\ \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t), & t \in [t_0, T]. \end{cases}$$

In order to use the transversality theorem, first we establish the priori estimates for the solutions of the integral equation and $\lambda \in (0, 1)$,

$$x(t) = \begin{cases} \lambda \varphi(t - t_0), & t \in [t_0 - r, t_0], \\ \lambda \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ \left. + \int_{\xi_k}^t \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t), & t \in [t_0, T], \end{cases}$$

Thus by $(H_2) - (H_4)$, we have

$$\begin{aligned} \|x(t)\|^p &\leq \lambda^p \left[\sum_{k=0}^{+\infty} \left[\left\| \prod_{i=1}^k b_i(\tau_i) \right\| \|\varphi(0)\| \right. \right. \\ &\quad \left. + \sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\{ \int_{\xi_{i-1}}^{\xi_i} \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right\} \right. \\ &\quad \left. + \int_{\xi_k}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \Big]^p \\ &\leq 2^{p-1} \left[\sum_{k=0}^{+\infty} \left[\left\| \prod_{i=1}^k b_i(\tau_i) \right\|^p \|\varphi(0)\|^p I_{[\xi_k, \xi_{k+1})}(t) \right] \right. \\ &\quad \left. + \left[\sum_{k=0}^{+\infty} \left[\sum_{i=1}^k \left\| \prod_{j=i}^k b_j(\tau_j) \right\| \left\{ \int_{\xi_{i-1}}^{\xi_i} \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right\} \right. \right. \right. \\ &\quad \left. \left. + \int_{\xi_k}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^p \right] \\ &\leq 2^{p-1} \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \|\varphi(0)\|^p \\ &\quad + 2^{p-1} \left[\max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^p \cdot \left(\int_{t_0}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right)^p \end{aligned}$$

Noting that the last term of the right hand side of the above inequality increases in t and choose $C^p \geq \frac{1}{2^{p-1}}$, we obtain that

$$\begin{aligned} \|x\|_t^p &\leq 2^{p-1} \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \|\varphi\|^p \\ &\quad + 2^{p-1} \left[\max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^p (T - t_0) \int_{t_0}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\|^p ds, \end{aligned}$$

then

$$\begin{aligned}
E\|x\|_t^p &\leq 2^{p-1}C^pE[\|\varphi\|^p] \\
&+ 2^{p-1}\max\{1, C^p\}(T-t_0)\int_{t_0}^t\left[\int_0^sE\|F(s,\mu,x(\sigma(\mu)))d\mu\|^p\right]ds \\
&\leq 2^{p-1}C^pE[\|\varphi\|^p] \\
&+ 2^{p-1}\max\{1, C^p\}(T-t_0)\int_{t_0}^t\left[\int_0^sp(s,\mu)H(E\|x(\sigma(\mu))\|_s^p)d\mu\right]ds \\
&\leq 2^{p-1}C^pE[\|\varphi\|^p] \\
&+ 2^{p-1}\max\{1, C^p\}(T-t_0)\int_{t_0}^t\left[\int_0^sp(s,\mu)H(E\|x(\mu)\|_s^p)d\mu\right]ds \\
&\leq 2^{p-1}C^pE[\|\varphi\|^p] + 2^{p-1}\max\{1, C^p\}(T-t_0)^2\int_{t_0}^tp(s,s)H(E\|x\|_s^p)ds
\end{aligned}$$

Because the last term of the right hand side of the above inequality also increases in t , we have

$$\begin{aligned}
&\sup_{t_0\leq v\leq t}E\|x\|_v^p \\
&\leq 2^{p-1}C^pE[\|\varphi\|^p] + 2^{p-1}\max\{1, C^p\}(T-t_0)^2\int_{t_0}^tp(s,s)H(E\|x\|_s^p)ds \\
&\leq 2^{p-1}C^pE[\|\varphi\|^p] + 2^{p-1}\max\{1, C^p\}(T-t_0)^2\int_{t_0}^tp(s,s)H\left(\sup_{t_0\leq v\leq s}E\|x\|_v^p\right)ds.
\end{aligned}$$

We consider the function $\ell(t)$ defined by

$$\ell(t) = \sup_{t_0\leq v\leq t}E\|x\|_v^p, \quad t \in [t_0, T].$$

Then, for any $t \in [t_0, T]$ it follows that

$$(3.2) \quad \ell(t) \leq 2^{p-1}C^pE[\|\varphi\|^p] + 2^{p-1}\max\{1, C^p\}(T-t_0)^2\int_{t_0}^tp(s,s)H(\ell(s))ds.$$

Denoting by $u(t)$ the right hand side of the above inequality (3.2), we obtain that

$$\begin{aligned}
\ell(t) &\leq u(t), \quad t \in [t_0, T], \\
u(t_0) &= 2^{p-1}C^pE\|\varphi\|^p = c_1
\end{aligned}$$

and

$$\begin{aligned}
u'(t) &= 2^{p-1}\max\{1, C^p\}(T-t_0)^2p(t,t)H(\ell(t)) \\
&\leq 2^{p-1}\max\{1, C^p\}(T-t_0)^2p(t,t)H(u(t)), \quad t \in [t_0, T].
\end{aligned}$$

Then

$$(3.3) \quad \frac{u'(t)}{H(u(t))} \leq 2^{p-1} \max \{1, C^p\} (T - t_0)^2 p(t, t), \quad t \in [t_0, T].$$

Integrating (3.3) from t_0 to t and by making use of the change of variable, we obtain

$$(3.4) \quad \begin{aligned} \int_{u(t_0)}^{u(t)} \frac{ds}{H(s)} &\leq 2^{p-1} \max \{1, C^p\} (T - t_0)^2 \int_{t_0}^t p(s, s) ds \\ &\leq 2^{p-1} \max \{1, C^p\} (T - t_0)^2 \int_{t_0}^T p(s, s) ds \\ &< \int_{u(t_0)}^{\infty} \frac{ds}{H(s)}, \quad t \in [t_0, T], \end{aligned}$$

where the last inequality is obtained by (3.1). From (3.4) and by mean value theorem, there is a constant η_1 such that $u(t) \leq \eta_1$ and hence $\ell(t) \leq \eta_1$. Since $\sup_{t_0 \leq v \leq t} E\|x\|_v^p = \ell(t)$ holds for every $t \in [t_0, T]$, we have $\sup_{t_0 \leq v \leq T} E\|x\|_v^p \leq \eta_1$, where η_1 only depends on T , the functions p and H , and consequently

$$E\|x\|_{\mathcal{B}}^p = \sup_{t_0 \leq v \leq T} E\|x\|_v^p \leq \eta_1.$$

In the next steps, we will prove that Φ is continuous and completely continuous.

Step 1. We prove that Φ is continuous.

Let $\{x_n\}$ be a convergent sequence of elements of x in \mathcal{B} . Then for each $t \in [t_0, T]$, we have

$$\begin{aligned} \Phi x_n(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x_n(\sigma(\mu))) d\mu ds \right. \\ &\quad \left. + \int_{\xi_k}^t \int_0^s F(s, \mu, x_n(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t). \end{aligned}$$

Thus,

$$\begin{aligned} &\Phi x_n(t) - \Phi x(t) \\ &= \sum_{k=0}^{+\infty} \left[\sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left\{ \int_0^s F(s, \mu, x_n(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\} ds \right. \\ &\quad \left. + \int_{\xi_k}^t \left\{ \int_0^s F(s, \mu, x_n(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\} ds \right] I_{[\xi_k, \xi_{k+1})}(t). \end{aligned}$$

and

$$\begin{aligned} &E\|\Phi x_n - \Phi x\|_t^p \\ &\leq \max \{1, C^p\} (T - t_0) \int_{t_0}^t E\|\{ \int_0^s F(s, \mu, x_n(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \}\|^p ds \\ &\longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus Φ is clearly continuous.

Step 2. We prove that Φ is completely continuous operator.

Denote

$$B_m = \left\{ x \in \mathcal{B} \mid \|x\|_{\mathcal{B}}^p \leq m \right\}$$

for some $m \geq 0$.

Step 2.1 We show that Φ maps B_m into an equicontinuous family.

Let $y \in B_m$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using hypotheses $(H_2) - (H_4)$ and condition (3.1), we have

$$\begin{aligned} & \Phi x(t_1) - \Phi x(t_2) \\ &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ & \quad \left. + \int_{\xi_k}^{t_1} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t_1) \\ & - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ & \quad \left. + \int_{\xi_k}^{t_2} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t_2). \end{aligned}$$

Thus,

$$\begin{aligned} & \Phi x(t_1) - \Phi x(t_2) \\ &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ & \quad \left. + \int_{\xi_k}^{t_1} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] \left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2) \right) \\ & + \sum_{k=0}^{+\infty} \left[\sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{t_1}^{t_2} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t_2). \end{aligned}$$

Then,

$$(3.5) \quad E \|\Phi x(t_1) - \Phi x(t_2)\|^p \leq 2^{p-1} E \|I_1\|^p + 2^{p-1} E \|I_2\|^p,$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ & \quad \left. + \int_{\xi_k}^{t_1} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] \left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2) \right) \end{aligned}$$

and

$$I_2 = \sum_{k=0}^{+\infty} \left[\sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{t_1}^{t_2} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t_2).$$

Furthermore,

$$\begin{aligned} (3.6) \quad E\|I_1\|^p &\leq 2^{p-1} C^p E\|\varphi(0)\|^p E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\quad + 2^{p-1} \max\{1, C^p\} (t_1 - t_0) E \int_{t_0}^{t_1} \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\|^p ds \\ &\quad E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\leq 2^{p-1} C^p E\|\varphi(0)\|^p E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\quad + 2^{p-1} \max\{1, C^p\} (t_1 - t_0) \int_{t_0}^{t_1} \int_0^s p(s, \mu) H(E\|x\|_s^p) d\mu ds \\ &\leq 2^{p-1} C^p E\|\varphi(0)\|^p E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\quad + 2^{p-1} \max\{1, C^p\} (t_1 - t_0)^2 \int_{t_0}^{t_1} p(s, s) H(E\|x\|_s^p) ds \\ &\quad E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\leq 2^{p-1} C^p E\|\varphi(0)\|^p E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\quad + 2^{p-1} \max\{1, C^p\} (t_1 - t_0)^2 E \int_{t_0}^{t_1} M^* H(E(m)) ds \\ &\quad E\left(I_{[\xi_k, \xi_{k+1})}(t_1) - I_{[\xi_k, \xi_{k+1})}(t_2)\right) \\ &\rightarrow 0 \quad \text{as } t_2 \rightarrow t_1, \end{aligned}$$

where $M^* = \sup\{p(t, t) : t \in [t_0, T]\}$, and

$$\begin{aligned} E\|I_2\|^p &\leq C^p (t_2 - t_1) E \int_{t_1}^{t_2} \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\|^p ds \\ (3.7) \quad &\leq C^p (t_2 - t_1)^2 \int_{t_1}^{t_2} M^* H(m) ds \\ &\rightarrow 0 \quad \text{as } t_2 \rightarrow t_1. \end{aligned}$$

The right hand side of (3.6) and (3.7) is independent of $x \in B_m$. It follows that the right hand side of (3.5) tends to zero as $t_2 \rightarrow t_1$. Thus, Φ maps B_m into an equicontinuous family of functions.

Step 2.2 We show that ΦB_m is uniformly bounded.

From (3.1), $\|x\|_{\mathcal{B}}^p \leq m$ and by $(H_2) - (H_4)$ it yields that

$$\begin{aligned} \|(\Phi x)(t)\|^p &\leq 2^{p-1} \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \|\varphi(0)\|^p \\ &\quad + 2^{p-1} \left[\max_{i,k} \{1, \prod_{i=1}^k \|b_i(\tau_i)\|\} \right]^p \\ &\quad \cdot \left(\sum_{k=0}^{+\infty} \int_{t_0}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds I_{[\xi_k, \xi_{k+1})}(t) \right)^p. \end{aligned}$$

Thus,

$$\begin{aligned} E\|(\Phi x)\|_t^p &\leq 2^{p-1} C^p E\|\varphi(0)\|^p \\ &\quad + 2^{p-1} \max\{1, C^p\} (T - t_0) \int_{t_0}^t E \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right\|^p ds \\ E\|(\Phi x)\|_t^p &\leq 2^{p-1} C^p E\|\varphi(0)\|^p + 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \|\alpha_m\|_{L^1}. \end{aligned}$$

This yields that the set $\{(\Phi x)(t), \|x\|_{\mathcal{B}}^p \leq m\}$ is uniformly bounded, so $\{\Phi B_m\}$ is uniformly bounded. We have already shown that ΦB_m is equicontinuous collection. Now it is sufficient, by the Arzela - Ascoli theorem, to show that Φ maps B_m into a precompact set in \mathfrak{R}^n .

Step 2.3. We show that ΦB_m is compact.

Let $t_0 < t \leq T$ be fixed and ϵ a real number satisfying $\epsilon \in (0, t - t_0)$, for $x \in B_m$. We define

$$\begin{aligned} (\Phi_\epsilon x)(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right. \\ (3.8) \quad &\quad \left. + \int_{\xi_k}^{t-\epsilon} \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in (t_0, t - \epsilon). \end{aligned}$$

The set

$$H_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_m\}$$

is precompact in \mathfrak{R}^n for every $\epsilon \in (0, t - t_0)$. By using $(H_2) - (H_4)$, (3.1) and $E\|x\|_{\mathcal{B}}^p \leq m$, we obtain

$$E\|(\Phi x) - (\Phi_\epsilon x)\|_t^p \leq \max\{1, C^p\} (T - t_0)^2 \int_{t-\epsilon}^t M^* H(m) ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_m\}$. Hence the set $\{(\Phi x)(t) : x \in B_m\}$ is precompact in \mathfrak{R}^n . Therefore, Φ is a completely continuous operator.

Moreover, the set $U(\Phi) = \{x \in \mathcal{B} : x = \lambda \Phi x, \text{ for some } 0 < \lambda < 1\}$ is bounded. Consequently, by Theorem 2.1, the operator Φ has a fixed point in \mathcal{B} . Therefore, the system (2.1) has a solution. Thus, the proof is completed. \square

Now, we give another existence result for the system (2.1) by means of Banach contraction principle.

Theorem 3.2. *If the hypothesis (H_1) , (H_3) and (H_4) holds then the initial value problem (2.1) has a unique solution on $[t_0, T]$. Proof.* Consider the nonlinear operator $\Phi : \mathcal{B} \rightarrow \mathcal{B}$ defined as in Theorem 3.1

$$\begin{aligned}
 & E \|\Phi x - \Phi y\|_t^p \\
 & \leq 2^{p-1} \max\{1, C^p\} (T - t_0) \int_0^t \left\{ \int_0^s E \|F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, y(\sigma(\mu))) d\mu\|^p \right\} ds \\
 & \leq 2^{p-1} \max\{1, C^p\} (T - t_0) \\
 & \int_{t_0}^t \left\{ \int_0^s L(s, \mu, E\|x(\sigma(\mu))\|^p, E\|y(\sigma(\mu))\|^p) E\|x(\sigma(\mu)) - y(\sigma(\mu))\|_s^p d\mu \right\} ds \\
 & \leq 2^{p-1} \max\{1, C^p\} (T - t_0) \int_{t_0}^t \left\{ \int_0^s L(s, \mu, E\|x\|^p, E\|y\|^p) E\|x(\mu) - y(\mu)\|_s^p d\mu \right\} ds \\
 & \leq 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) E\|x - y\|_s^p ds.
 \end{aligned}$$

Taking supremum over t , we get,

$$\|\Phi x - \Phi y\|_{\mathcal{B}}^p \leq \Lambda(T) \|x - y\|_{\mathcal{B}}^p,$$

with $\Lambda(T) = 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) ds$.

Then we can take a suitable $0 < T_1 < T$ sufficient small such that $\Lambda(T_1) < 1$, and hence Φ is a contraction on \mathcal{B}_{T_1} (\mathcal{B}_{T_1} denotes \mathcal{B} with T substituted by T_1). Thus, by the well-known Banach fixed point theorem we obtain a unique fixed point $x \in \mathcal{B}_{T_1}$ for operator Φ , and hence $\Phi x = x$ is a solution of (2.1). This procedure can be repeated to extend the solution to the entire interval $[-r, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of solutions on the whole interval $[-r, T]$. \square

4. Continuous Dependence

In this section, we study the stability of the system (2.1) through the continuous dependence of solutions on initial condition.

Theorem 4.1. *Let $x(t)$ and $\bar{x}(t)$ be solutions of the system (2.1) with initial values $\varphi(0)$ and $\bar{\varphi}(0) \in \mathcal{B}$ respectively. If the assumptions of Theorem 3.2 is satisfied, then the solution of the system (2.1) is stable in the p^{th} mean. Proof.* By the assumptions, x and \bar{x} are the two solutions of the system (2.1) for $t \in [t_0, T]$. Then,

$$\begin{aligned}
 x(t) - \bar{x}(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) [\varphi(0) - \bar{\varphi}(0)] \right] \\
 &+ \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right] ds \\
 &+ \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right] ds \Big] I_{[\xi_k, \xi_{k+1})}(t).
 \end{aligned}$$

By using the hypotheses $(H_1), (H_3), (H_4)$, we get

$$\begin{aligned}
E\|x - \bar{x}\|_t^p &\leq 2^{p-1} \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k \|b_i(\tau_i)\|^p E\|\varphi(0) - \overline{\varphi(0)}\|^p I_{[\xi_k, \xi_{k+1})}(t) \right] \\
&+ 2^{p-1} E \left[\sum_{k=0}^{+\infty} \left[\sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \right. \right. \\
&\quad \times \left. \left. \int_{\xi_{i-1}}^{\xi_i} \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right\| ds \right. \right. \\
&\quad \left. \left. + \int_{\xi_k}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^p \\
&\leq 2^{p-1} E \left\{ \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \right\} E\|\varphi(0) - \overline{\varphi(0)}\|^p \\
&+ 2^{p-1} E \left[\max_{i,k} \left\{ 1, \prod_{j=i}^k \|b_j(\tau_j)\| \right\} \right]^p \\
&\quad \times E \left(\int_{t_0}^t \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right\| ds I_{[\xi_k, \xi_{k+1})}(t) \right)^p \\
&\leq 2^{p-1} C^p E\|\varphi(0) - \overline{\varphi(0)}\|^p \\
&+ 2^{p-1} \max\{1, C^p\} (t - t_0) \int_{t_0}^t E \left\| \int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right. \\
&\quad \left. - \int_0^s F(s, \mu, \bar{x}(\sigma(\mu))) d\mu \right\|^p ds \\
\sup_{t \in [t_0, T]} E\|x - \bar{x}\|_t^p &\leq 2^{p-1} C^p E\|\varphi(0) - \overline{\varphi(0)}\|^p \\
&+ 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \\
&\quad \int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) \sup_{s \in [t_0, t]} E\|x - \bar{x}\|_s^p ds.
\end{aligned}$$

By applying Grownwall's inequality, we have

$$\begin{aligned}
\sup_{t \in [t_0, T]} E\|x - \bar{x}\|_t^p &\leq 2^{p-1} C^p E\|\varphi(0) - \overline{\varphi(0)}\|^p \\
&\quad \times \exp(2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) ds) \\
&\leq \Gamma E\|\varphi(0) - \overline{\varphi(0)}\|^2,
\end{aligned}$$

where, $\Gamma = 2^{p-1} C^p \exp(2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) ds)$.

Now given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{\Gamma}$ such that $E\|\varphi(0) - \overline{\varphi(0)}\|^p < \delta$. Then

$$\sup_{t \in [t_0, T]} E\|x - \bar{x}\|_t^p \leq \epsilon.$$

This completes the proof. \square

5. Ulam -Hyers- Rassias Type Stability

In this section, we study the Ulam- Hyers stability for random impulsive nonlinear delay integro-differential equation (2.1). Let $\epsilon > 0$, $\mu \geq 0$ and $\phi : [t_0, T] \rightarrow \mathbb{R}^+$ be a piecewise continuous function. We consider the following inequalities

$$(5.1) \quad \begin{cases} E\|x'(t) - \int_0^t F(t, s, x(\sigma(s)))ds\|^p & \leq \epsilon, \quad t \neq \xi_k, \quad t \geq t_0. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p & \leq \epsilon, \quad k = 1, 2, \dots \end{cases}$$

$$(5.2) \quad \begin{cases} E\|x'(t) - \int_0^t F(t, s, x(\sigma(s)))ds\|^p & \leq \phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p & \leq \mu, \quad k = 1, 2, \dots \end{cases}$$

$$(5.3) \quad \begin{cases} E\|x'(t) - \int_0^t F(t, s, x(\sigma(s)))ds\|^p & \leq \epsilon\phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p & \leq \epsilon\mu, \quad k = 1, 2, \dots \end{cases}$$

Definition 5.1. The system (2.1) is Ulam- Hyers stable in the p^{th} mean if there exists a real number $\kappa > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathcal{B}$ of the inequality (5.1) there exists a solution $y \in \mathcal{B}$ of the system (2.1) with

$$E\|x(t) - y(t)\|^p \leq \kappa\epsilon, \quad t \in [t_0, T].$$

Definition 5.2. The system (2.1) is generalized Ulam- Hyers stable in the p^{th} mean if there exists a real number $\eta \in \mathcal{B}, \eta(0) = 0$ such that for each solution $x \in \mathcal{B}$ of the inequality (5.1) there exists a solution $y \in \mathcal{B}$ of the system (2.1) with

$$E\|x(t) - y(t)\|^p \leq \eta(\epsilon), \quad t \in [t_0, T].$$

Definition 5.3. The system (2.1) is Ulam- Hyers- Rassias stable in the p^{th} mean with respect to (ϕ, μ) if there exists a real number $\zeta > 0$ such that for each $\epsilon > 0$ and for each solution $x \in \mathcal{B}$ of the inequality (5.3) there exists a solution $y \in \mathcal{B}$ of the system (2.1) with

$$E\|x(t) - y(t)\|^p \leq \zeta\epsilon(\phi(t) + \mu), \quad t \in [t_0, T].$$

Definition 5.4. The system (2.1) is generalized Ulam- Hyers- Rassias stable in the p^{th} mean with respect to (ϕ, μ) if there exists a real number $\zeta > 0$ such that for each solution $x \in \mathcal{B}$ of the inequality (5.2) there exists a mild solution $y \in \mathcal{B}$ of the system (2.1) with

$$E\|x(t) - y(t)\|^p \leq \zeta(\phi(t) + \mu), \quad t \in [t_0, T].$$

Remark 5.1. It is clear that

1. Definition (5.1) \Rightarrow Definition (5.2)
2. Definition (5.3) \Rightarrow Definition (5.4)

3. Definition (5.3) for $\phi(t) = \mu = 1 \Rightarrow$ Definition (5.1).

Remark 5.2. A function $x \in \mathcal{B}$ is a solution of the inequality (5.3) if and only if there exists a function $h \in \mathcal{B}$ and the sequence $h_k, k = 1, 2, \dots$ (which depend on x) such that

(i): $E\|h(t)\|^p \leq \epsilon\phi(t), t \in [t_0, T]$ and $E\|h_k\|^p \leq \epsilon\mu, k = 1, 2, \dots$;

(ii): $x'(t) = \int_0^t F(t, s, x(\sigma(s)))ds + h(t), \quad t \neq \xi_k, \quad t \geq t_0$;

(iii): $x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, \quad k = 1, 2, \dots$

One can have similar remarks for the inequalities (5.1) and (5.2).

Remark 5.3. If $x \in \mathcal{B}$ is a solution of the inequality (5.3) then x is a solution of the following integral inequality

$$\begin{aligned} E\left\|x(t) - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i)\varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu)))d\mu \right] ds \right. \right. \\ \left. \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu)))d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\ \leq 2^{p-1}\epsilon \left\{ \mathcal{C}^p\mu + \max\{1, \mathcal{C}^p\}(T - t_0) \int_{t_0}^t \phi(s)ds \right\}, \quad t \in [t_0, T]. \end{aligned}$$

From the Remark we have

$$(5.4) \quad \begin{cases} x'(t) = \int_0^t F(t, s, x(\sigma(s)))ds + h(t), & t \neq \xi_k, \quad t \geq t_0. \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, & k = 1, 2, \dots \end{cases}$$

Then

$$\begin{aligned} x(t + t_0) &= \varphi(t), \text{ for } t \in [-r, 0], \\ x(t) &= \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i)\varphi(0) + \prod_{i=1}^k b_i(\tau_i)h_i \right. \\ &\quad \left. + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu)))d\mu \right] ds \right. \\ &\quad \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu)))d\mu \right] ds \right. \\ &\quad \left. + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} h(s)ds + \int_{\xi_k}^t h(s)ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T]. \end{aligned}$$

Therefore,

$$\begin{aligned}
& E \left\| x(t) - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right. \right. \\
& \quad \left. \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
&= E \left\| \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) h_i + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} h(s) ds + \int_{\xi_k}^t h(s) ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
&\leq 2^{p-1} E \left\{ \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \right\} E \|h_i\|^p \\
&+ 2^{p-1} E \left[\max_{i,k} \left\{ 1, \prod_{j=1}^k \|b_i(\tau_i)\| \right\} \right]^p (T - t_0) \int_{t_0}^t E \|h(s)\|^p ds \\
&\leq 2^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \int_{t_0}^t \phi(s) ds \right\}.
\end{aligned}$$

We have similar remarks for the solutions of the inequalities (5.1) and (5.2). Now, we give the main results, Ulam-Hyers-Rassias results, in this section.

Theorem 5.1. *Assumption (H_1) , (H_3) and (H_4) hold. Suppose there exists $\lambda > 0$ such that*

$$\int_{t_0}^t \phi(s) ds \leq \lambda \phi(t), \quad \text{for each } t \in [t_0, T],$$

where $\phi : [t_0, T] \rightarrow \mathbb{R}^+$ is a continuous nondecreasing function. Then the system (2.1) is Ulam - Hyers- Rassias stable in the p^{th} mean.

Proof. Let $x \in \mathcal{B}$ be a solution of the inequality (5.3). By Theorem 3.2 there exist a unique solution y of the random impulsive delay integro-differential system

$$\begin{aligned}
(5.5) \quad & y'(t) = \int_0^t F(t, s, y(\sigma(s))) ds, \quad t \neq \xi_k, \quad t \geq t_0 \\
& y(\xi_k) = b_k(\tau_k) y(\xi_k^-), \quad k = 1, 2, \dots \\
& y_{t_0} = \varphi.
\end{aligned}$$

Then we have

$$\begin{aligned}
& y(t + t_0) = \varphi(t), \quad \text{for } t \in [-r, 0], \\
& y(t) = \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right. \\
& \quad \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T].
\end{aligned}$$

By differential inequality (5.3), we have

$$\begin{aligned}
& E \left\| x(t) - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right. \right. \\
& \quad \left. \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
&= E \left\| \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) h_i + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} h(s) ds + \int_{\xi_k}^t h(s) ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
&\leq 2^{p-1} E \left\{ \max_k \left\{ \prod_{i=1}^k \|b_i(\tau_i)\|^p \right\} \right\} E \|h_i\|^p + 2^{p-1} E \left[\max_{i,k} \left\{ 1, \prod_{j=1}^k \|b_j(\tau_j)\|^p \right\} \right] \\
&\quad (T - t_0) \times \epsilon \int_{t_0}^t \phi(s) ds \\
&\leq 2^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \lambda \phi(t) \right\}, \quad t \in [t_0, T].
\end{aligned}$$

Hence for each $t \in [t_0, T]$, we have

$$\begin{aligned}
E \|x(t) - y(t)\|^p &= E \left\| x(t) - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) \right. \right. \\
&\quad \left. \left. + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right. \right. \\
&\quad \left. \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
E \|x - y\|_t^p &\leq 2^{p-1} E \left\| x(t) - \sum_{k=0}^{+\infty} \left[\prod_{i=1}^k b_i(\tau_i) \varphi(0) \right. \right. \\
&\quad \left. \left. + \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right. \right. \\
&\quad \left. \left. + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^p \\
&\quad + 2^{p-1} E \left\| \sum_{i=1}^k \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right. \right. \\
&\quad \left. \left. - \int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds + \int_{\xi_k}^t \left[\int_0^s F(s, \mu, x(\sigma(\mu))) d\mu \right. \right. \\
&\quad \left. \left. - \int_0^s F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^2 \\
&\leq 4^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \lambda \phi(t) \right\} \\
&\quad + 2^{p-1} \max \{1, \mathcal{C}^p\} (T - t_0)^2 \int_{t_0}^t L(s, s, E \|x\|^p, E \|y\|^p) E \|x - y\|_s^2 ds.
\end{aligned}$$

$$\begin{aligned}
\sup_{t \in [t_0, T]} E\|x - y\|_t^p &\leq 4^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \lambda \phi(t) \right\} \\
&+ 2^{p-1} \max \{1, \mathcal{C}^p\} (T - t_0)^2 \\
&\int_{t_0}^t L(s, s, E\|x\|^p, E\|y\|^p) \sup_{s \in [t_0, t]} E\|x - y\|_s^2 ds. \\
&\leq 4^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \lambda \phi(t) \right\} \\
&+ 2^{p-1} \max \{1, \mathcal{C}^p\} (T - t_0)^2 \\
&\int_{t_0}^T L(s, s, E\|x\|^p, E\|y\|^p) ds \sup_{t \in [t_0, T]} E\|x - y\|_t^2.
\end{aligned}$$

There exists a constant $\hbar = \frac{1}{1 - 2^{p-1} \max \{1, \mathcal{C}^p\} (T - t_0)^2 \int_{t_0}^T L(s, s, E\|x\|^p, E\|y\|^p) ds} > 0$ independent of $\lambda \phi(t)$ such that

$$\sup_{t \in [t_0, T]} E\|x - y\|_t^p \leq \hbar 4^{p-1} \epsilon \left\{ \mathcal{C}^p \mu + \max \{1, \mathcal{C}^p\} (T - t_0) \lambda \phi(t) \right\}, \quad t \in [t_0, T].$$

Thus, the system (2.1) is Ulam - Hyers-Rassias stable in the p^{th} mean. Hence the proof. \square

Remark 5.4.

1. Under the assumption of Theorem , we consider the system (2.1) and the inequality (5.1). One can repeat the same process to verify that the system (2.1) is Ulam - Hyers stable in the p^{th} mean.
2. Under the assumption of Theorem , we consider the system (2.1) and the inequality (5.2). One can repeat the same process to verify that the system (2.1) is generalized Ulam - Hyers-Rassias stable in the p^{th} mean.

Acknowledgements. The authors sincerely thank the anonymous reviewers for his careful reading, constructive comments and fruitful suggestions to improve the quality of the manuscript. We also thank Dr. M. Haridass, Department of Mathematics, PSG College of Technology, for his suggestions on preparing the manuscript.

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