Bull. Korean Math. Soc. **53** (2016), No. 4, pp. 1259–1280
http://dx.doi.org/10.4134/BKMS.b150666
pISSN: 1015-8634 / eISSN: 2234-3016

EQUIVALENCE CONDITIONS OF SYMMETRY PROPERTIES IN LIGHTLIKE HYPERSURFACES OF INDEFINITE KENMOTSU MANIFOLDS

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Abstract. The paper deals with lightlike hypersurfaces which are locally symmetric, semi-symmetric and Ricci semi-symmetric in indefinite Kenmotsu manifold having constant $\overline{\phi}$-holomorphic sectional curvature c. We obtain that these hypersurfaces are totally goedesic under certain conditions. The non-existence condition of locally symmetric lightlike hypersurfaces are given. Some Theorems of specific lightlike hypersurfaces are established. We prove, under a certain condition, that in lightlike hypersurfaces of an indefinite Kenmotsu space form, tangent to the structure vector field, the parallel, semi-parallel, local symmetry, semi-symmetry and Ricci semi-symmetry notions are equivalent.

1. Introduction

It is natural to impose condition on semi-Riemannian manifold that its Riemannian curvature tensor R be parallel, that is, have vanishing covariant differential, ∇R, where ∇ is the Levi-Civita connection on semi-Riemannian manifold and R is the corresponding curvature tensor. Such a manifold is said to be locally symmetric. This class of manifolds contains one of manifolds of constant curvature. A semi-Riemannian manifold is called semi-symmetric, if $R \cdot R = 0$, which is the integrability condition of $\nabla R = 0$. The semi-symmetric manifolds have been classified, in Riemannian case, by Szabo in [20] and [21]. A semi-Riemannian manifold is called Ricci semi-symmetric, if $R \cdot Ric = 0$.

We are interested to answer to the following question: "Are conditions $\nabla R = 0$, $R \cdot R = 0$ and $R \cdot Ric$ equivalent on lightlike hypersurfaces of semi-Riemannian manifolds?" These equivalences are not true in general. Ryan [18] raised the following question for hypersurfaces of Euclidean spaces in 1972: Are conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces? Although there are many results which contributed to the solution of the above

Received August 17, 2015.
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C40, 53C50.
Key words and phrases. indefinite Kenmotsu space form, locally symmetric lightlike hypersurface, semi-symmetric lightlike hypersurface, Ricci semi-symmetric lightlike hypersurface.

question in the affirmative under some conditions (see [6], [7], [17] and references therein). In [1], the authors gave an explicit example of a hypersurface in Euclidean $E^{n+1}(n \geqslant 4)$ that is Ricci semi-symmetric but not semi-symmetric (see [5] for another example). This result shows that the conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ also are not equivalent for hypersurfaces of Euclidean space in general. In [5] a survey on Ricci semi-symmetric spaces and contributions to the solution of above problem are given.

In virtue of results given by Günes, Sahin and Kilic ([10], Theorem 3.1) and Sahin ([19], Theorem 4.2), we see that the conditions $\nabla R = 0$ and $R \cdot R = 0$ are equivalent for lightlike hypersurfaces of semi-Euclidean space under conditions $Ric(E, X) = 0$ and $A_N E$ a vector field non-null. In [14], the authors show that $\nabla R = 0$ and $R \cdot R = 0$ are equivalent for lightlike hypersurfaces of indefinite Sasakian space form under condition $A_N E$ a vector field non-null. Also in [14], this equivalence is extended to the Ricci semi-symmetric notion when the lightlike hypersurfaces are considered to be η-totally umbilical. In [16], the author proved that, in the null Einstein hypersurfaces of an indefinite Kenmotsu space form, tangent to the structure vector field, the local symmetry, semi-symmetry and Ricci semi-symmetry notions are equivalent.

In the present paper we give an affirmative answer to the equivalence between $\nabla R = 0$ and $R \cdot R = 0$ for lightlike hypersurfaces of an indefinite Kenmotsu space form $\overline{M}(c)$, under condition $Ric(E, \zeta) \neq 0$, for some $\zeta \in \Gamma(S(TM)) - \langle \xi \rangle$ (Theorem 5.8). This equivalence is extended to the parallel, semi-parallel and Ricci semi-symmetric notions under condition $Ric(E, A_N E) \neq 0$ (Theorem 6.5).

The general theory of lightlike submanifolds was introduced and presented in [9] by K. L. Duggal and A. Bejancu. The theory of lightlike submanifolds is a new area of differential geometry and it is very different from Riemannian geometry as well as semi-Riemannian geometry.

In the present paper, we study the symmetry properties of lightlike hypersurfaces in indefinite Kenmotsu manifolds $\overline{M}^c$, tangent to the structure vector field, by particularly paying attention to the locally symmetric, semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces. The paper is organised as follows. In Section 2, we recall some basic definitions and formulas for indefinite Kenmotsu manifolds supported by an example and also for lightlike hypersurfaces of semi-Riemannian manifolds. In Section 3, we give the decomposition of screen distribution and tangent bundle on lightlike hypersurfaces of indefinite Kenmotsu manifolds which are tangential to the structure vector field. In Section 4, we consider a lightlike hypersurface M of an indefinite Kenmotsu manifold $\overline{M}^c$, with constant $\overline{\phi}$-holomorphic sectional curvature c and study local symmetry conditions on this hypersurface. It is known in [10] that in locally symmetric semi-Riemannian manifold $\overline{M}$, the locally symmetric lightlike hypersurfaces are totally geodesic, under condition that the vector field $A_N E$

is non-null. Here we show that there are no locally symmetric lightlike hypersurfaces in indefinite Kenmotsu manifold $(\overline{M}^c, c \neq -1)$. On the other hand we prove that, in indefinite Kenmotsu space form $\overline{M}$ $(c = -1)$, any locally symmetric lightlike hypersurface is totally geodesic (Theorem 4.5). An example of locally symmetric lightlike hypersurface is given. We also prove, in the same section, that totally contact umbilical lightlike hypersurfaces of an indefinite Kenmotsu manifold which are locally symmetric are totally geodesic (Theorem 4.8). We obtain equivalence between parallel and locally symmetry notions on lightlike hypersurfaces of an indefinite Kenmotsu manifold $\overline{M}^c$ (Theorem 4.10). In Section 5, we study semi-symmetric lightlike hypersurfaces of indefinite Kenmotsu manifolds $\overline{M}^c$. We give a characterization of semi-symmetric lightlike hypersurfaces and We prove, under a certain condition, that in lightlike hypersurfaces of an indefinite Kenmotsu space form, tangent to the structure vector field, the local symmetry and semi-symmetry notions are equivalent (Theorem 5.8). Also this equivalence is extended to the semi-parallel notion (Theorem 5.10). We also give a sufficient condition on lightlike hypersurface of indefinite Kenmotsu space form to be not semi-symmetric (Corollary 5.7). Finally, in Section 6, we give a characterization of Ricci semi-symmetric lightlike hypersurfaces of an indefinite Kenmotsu manifold $\overline{M}^c$, tangent to the structure vector field. We show that, under a certain condition, the Ricci semi-symmetric lightlike hypersurfaces of indefinite Kenmotsu space form $\overline{M}(c)$ are totaly geodesic (Theorem 6.2). In Theorem 6.5, under a certain condition, we extend the equivalence given in Theorem 5.8 to Ricci semi-symmetry notion. Finally, we obtain under certain condition, the equivalence between the parallel, semi-parallel, local symmetry, semi-symmetry and Ricci semi-symmetry notions in hypersurfaces of an indefinite Kenmotsu space form (Theorem 6.5).

2. Preliminaries

Let $\overline{M}$ a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e., $\overline{\phi}$ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \overline{\phi} = 0, \ \overline{\phi}(\xi) = 0 \text{ and rank } \overline{\phi} = 2n. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that for any vector field $\overline{X}, \overline{Y}$ on $\overline{M}$

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \ \overline{g}(\overline{\phi X}, \overline{\phi Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}). \tag{2.2}$$

If moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi X}$, therefore $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$, and $(\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Kenmotsu manifold [12].

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. Since $\overline{\phi}\sigma = \sigma$, the $\overline{\phi}$-section σ is a holomorphic $\overline{\phi}$-section and the sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-holomorphic sectional curvature (see [3], [11] and references therein for more details). If a Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of the Proposition 12 in [12], the Riemann curvature tensor $\overline{R}$ of $\overline{M}$ is given by, for any $\overline{X}, \overline{Y}, \overline{Z} \in \Gamma(TM)$,

$$\begin{aligned} \overline{R}(\overline{X},\overline{Y})\overline{Z} = {} & \frac{c-3}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\} + \frac{c+1}{4}\{\eta(\overline{X})\eta(\overline{Z})\overline{Y} \\ & - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi \\ & + \overline{g}(\overline{\phi Y},\overline{Z})\overline{\phi X} - \overline{g}(\overline{\phi X},\overline{Z})\overline{\phi Y} - 2\overline{g}(\overline{\phi X},\overline{Y})\overline{\phi Z}\}. \end{aligned} \tag{2.3}$$

A Kenmotsu manifold is a typical example of $C(\alpha)$-manifold, with $\alpha = -1$, introduced by Janssens and Vanhecke [11].

Note that the $\overline{\phi}$-holomorphic sectional curvature of an indefinite $C(\alpha)$-manifold does not satisfy, in general, a "Schur Lemma" although it holds for co-Kähler and indefinite Sasakian manifolds (see [3] for details).

An indefinite Kenmotsu manifold $\overline{M}$ which has constant $\overline{\phi}$-holomorphic sectional curvature c will be denoted by $\overline{M}^c$. A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-holomorphic sectional curvature c will be called *Kenmotsu space form* and denoted by $\overline{M}(c)$. Here $\overline{M}^c$ is different from $\overline{M}(c)$ and this is well specified in [12] through Proposition 12 and Theorem 13.

If a $(2n+1)$-dimensional Kenmotsu manifold $\overline{M}$ has a constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of Proposition 12 [12], the Ricci tensor $\overline{Ric}$ and the scalar curvature $\overline{r}$ are given by [12]

$$\overline{Ric} = \frac{1}{2}(n(c-3)+c+1)\overline{g} - \frac{1}{2}(n+1)(c+1)\eta \otimes \eta, \tag{2.4}$$

$$\overline{r} = \frac{1}{2}(n(2n+1)(c-3) - n(c+1)). \tag{2.5}$$

This means that $\overline{M}^c$ is η-Einstein. But if $\overline{M}$ becomes a space of constant $\overline{\phi}$-holomorphic sectional curvature c, that is, a Kenmotsu space form $\overline{M}(c)$, the Riemann curvature tensor of $\overline{M}(c)$ has also the form given in (2.3) with c constant which implies, through the Eq. (2.4), that $\overline{M}(c)$ is η-Einstein. Since the coefficients of $\overline{Ric}$ are constant on $\overline{M}(c)$, by Corollary 9 in [12], $\overline{M}$ is an Einstein one and consequently $c+1=0$, that is $c=-1$. So, the Ricci tensor becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n+1)$.

Thus, if a Kenmotsu manifold $\overline{M}$ is a space form, then it is an Einstein and $c=-1$. This means that it is a space of constant curvature -1 so, locally it is isometric to the hyperbolic space.

Example 2.1. We consider the 7-dimensional manifold $\overline{M} = \{(x_1, x_2, \dots, x_7) \in \mathbb{R}^7\}$, where $x = (x_1, x_2, \dots, x_7)$ are cartesian coordinates on $\mathbb{R}^7$. We define with respect to the natural field of frames $\{\frac{\partial}{\partial x_i}\}$, the differential 1-form η and the vector field ξ by

$$\eta = dx_7, \qquad \xi = \frac{\partial}{\partial x_7}, \tag{2.6}$$

the semi-Riemannian metric $\overline{g}$ of index $\nu = 2$ on $\overline{M}$ by

$$\overline{g} = \eta \otimes \eta + e^{2x_7}\{ - dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 + dx_5^2 + dx_6^2\}. \tag{2.7}$$

The vector fields

$$\begin{aligned} &e_1 = e^{-x_7}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_7}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_7}\frac{\partial}{\partial x_3}, \quad e_4 = e^{-x_7}\frac{\partial}{\partial x_4}, \\ &e_5 = e^{-x_7}\frac{\partial}{\partial x_5}, \quad e_6 = e^{-x_7}\frac{\partial}{\partial x_6}, \quad e_7 = \frac{\partial}{\partial x_7}. \end{aligned} \tag{2.8}$$

In local field of frames $\{e_i\}$, the metric $\overline{g}$ is given by $\overline{g}(e_i, e_j) = 0$, $\forall i \neq j$, $i, j = 1, 2, \dots, 7$, $\overline{g}(e_k, e_k) = 1$, $\forall k = 2, 3, 5, 6, 7$ and $\overline{g}(e_m, e_m) = -1$, $\forall m = 1, 4$. The 1-form η is given by $\eta(\overline{X}) = \overline{g}(\overline{X}, e_7)$ for any $\overline{X} \in \Gamma(T\overline{M})$. Let $\overline{\phi}$ be the (1.1)-tensor field defined by

$$\overline{\phi}e_1 = e_4,\ \overline{\phi}e_2 = -e_5,\ \overline{\phi}e_3 = e_6,\ \overline{\phi}e_4 = -e_1,\ \overline{\phi}e_5 = e_2,\ \overline{\phi}e_6 = -e_3,\ \overline{\phi}e_7 = 0.$$

Then using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\eta(e_7) = 1$, $\overline{\phi}^2\overline{X} = -\overline{X} + \eta(\overline{X})e_7$, $\overline{g}(\overline{\phi X}, \overline{\phi Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$ for any $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$. Thus, for $e_7 = \xi$, $(\overline{\phi}, \xi, \eta, \overline{g})$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric $\overline{g}$. Then, we have $[e_i, e_7] = e_i$, $\forall i = 1, 2, \dots, 6$ and $[e_i, e_j] = 0$, $\forall i \neq j$, $i, j = 1, 2, \dots, 6$. Using the Koszul's formula given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) = {} & \overline{X} \cdot \overline{g}(\overline{Y}, \overline{Z}) + \overline{Y} \cdot \overline{g}(\overline{Z}, \overline{X}) - \overline{Z} \cdot \overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) \\ & - \overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]), \end{aligned}$$

the non-vanishing covariant derivative are given by $\overline{\nabla}_{e_1}e_1 = e_7$, $\overline{\nabla}_{e_2}e_2 = -e_7$, $\overline{\nabla}_{e_3}e_3 = -e_7$, $\overline{\nabla}_{e_4}e_4 = e_7$, $\overline{\nabla}_{e_5}e_5 = -e_7$, $\overline{\nabla}_{e_6}e_6 = -e_7$, $\overline{\nabla}_{e_1}e_7 = e_1$, $\overline{\nabla}_{e_2}e_7 = e_2$, $\overline{\nabla}_{e_3}e_7 = e_3$, $\overline{\nabla}_{e_4}e_7 = e_4$, $\overline{\nabla}_{e_5}e_7 = e_5$, $\overline{\nabla}_{e_6}e_7 = e_6$. From these relations, it follows that the manifold $\overline{M}$ satisfies $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi X}$. Hence, $\overline{M}$ is indefinite Kenmotsu manifold. Also, it is easy to check that $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ is an indefinite Kenmotsu manifold of constant $\overline{\phi}$-holomorphic sectional curvature $c = -1$.

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with constant index ν, $0 < \nu < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is said to be a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the orthogonal vector bundle $TM^{\perp}$ to tangent vector bundle TM, defined as

$$TM^{\perp} = \bigcup_{p \in M} \{Y_p \in T_p\overline{M} : \overline{g}_p(X_p, Y_p) = 0, \forall X_p \in T_pM\} \tag{2.9}$$

is a distribution of rank 1 on M [9]: $TM^\perp \subset TM$ and then coincides with the radical distribution $RadTM = TM \cap TM^\perp$. A complementary bundle of $TM^\perp$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a screen distribution and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.2 ([9]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section E of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall W \in \Gamma(S(TM)_{|\mathcal{U}}). \tag{2.10}$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(F)$ the smooth sections of the vector bundle F. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.2 we may write down the following decomposition

$$TM = S(TM) \perp TM^\perp, \tag{2.11}$$

$$T\overline{M}_{|M} = S(TM) \perp \left\{TM^\perp \oplus tr(TM)\right\} = TM \oplus tr(TM). \tag{2.12}$$

Let ∇ be the Levi-Civita connection on (M, g), then by using the second decomposition of (2.12) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.2, we have Gauss and Weingarten formulae in the form, for any $X, Y \in \Gamma(TM_{|\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \text{ and } \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.13}$$

where $\nabla_X Y, A_N X \in \Gamma(TM)$. ∇ is an induced a symmetric linear connection on M, $\nabla^\perp$ is a linear connection on the vector bundle $tr(TM)$, h is a symmetric bilinear form and A_N is the shape operator of M.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.2. Then (2.13) takes the form, for any $X, Y \in \Gamma(TM_{|\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \text{ and } \overline{\nabla}_X N = -A_N X + \tau(X)N. \tag{2.14}$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.14), we obtain $X, Y \in \Gamma(TM_{|\mathcal{U}})$

$$B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E) \text{ and } \tau(X) = \overline{g}(\nabla^\perp_X N, E). \tag{2.15}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition (2.11). We have for any $X, Y \in \Gamma(TM_{|\mathcal{U}})$,

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)E \text{ and } \nabla_X E = -A^*_E X - \tau(X)E, \tag{2.16}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, $\overset{*}{A}_E$ and $\overset{*}{\nabla}$ are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \quad \forall X,Y \in \Gamma(TM_{|\mathcal{U}}), \tag{2.17}$$

where θ is a differential 1-form locally defined on M by $\theta(X) := \overline{g}(N,X)$, $\forall X \in \Gamma(TM)$. The local second fundamental form of M satisfies $B(X,PY) = g(\overset{*}{A}_E X, PY)$ and $B(X,E) = 0$, also $B(\overset{*}{A}_E X, Y) = B(X, \overset{*}{A}_E Y)$ and $g(\overset{*}{A}_E X, N) = 0$. The local second fundamental form of $S(TM)$ satisfies $C(X,PY) = g(A_N X, PY)$.

Denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{M}$ and M, respectively. From Gauss equation [9], we have the following, for any $X,Y,Z \in \Gamma(TM_{|\mathcal{U}})$,

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + \{(\nabla_X B)(Y,Z) \\ &\quad - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{2.18}$$

where $(\nabla_X B)(Y,Z) = X \cdot B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$.

3. Lightlike hypersurfaces of indefinite Kenmotsu manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and (M,g) be its lightlike hypersurface, tangent to the structure vector field ξ, i.e., $\xi \in \Gamma(TM)$. If E is a local section of $TM^\perp$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^\perp)$ as vector subbundle. We consider local section N of $tr(TM)$. Since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}N$ is also tangent to M. On the other hand, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the component of $\overline{\phi}N$ with respect to E vanishes. Thus $\overline{\phi}N \in \Gamma(S(TM))$. From the second equation of (2.2) we have $\overline{g}(\overline{\phi}N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(tr(TM))$ (direct sum but not orthogonal) is a non-degenerate vector subbundle of $S(TM)$ of rank two.

It is known [4] that if M is tangent to the structure vector field ξ, then ξ belongs to $S(TM)$. using this and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non-degenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(tr(TM))\right\} \perp D_0 \perp \langle\xi\rangle, \tag{3.1}$$

where $\langle\xi\rangle = \mathrm{Span}\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\overline{\phi}$, i.e., $\overline{\phi}(D_0) = D_0$.

Example 3.1. Let M be a hypersurface of $(\overline{M} = \mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$ (indefinite Kenmotsu space form defined in Example 1) given by

$$M = \left\{(x_1, \dots, x_7) \in \mathbb{R}^7 : x_5 - x_4 = 0\right\}, \tag{3.2}$$

where $(x_1, \dots, x_7)$ is a local coordinate system on $\mathbb{R}^7$. Thus the tangent space TM is spanned by $\{U_i\}_{1 \leqslant i \leqslant 6}$, where $U_1 = e_1$, $U_2 = e_2$, $U_3 = e_3$, $U_4 = e_4 + e_5$, $U_5 = e_6$, $U_6 = e_7$ and the distribution $TM^\perp$ of rank 1 is spanned by $E = e_4 + e_5$.

It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\mathbb{R}^7$. Also, the transversal bundle $tr(TM)$ is spanned by $N = \frac{1}{2}(e_5 - e_4)$. On the other hand, by using the almost contact structure $(\overline{\phi}, \xi, \eta)$ of $\mathbb{R}^7$ and also by taking into account of the decomposition of screen distribution $S(TM)$ given in (3.1), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_3$, $\overline{\phi}F = U_5$, and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(tr(TM))$ are spanned, respectively by $\xi = U_6$, $\overline{\phi}E = U_2 - U_1$ and $\overline{\phi}N = \frac{1}{2}(U_1 + U_2)$. Hence M is a lightlike hypersurface of an indefinite Kenmotsu space form $(\mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$.

Moreover, from (2.11), (2.12) and (3.1) we obtain the decomposition

$$TM = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(tr(TM))\right\} \perp D_0 \perp \langle\xi\rangle \perp TM^\perp, \tag{3.3}$$

$$T\overline{M}_{|M} = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(tr(TM))\right\} \perp D_0 \perp \langle\xi\rangle \perp (TM^\perp \oplus tr(TM)). \tag{3.4}$$

Now, we consider the distributions on M,

$$D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0, \text{ and } D' := \overline{\phi}(tr(TM)). \tag{3.5}$$

Then D is invariant under $\overline{\phi}$, i.e., $\overline{\phi}(D) = D$ So we have the decomposition

$$TM = (D \oplus D') \perp \langle\xi\rangle. \tag{3.6}$$

Let us consider the local null vector fields $U := -\overline{\phi}N$, $V := -\overline{\phi}E$. Then, from (3.6), any $X \in \Gamma(TM)$ is written as

$$X = RX + QX + \eta(X)\xi, \; QX = u(X)U, \tag{3.7}$$

where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(X) = g(X, V)$. Applying $\overline{\phi}$ to (3.7) and (2.1), note that $\overline{\phi}^2 N = -N$, we obtain

$$\overline{\phi}X = \phi X + u(X)N, \tag{3.8}$$

where ϕ is a tensor field of type $(1, 1)$ defined on M by $\phi X := \overline{\phi}RX$, for any $X \in \Gamma(TM)$. In addition, we obtain, for any $X \in \Gamma(TM)$,

$$B(X, \xi) = 0, \tag{3.9}$$

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \quad \text{and} \tag{3.10}$$

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3.11}$$

By using (2.2) and (3.8) we derive that, for any $X, Y \in \Gamma(TM)$,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.12}$$

where v is a 1-form locally defined on M by $v(X) = g(X, U)$, for any $X \in \Gamma(TM)$. we note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \tag{3.13}$$

For future use, we have the following identities: for any $X, Y \in \Gamma(TM)$,

$$C(X, \xi) = \theta(X), \tag{3.14}$$

$$B(X, U) = C(X, V), \tag{3.15}$$

$$(\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X) - \eta(Y)u(X), \tag{3.16}$$

$$(\nabla_X \phi)Y = \overline{g}(\overline{\phi}X, Y)\xi - \eta(Y)\phi X - B(X, Y)U + u(Y)A_N X. \tag{3.17}$$

4. Locally symmetric lightlike hypersurfaces in indefinite Kenmotsu manifolds

Let M be a lightlike hypersurfaces in indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in \Gamma(TM)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (Theorem 2.2). By using (2.3), (2.18) and (3.8), and comparing the tangential and transversal parts of the both sides, we have, for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y)Z = \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \overline{g}(\overline{\phi}Y,Z)\phi X \\ - \overline{g}(\overline{\phi}X,Z)\phi Y - 2\overline{g}(\overline{\phi}X,Y)\phi Z\} + B(Y,Z)A_N X - B(X,Z)A_N Y \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \tau(Y)B(X,Z) - \tau(X)B(Y,Z) \\ + \frac{c+1}{4}\{\overline{g}(\overline{\phi}Y,Z)u(X) \\ - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\}. \end{aligned} \tag{4.2}$$

A lightlike hypersurface $(M, g, S(TM))$ of a semi Riemannian manifold $(\overline{M}, \overline{g})$ is said locally symmetric [10], if and only if for any $X, Y, Z, T, W \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$ the following hold

$$g((\nabla_W R)(X,Y)Z, PT) = 0 \text{ and } \overline{g}((\nabla_W R)(X,Y)Z, N) = 0. \tag{4.3}$$

That is $(\nabla_W R)(X,Y)Z = 0$. Using Lemma 3.2 of [10], for any $W, X, Y, Z \in \Gamma(TM)$, $T \in \Gamma(S(TM))$ and $N \in \Gamma(tr(TM))$, we have

$$\begin{aligned} \overline{g}((\overline{\nabla}_W \overline{R})(X,Y)Z,T) = g((\nabla_W R)(X,Y)Z,T) + (\nabla_W B)(X,Z)C(Y,T) \\ + B(X,Z)g((\nabla_W A_N)Y,T) - (\nabla_W B)(Y,Z)C(X,T) \\ - B(Y,Z)g((\nabla_W A_N)X,T) - B(Y,Z)\tau(X)C(W,T) \\ + (\nabla_Y B)(X,Z)C(W,T) - (\nabla_X B)(Y,Z)C(W,T) \\ + B(X,Z)\tau(Y)C(W,T) - B(W,X)\overline{R}(N,Y,Z,T) \\ - B(W,Y)\overline{R}(X,N,Z,T) - B(W,Z)\overline{R}(X,Y,N,T), \end{aligned} \tag{4.4}$$

and,

$$\begin{aligned} \overline{g}((\overline{\nabla}_W \overline{R})(X,Y)Z,N) = g((\nabla_W R)(X,Y)Z,N) + B(X,Z)g(\nabla_W(A_N Y),N) \\ - B(Y,Z)g(\nabla_W(A_N X),N) - B(W,X)\overline{R}(N,Y,Z,N) \\ - B(W,Y)\overline{R}(X,N,Z,N). \end{aligned} \tag{4.5}$$

Suppose that M is a lightlike hypersurface in indefinite Kenmotsu manifold $(\overline{M}^c, c = 3)$ with $\xi \in TM$. Then the relation (4.1) becomes

$$\begin{aligned} R(X,Y)Z = {} & \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & - g(Y,Z)\eta(X)\xi + \overline{g}(\overline{\phi}Y,Z)\phi X - \overline{g}(\overline{\phi}X,Z)\phi Y \\ & - 2\overline{g}(\overline{\phi}X,Y)\phi Z + B(Y,Z)A_N X - B(X,Z)A_N Y. \end{aligned} \tag{4.6}$$

From decomposition (2.11), the curvature tensor R is written as

$$\begin{aligned} R(X,Y)Z = {} & R(PX,PY)PZ + \theta(X)R(E,PY)PZ \\ & + \theta(Y)R(PX,E)PZ + \theta(Z)R(PX,PY)E \\ & + \theta(X)\theta(Z)R(E,PY)E + \theta(Y)\theta(Z)R(PX,E)E, \end{aligned} \tag{4.7}$$

where, in particular and using (4.6), the component $R(E,\cdot)E$ is given by

$$R(E,PY)E = -3u(PY)V. \tag{4.8}$$

By using (4.6), the covariant derivative of R is given by, for any $W \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_W R)(E,PY)E = {} & \nabla_W R(E,PY)E - R(\nabla_W E,PY)E - R(E,\nabla_W PY)E \\ & - R(E,PY)\nabla_W E \\ = {} & -3W\cdot u(PY)V + 3u(PY)u(W)\xi + 3u(PY)\phi\nabla_W E \\ & - u(PY)\phi\nabla_W E + u(\nabla_W E)\phi PY - 2\overline{g}(\overline{\phi}\nabla_W E,PY)V \\ & + 3u(\nabla_W PY)V + \eta(PY)\eta(\nabla_W E)E + \overline{g}(\overline{\phi}PY,\nabla_W E)V \\ & - u(\nabla_W E)\phi PY - 2u(PY)\phi\nabla_W E - B(PY,\nabla_W E)A_N E, \end{aligned} \tag{4.9}$$

which implies

$$\overline{g}((\nabla_W R)(E,PY)E,N) = -\eta(PY)u(W). \tag{4.10}$$

Taking $PY = \xi$ and $W = U$ in (4.10), we obtain $\overline{g}((\nabla_U R)(E,\xi)E,N) = -1$. This means that a lighlike hypersurface of indefinite Kenmotsu manifold $(\overline{M}^c, c = 3)$ with $\xi \in TM$ cannot be locally symmetric.

Lemma 4.1. *There are no lightlike hypersurfaces of indefinite Kenmotsu manifold $(\overline{M}^c, c = 3)$ tangent to the structure vector field ξ, which are locally symmetric.*

Lemma 4.2. *Let $(\overline{M}^c, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and $\overline{R}$ the Riemann curvature tensor of Levi-Civita connection $\overline{\nabla}$. Then we have, for any $W, X, Y, Z \in \Gamma(T\overline{M})$,*

$$\begin{aligned} & (\overline{\nabla}_W \overline{R})(X,Y)Z \\ = {} & \frac{c+1}{4}\{\overline{g}(\overline{\phi}W,\overline{\phi}X)\eta(Z)Y + \overline{g}(\overline{\phi}W,\overline{\phi}Z)\eta(X)Y - \overline{g}(\overline{\phi}W,\overline{\phi}Y)\eta(Z)X \\ & - \overline{g}(\overline{\phi}W,\overline{\phi}Z)\eta(Y)X + \overline{g}(X,Z)\overline{g}(\overline{\phi}W,\overline{\phi}Y)\xi + \overline{g}(X,Z)\eta(Y)W \\ & - \overline{g}(X,Z)\eta(Y)\eta(W)\xi - \overline{g}(Y,Z)\overline{g}(\overline{\phi}W,\overline{\phi}X)\xi - \overline{g}(Y,Z)\eta(X)W \end{aligned}$$

$$
\begin{aligned}
&+ \overline{g}(Y,Z)\eta(X)\eta(W)\xi + \overline{g}(\overline{\phi}W,Y)\eta(Z)\overline{\phi}X - \overline{g}(\overline{\phi}W,Z)\eta(Y)\overline{\phi}X \\
&+ \overline{g}(\overline{\phi}Y,Z)\overline{g}(\overline{\phi}W,X)\xi - \overline{g}(\overline{\phi}Y,Z)\eta(X)\overline{\phi}W - \overline{g}(\overline{\phi}W,X)\eta(Z)\overline{\phi}Y \\
&+ \overline{g}(\overline{\phi}W,Z)\eta(X)\overline{\phi}Y - \overline{g}(\overline{\phi}X,Z)\overline{g}(\overline{\phi}W,Y)\xi + \overline{g}(\overline{\phi}X,Z)\eta(Y)\overline{\phi}W \\
&- 2\overline{g}(\overline{\phi}W,X)\eta(Y)\overline{\phi}Z + 2\overline{g}(\overline{\phi}W,Y)\eta(X)\overline{\phi}Z - 2\overline{g}(\overline{\phi}X,Y)\overline{g}(\overline{\phi}W,Z)\xi \\
&+ 2\overline{g}(\overline{\phi}X,Y)\eta(Z)\overline{\phi}W\}. \qquad (4.11)
\end{aligned}
$$

Proof. By using the relation (2.3), let decompose the Riemann curvature $\overline{R}$ on $\overline{M}(c)$ by

$$\overline{R} = \overline{R}_1 + \overline{R}_2, \qquad (4.12)$$

where for any $X, Y, Z \in \Gamma(TM)$,

$$\overline{R}_1(X,Y)Z = \frac{c-3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \qquad (4.13)$$

$$
\begin{aligned}
\overline{R}_2(X,Y)Z = \frac{c+1}{4}\{&\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)\xi \\
&- \overline{g}(Y,Z)\eta(X)\xi + \overline{g}(\overline{\phi}Y,Z)\overline{\phi}X - \overline{g}(\overline{\phi}X,Z)\overline{\phi}Y - 2\overline{g}(\overline{\phi}X,Y)\overline{\phi}Z\}. \qquad (4.14)
\end{aligned}
$$

By covariant derivation of $\overline{R}$, we have, $(\overline{\nabla}_W\overline{R})(X,Y)Z = (\overline{\nabla}_W\overline{R}_2)(X,Y)Z = \overline{\nabla}_W(\overline{R}_2(X,Y)Z) - \overline{R}_2(\overline{\nabla}_WX,Y)Z - \overline{R}_2(X,\overline{\nabla}_WY)Z - \overline{R}_2(X,Y)\overline{\nabla}_WZ$. By direct calculation, using (4.14) and the definition of covariant derivative of differential forms, we obtain the result. □

Theorem 4.3. *There are no locally symmetric lightlike hypersurfaces of indefinite Kenmotsu manifold ($\overline{M}^c, c \neq -1$), tangent to the structure vector field ξ.*

Proof. Let M be a locally symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$. Suppose $c \neq -1$. From (4.11), we have, for any $W, X, Y, Z \in \Gamma(TM)$

$$
\begin{aligned}
&\overline{g}((\overline{\nabla}_W\overline{R})(X,Y)Z, U) \\
= &\frac{c+1}{4}\{\overline{g}(\overline{\phi}W,\overline{\phi}X)\eta(Z)v(Y) + \overline{g}(\overline{\phi}W,\overline{\phi}Z)\eta(X)v(Y) \\
&- \overline{g}(\overline{\phi}W,\overline{\phi}Y)\eta(Z)v(X) - \overline{g}(\overline{\phi}W,\overline{\phi}Z)\eta(Y)v(X) + \overline{g}(X,Z)\eta(Y)v(W) \\
&- \overline{g}(Y,Z)\eta(X)v(W) - \overline{g}(\overline{\phi}W,Y)\eta(Z)\theta(X) + \overline{g}(\overline{\phi}W,Z)\eta(Y)\theta(X) \\
&+ \overline{g}(\overline{\phi}Y,Z)\eta(X)\theta(W) + \overline{g}(\overline{\phi}W,X)\eta(Z)\theta(Y) - \overline{g}(\overline{\phi}W,Z)\eta(X)\theta(Y) \\
&- \overline{g}(\overline{\phi}X,Z)\eta(Y)\theta(W) + 2\overline{g}(\overline{\phi}W,X)\eta(Y)\theta(Z) - 2\overline{g}(\overline{\phi}W,Y)\eta(X)\theta(Z) \\
&- 2\overline{g}(\overline{\phi}X,Y)\eta(Z)\theta(W)\}. \qquad (4.15)
\end{aligned}
$$

From relation (2.3), we have $\overline{R}(E,Y,E,U) = -\frac{3}{4}(c+1)u(Y)$. By taking $X = E$, $Z = E$ and $T = U$ in (4.4) and (4.15), we obtain for any $W, Y \in \Gamma(TM)$

$$-(\nabla_W B)(Y,E)C(Y,U) + \frac{3}{4}(c+1)B(W,Y)u(Y) = \frac{3}{4}(c+1)\eta(Y)u(W). \tag{4.16}$$

Then, by taking $Y = \xi$ and $W = U$ in (4.16), we have $c = -1$ which is a contradiction. Hence, the claim hold. □

Lemma 4.4. *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$. If M is locally symmetric, then it is totally geodesic.*

Proof. Let M be a locally symmetric lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}^c$, with $\xi \in \Gamma(TM)$. From relation (4.11), we have, for any $W, X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned}
&\overline{g}((\overline{\nabla}_W \overline{R})(X,Y)Z, N) \\
&= \frac{c+1}{4}\{\overline{g}(\overline{\phi}W, \overline{\phi}X)\eta(Z)\theta(Y) + \overline{g}(\overline{\phi}W, \overline{\phi}Z)\eta(X)\theta(Y) \\
&\quad - \overline{g}(\overline{\phi}W, \overline{\phi}Y)\eta(Z)\theta(X) - \overline{g}(\overline{\phi}W, \overline{\phi}Z)\eta(Y)\theta(X) + \overline{g}(X,Z)\eta(Y)\theta(W) \\
&\quad - \overline{g}(Y,Z)\eta(X)\theta(W) + \overline{g}(\overline{\phi}W, Y)\eta(Z)v(X) - \overline{g}(\overline{\phi}W, Z)\eta(Y)v(X) \\
&\quad - \overline{g}(\overline{\phi}Y, Z)\eta(X)v(W) - \overline{g}(\overline{\phi}W, X)\eta(Z)v(Y) + \overline{g}(\overline{\phi}W, Z)\eta(X)v(Y) \\
&\quad + \overline{g}(\overline{\phi}X, Z)\eta(Y)v(W) - 2\overline{g}(\overline{\phi}W, X)\eta(Y)v(Z) + 2\overline{g}(\overline{\phi}W, Y)\eta(X)v(Z) \\
&\quad + 2\overline{g}(\overline{\phi}X, Y)\eta(Z)v(W)\}.
\end{aligned} \tag{4.17}$$

From relation (2.3), we have $\overline{R}(E,N,E,N) = \frac{c-3}{4}$. By taking $X = E$, $Z = E$ in (4.5) and (4.17), we obtain for any $W, Y \in \Gamma(TM)$

$$-\frac{c-3}{4}B(W,Y) = \overline{g}((\overline{\nabla}_W \overline{R})(E,Y)E,N) = 0. \tag{4.18}$$

In virtue of lemma 4.1, $c \neq 3$, so, the relation (4.18) lead to $B(W,Y) = 0$, $\forall W, Y \in \Gamma(TM)$, that is M is totally geodesic. □

It is known, that in locally symmetric semi-Riemannian manifold $\overline{M}$, the locally symmetric lightlike hypersurfaces are totally geodesic lightlike hypersurfaces if the vector field $A_N E$ is non-null (see [10]). Also in [14], the authors have proved that in indefinite Sasakian space form, the locally symmetric lightlike hypersurfaces tangent to the vector structure are totally geodesic. So, in indefinite Kenmotsu space form $\overline{M}(c)$ we have the following.

Theorem 4.5. *Let M be a lightlike hypersurface of indefinite Kenmotsu space form $\overline{M}(c)$, tangent to the structure vector field ξ. Then M is locally symmetric if and only if it is totally geodesic.*

Proof. Let M be a totally geodesic lightlike hypersurface of indefinite Kenmotsu space form $(\overline{M}(c)$. Since $c = -1$, by using (4.4), (4.5) and (4.11), we obtain $g((\nabla_W R)(X,Y)Z, PT) = 0$ and $\overline{g}((\nabla_W R)(X,Y)Z, N) = 0$, that is M is locally symmetric. The converse is given by Lemma 4.4. □

Example 4.6. Let M be a hypersurface of an indefinite Kenmotsu space form $(\overline{M} = \mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$ of Example 2, given by

$$M = \big\{(x_1, \ldots, x_7) \in \mathbb{R}^7 : x_5 - x_4 = 0\big\},$$

where $(x_1, \ldots, x_7)$ is a local coordinate system on $\mathbb{R}^7$. As explained in Example 2, M is a lightlike hypersurface of $\overline{M}$ having a local quasi-orthonormal field of frames $\big\{U_1 = e_1,\ U_2 = e_2,\ U_3 = e_3,\ U_4 = E = e_4 + e_5,\ U_5 = e_6,\ U_6 = \xi = e_7,\ N = \frac{1}{2}(e_5 - e_4)\big\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$. Then, using non-vanishing components of $\overline{\nabla}$ given in Example 1, we obtain

$$\overline{\nabla}_E N = -\xi \text{ and } \overline{\nabla}_X N = 0,\ \forall X \in \Gamma(S(TM)).$$

Thus, the differential 1-form τ vanish, that is $\tau(X) = 0,\ \forall X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae we have

$$A_N E = -\xi, \quad A_N X = 0,\ \forall X \in \Gamma(S(TM)), \tag{4.19}$$

$$\overset{*}{A}_E X = 0, \quad \nabla_X E = 0,\ \forall X \in \Gamma(TM). \tag{4.20}$$

From (4.20), we infer that, the lightlike hypersurface M of $\overline{M}(c = -1)$ is totally geodesic. Let consider the induced Riemannian curvature R on M given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ \forall X, Y, Z \in \Gamma(TM).$$

By straightforward calculation, the non-vanishing local components of R are given by

$$\begin{aligned}
&R(U_1, U_i)U_1 = -U_i,\ i = 2, \ldots, 6; \quad R(U_1, U_i)U_i = -U_1,\ i = 2, 3, 5, 6;\\
&R(U_2, U_i)U_2 = U_i,\ i = 3, \ldots, 6; \quad R(U_2, U_i)U_i = -U_2,\ i = 3, 5, 6;\\
&R(U_3, U_i)U_3 = U_i,\ i = 4, 5, 6; \quad R(U_3, U_i)U_i = -U_3,\ i = 5, 6;\\
&R(U_4, U_i)U_i = -U_4,\ i = 5, 6;\\
&R(U_5, U_6)U_5 = U_6; \quad R(U_5, U_6)U_6 = -U_5.
\end{aligned} \tag{4.21}$$

By direct calculation, using relations (4.21) above, we obtain

$$(\nabla_{U_i} R)(U_j, U_k)U_l = 0, \quad i, j, k, l = 1, \ldots, 6.$$

Therefore, the lightlike hypersurface M of $\overline{M}(c = -1)$ is locally symmetric.

A submanifold M is said to be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies

$$B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{4.22}$$

where ρ is the smooth function on $\mathcal{U} \subset M$.

If we assume that M is a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$, then we have $B(X, Y) = \rho g(X, Y)$ for any $X, Y \in \Gamma(TM)$, which implies, by using (3.9), that $0 = B(\xi, \xi) = \rho$. Hence M is totally geodesic.

It follows that an indefinite Kenmotsu manifold $\overline{M}$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [2] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [13].

A submanifold M is said to be totally contact umbilical if its second fundamental form h of M satisfies [2]

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \tag{4.23}$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$). The totally contact umbilical condition (4.23) can be rewritten as,

$$h(X,Y) = B(X,Y)N = \{B_1(X,Y) + B_2(X,Y)\}N,$$

where $B_1(X,Y) = \lambda\{g(X,Y) - \eta(X)\eta(Y)\}$ and $B_2(X,Y) = \eta(X)B(Y,\xi) + \eta(Y)B(X,\xi)$.

If the $\lambda = 0$ (that is $B_1 = 0$), then the lightlike hypersurface M is said to be *totally contact geodesic* and if $B_2 = 0$, M is said to be *η-totally umbilical*. It is easy to check that a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold is η-totally umbilical. So, as was proved in [15], There exist no totally contact umbilical lightlike hypersurfaces of an indefinite Kenmotsu manifold $(\overline{M}^c, c \neq -1)$, tangent to the structure vector field ξ.

If M is totally contact umbilical of an indefinite Kenmotsu manifold $\overline{M}^c$, then, by Corollary 3.7 in [15], $c = -1$ and from (4.1), the induced curvature tensor R on M is given by, for any $X, Y, Z \in \Gamma(TM)$,

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X + B(Y,Z)A_N X - B(X,Z)A_N Y. \tag{4.24}$$

The covariant derivative of R is given by

$$\begin{aligned}
&(\nabla_W R)(X,Y)Z\\
&= (\nabla_W g)(X,Z)Y - (\nabla_W g)(Y,Z)X - (\nabla_W B)(X,Z)A_N Y\\
&\quad + (\nabla_W B)(Y,Z)A_N X - B(X,Z)(\nabla_W A_N)Y + B(Y,Z)(\nabla_W A_N)X\\
&= \{B(W,X)\theta(Z) + B(W,Z)\theta(X)\}Y - \{B(W,Y)\theta(Z) + B(W,Z)\theta(Y)\}X\\
&\quad - (\nabla_W B)(X,Z)A_N Y + (\nabla_W B)(Y,Z)A_N X - B(X,Z)(\nabla_W A_N)Y\\
&\quad + B(Y,Z)(\nabla_W A_N)X.
\end{aligned} \tag{4.25}$$

Taking $X = Z = E$, $W = U$ and $Y = V$ into (4.25) and using (4.23), we have

$$\overline{g}((\nabla_U R)(E,V)E, N) = -\lambda. \tag{4.26}$$

Thus, we have the following results.

Theorem 4.7. *There are no non-totally geodesic lightlike hypersurfaces of indefinite Kenmotsu manifolds $\overline{M}^c$, with $\xi \in TM$ and $(\nabla_U R)(E,V)E \in \Gamma(S(TM))$ which is totally contact umbilical.*

Theorem 4.8. *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$. If M is totally contact umbilical, then M is locally symmetric if and only if it is totally geodesic.*

Let M be a submanifold of a semi-Riemannian manifold $\overline{M}$. The second fundamental form $h = B \otimes N$ of M is said to be parallel if

$$(\nabla_X h)(Y, Z) = \{(\nabla_X B)(Y, Z) + \tau(X)B(Y, Z)\}N = 0. \tag{4.27}$$

A submanifold of a semi-Riemannian manifold with parallel fundamental form h is called a *parallel submanifold*. So, as was proved in [15], there are no parallel lightlike hypersurfaces of indefinite Kenmotsu manifold $(\overline{M}^c, c \neq -1)$, tangent to the structure vector field ξ. Also, in virtue of Theorem 3.3 in [15], M is totally geodesic. Thus using (4.27), we have:

Lemma 4.9. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in TM$. Then M is parallel if and only if it is totally geodesic.*

In virtue of Theorem 4.5 and Lemma 4.9, we have the following result.

Theorem 4.10. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$. Then M is locally symmetric if and only if it is parallel.*

5. Semi-symmetric lightlike hypersurfaces in indefinite Kenmotsu manifold

Definition 5.1. Let M be a lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$. We say that M is semi-symmetric [19], if the following condition is satisfied,

$$(R(W_1, W_2) \cdot R)(X, Y, Z, T) = 0 \quad \forall W_1, W_2, X, Y, Z, T \in \Gamma(TM), \tag{5.1}$$

where R is the induced Riemann curvature on M.

This is equivalent to

$$-R(R(W_1, W_2)X, Y, Z, T) - \cdots - R(X, Y, Z, R(W_1, W_2)T) = 0.$$

In general the condition (5.1) is not equivalent to $(R(W_1, W_2) \cdot R)(X, Y)Z = 0$ as in the non-degenerate setting. Indeed, by direct calculation we have for any $W_1, W_2, X, Y, Z, T \in \Gamma(TM)$

$$\begin{aligned}(R(W_1, W_2) \cdot R)(X, Y, Z, T) &= g((R(W_1, W_2) \cdot R)(X, Y)Z, T) \\ &\quad + (R(W_1, W_2) \cdot g)(R(X, Y)Z, T).\end{aligned} \tag{5.2}$$

In the sequel, we need the following proposition

Proposition 5.2. *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$. Then for any $W_1, W_2, Y, T \in \Gamma(TM)$ and $E \in \Gamma(TM^\perp)$, we have*

$$\begin{aligned}&(\overline{R}(W_1, W_2) \cdot \overline{R})(E, Y, E, T) \\ &= (R(W_1, W_2).R)(E, Y, E, T) - B(W_1, Y)R(E, A_N W_2, E, PT) \\ &\quad + B(W_2, Y)R(E, A_N W_1, E, PT) + B(Y, R(W_1, W_2)E)g(A_N E, PT) \\ &\quad - B(W_1, PT)R(E, Y, E, A_N W_2) + B(W_2, PT)R(E, Y, E, A_N W_1)\end{aligned}$$

$$
\begin{aligned}
&- \{(\nabla_{W_1} B)(W_2, PT) - (\nabla_{W_2} B)(W_1, PT) + \tau(W_1)B(W_2, PT) \\
&- \tau(W_2)B(W_1, PT)\}\overline{R}(E, Y, E, N) - \{(\nabla_{W_1} B)(W_2, Y) \\
&- (\nabla_{W_2} B)(W_1, Y) + \tau(W_1)B(W_2, Y) \\
&- \tau(W_2)B(W_1, Y)\}\overline{R}(E, N, E, T) - \theta(T)\{(\nabla_E B)(Y, R(W_1, W_2)E) \\
&- (\nabla_Y B)(E, R(W_1, W_2)E) + \tau(E)B(Y, R(W_1, W_2)E)\}. \qquad (5.3)
\end{aligned}
$$

Proof. The proof follows from direct calculation by using $T = PT + \theta(T)E$, $(\nabla_X B)(Y, E) = (\nabla_Y B)(X, E)$ and definition of $\overline{R} \cdot \overline{R}$. □

Next we investigate the effect of semi-symmetry condition on geometry of lightlike hypersurfaces in indefinite Kenmotsu manifold $\overline{M}^c$.

A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be $(\overline{\phi}(TM^\perp), D\oplus D')$-mixed totally geodesic if its second fundamental form h satisfies $h(X, Y) = 0$ (equivalently $B(X, Y) = 0$) for any $X \in \Gamma(\overline{\phi}(TM^\perp))$ and $Y \in \Gamma(D \oplus D')$.

Theorem 5.3. *Let M be a semi-symmetric lightlike hypersurface of indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in \Gamma(TM)$. Then at least one of the following holds*:

(i) $c = -1$.
(ii) $A_N E = 0$.
(iii) *M is $(\overline{\phi}(TM^\perp), D \oplus D')$-mixed totally geodesic.*

Proof. Let M be a semi-symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in \Gamma(TM)$. From (2.3) we have $\overline{R}(E, N, E, X) = 0$ and $\overline{R}(E, X, E, N) = 0, \forall X \in \Gamma(TM)$. By using relation (5.3) we obtain, for any $W_1, W_2, Y, T \in \Gamma(TM)$,

$$
\begin{aligned}
&(\overline{R}(W_1, W_2) \cdot \overline{R})(E, Y, E, T) \\
= &- B(W_1, Y)R(E, AW_2, E, PT) \\
&+ B(W_2, Y)R(E, AW_1, E, PT) + B(Y, R(W_1, W_2)E)g(AE, PT) \\
&- B(W_1, PT)R(E, Y, E, AW_2) + B(W_2, PT)R(E, Y, E, AW_1) \\
&- \theta(T)\{(\nabla_E B)(Y, R(W_1, W_2)E) - (\nabla_Y B)(E, R(W_1, W_2)E) \\
&+ \tau(E)B(Y, R(W_1, W_2)E) - \tau(Y)B(E, R(W_1, W_2)E)\}. \qquad (5.4)
\end{aligned}
$$

By direct calculation, using (2.3), the left hand side is given by

$$
\begin{aligned}
&(\overline{R}(W_1, W_2) \cdot \overline{R})(E, Y, E, T) \\
= &- \frac{3(c+1)}{4}\big\{g(W_1, Y)u(W_2)u(T) - g(W_2, Y)u(W_1)u(T) \\
&+ g(W_1, T)u(Y)u(W_2) - g(W_2, T)u(Y)u(W_1)\big\}. \qquad (5.5)
\end{aligned}
$$

From (4.1) we have $R(E, X)E = \frac{3(c+1)}{4}u(X)\overline{\phi}E$. By taking $W_1 = E$ and $W_2 = U$, into (5.4) and (5.5), we obtain, for any $Y, T \in \Gamma(TM)$

$$
-\frac{3}{4}(c+1)B(Y, V)g(A_N E, PT) = 0. \qquad (5.6)
$$

This equation completes the proof. □

Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ having constant $\overline{\phi}$-holomorphic sectional curvature c. Let Consider a local quasi-orthonormal frame $\{E, \overline{\phi}E, \overline{\phi}N, \xi, F_i, N\}_{1\leqslant i\leqslant 2n-4}$ on $\overline{M}^c$, where $\{E, \overline{\phi}E, \overline{\phi}N, \xi, F_i\}$ is a local frame field on M with respect to the decomposition (3.3). By definition $Ric(X,Y) = trace(Z \longrightarrow R(Z,X)Y)$, we have, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} Ric(X,Y) = & \sum_{i=1}^{2n-4} \varepsilon_i g(R(F_i,X)Y,F_i) + g(R(\overline{\phi}E,X)Y,\overline{\phi}N) \\ & + g(R(\overline{\phi}N,X)Y,\overline{\phi}E) + g(R(\xi,X)Y,\xi) + \overline{g}(R(E,X)Y,N), \end{aligned} \tag{5.7}$$

where, ε_i is the causal character of the vector field F_i, of the orthonormal frame field $\{F_i\}_{1\leqslant i\leqslant 2n-4}$ of non-degenerate distribution D_0. From relation (2.3) and Gauss-Codazzi equations, we obtain

$$\begin{aligned} g(R(F_i,X)Y,F_i) = & \frac{c-3}{4}\{\varepsilon_i g(X,Y) - g(X, g(Y,F_i)F_i)\} \\ & + \frac{c+1}{4}\{-\varepsilon_i \eta(X)\eta(Y) + \overline{g}(\overline{\phi}Y, g(\phi X, F_i)F_i) \\ & + 2\overline{g}(\overline{\phi}X, g(\phi Y, F_i)F_i)\} + B(X,Y)C(F_i,F_i) \\ & - B(F_i,Y)C(X,F_i), \end{aligned} \tag{5.8}$$

$$\begin{aligned} g(R(\overline{\phi}E,X)Y,\overline{\phi}N) = & \frac{c-3}{4}\{g(X,Y) - g(X, g(Y,\overline{\phi}E)\overline{\phi}N)\} \\ & - \frac{c+1}{4}\eta(X)\eta(Y) + B(X,Y)C(\overline{\phi}E,\overline{\phi}N) \\ & - B(\overline{\phi}E,Y)C(X,\overline{\phi}N), \end{aligned} \tag{5.9}$$

$$\begin{aligned} g(R(\overline{\phi}N,X)Y,\overline{\phi}E) = & \frac{c-3}{4}\{g(X,Y) - g(X, g(Y,\overline{\phi}N)\overline{\phi}E)\} \\ & - \frac{c+1}{4}\eta(X)\eta(Y) + B(X,Y)C(\overline{\phi}N,\overline{\phi}E) \\ & - B(\overline{\phi}N,Y)C(X,\overline{\phi}E), \end{aligned} \tag{5.10}$$

$$\begin{aligned} g(R(\xi,X)Y,\xi) = & \frac{c-3}{4}\{g(X,Y) - g(X,\eta(Y)\xi)\} \\ & + \frac{c+1}{4}\{-g(X,Y) + \eta(X)\eta(Y)\} \\ & + B(X,Y)C(\xi,\xi) - B(\xi,Y)C(X,\xi), \end{aligned} \tag{5.11}$$

$$\begin{aligned} g(R(E,X)Y,N) = & \frac{c-3}{4}g(X,Y) + \frac{c+1}{4}\{-\eta(X)\eta(Y) \\ & + \overline{g}(\overline{\phi}Y, \theta(\phi X)E) + 2\overline{g}(\overline{\phi}X, \theta(\phi Y)E)\}. \end{aligned} \tag{5.12}$$

So substituting (5.8), (5.9), (5.10), (5.11), and (5.12) in (5.7) and regrouping like terms, we have the following result.

Lemma 5.4. *Let M be a lightlike hypersurface of indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in \Gamma(TM)$. Then the Ricci tensor Ric on M is given by, for any $X, Y \in \Gamma(TM)$,*

$$Ric(X,Y) = ag(X,Y) - b\eta(X)\eta(Y) + B(X,Y)trA_N - B(A_N X, Y), \tag{5.13}$$

where $a = \frac{(2n+1)(c-3)+8}{4}$, and $b = \frac{(2n+1)(c+1)}{4}$ and trA_N is written with respect to g restricted to $S(TM)$.

Theorem 5.5. *Let M be a semi-symmetric lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$. Then either M is totally geodesic or $Ric(E,X) = 0$, for any $X \in \Gamma(S(TM)) - \langle\xi\rangle$, where Ric is the Ricci tensor of M.*

Proof. Suppose that M is a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$. Since $c = -1$, from (4.24), the induced Riemann curvature tensor R satisfies $R(X,Y)Z = g(X,Z)Y - g(Y,Z)X + B(Y,Z)A_N X - B(X,Z)A_N Y$. By direct calculation, we obtain, for any $X, Y, Z, T \in \Gamma(TM)$

$$\begin{aligned}(R(E,X)\cdot R)(E,Y,Z,T) = & - B(X,Y)B(A_N E,Z)g(A_N E,T) \\ & - B(Y,A_N E)B(X,Z)g(A_N E,T) \\ & - B(Y,Z)B(X,T)g(A_N E,A_N E).\end{aligned} \tag{5.14}$$

If M is semi-symmetric, the left hand of (5.14) vanishes and by taking $T = \xi$, we obtain

$$B(X,Y)B(A_N E,Z) + B(Y,A_N E)B(X,Z) = 0. \tag{5.15}$$

Since $B(A_N E,X) = -Ric(E,X)$ for any $X \in \Gamma(TM)$, by taking $Y = Z$ in (5.15) we obtain $-B(X,Y)Ric(E,Y) = 0$, which leads $B(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$ or $Ric(E,Y) = 0$ for any $Y \in \Gamma(S(TM)) - \langle\xi\rangle$. This complete the proof. □

Theorem 5.6. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E,\zeta) \neq 0$ for some $\zeta \in \Gamma(S(TM)) - \langle\xi\rangle$. Then, M is semi-symmetric if and only if it is totally geodesic.*

Since $Ric(E,X) = -B(A_N E,X)$, from Theorem 5.6, the following hold.

Corollary 5.7. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E,\zeta) \neq 0$ for some $\zeta \in \Gamma(S(TM)) - \langle\xi\rangle$, then M is not semi-symmetric.*

In virtue of Theorem 4.5 and Theorem 5.6, we have the following result.

Theorem 5.8. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E,\zeta) \neq 0$ for some $\zeta \in \Gamma(S(TM)) - \langle\xi\rangle$. Then, M is locally symmetric if and only if it is semi-symmetric.*

A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be semi-parallel [8] if its second fundamental form h satisfies, for any $W_1, W_2, X, Y \in \Gamma(TM)$,

$$(R(W_1, W_2) \cdot h)(X, Y) = -h(R(W_1, W_2)X, Y) - h(X, R(W_1, W_2)Y) = 0. \tag{5.16}$$

Proposition 5.9. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E, \zeta) \neq 0$ for some $\zeta \in \Gamma(S(TM)) - \langle \xi \rangle$. Then, M is semi-parallel if and only if it is totally geodesic.*

Proof. Let consider M a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$. Since $c = -1$, the curvature tensor R satisfies (4.24) and we have for any $W_1, W_2, X, Y \in \Gamma(TM)$

$$\begin{aligned}(R(W_1, W_2) \cdot h)(X, Y) = \big\{ & g(W_2, X)B(W_1, Y) - g(W_1, X)B(W_2, Y) \\ & - B(W_2, X)B(A_N W_1, Y) + B(W_1, X)B(A_N W_2, Y) \\ & + g(W_2, Y)B(W_1, X) - g(W_1, Y)B(W_2, X) \\ & - B(W_2, Y)B(A_N W_1, X) \\ & + B(W_1, Y)B(A_N W_2, X)\big\}N. \end{aligned} \tag{5.17}$$

Then, by taking $W_2 = E$ and $X = Y$ into (5.17), we obtain

$$(R(W_1, E) \cdot h)(X, X) = 2B(W_1, X)B(A_N E, X), \ \forall W_1, X \in \Gamma(TM). \tag{5.18}$$

If M is semi-parallel, since $B(A_N E, \zeta) = -Ric(E, \zeta) \neq 0$ for $\zeta \in \Gamma(S(TM))$ and $\zeta \neq \xi$, by (5.18), we infer that $B(W_1, X) = 0$, $\forall W_1, X \in \Gamma(TM)$. The converse is obtain by using (5.17). □

In virtue of Theorem 5.6 and Proposition 5.9, we have the following result.

Theorem 5.10. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E, \zeta) \neq 0$ for some $\zeta \in \Gamma(S(TM)) - \langle \xi \rangle$. Then, M is semi-symmetric if and only if it is semi-parallel.*

6. Ricci semi-symmetric lightlike hypersurfaces in indefinite Kenmotsu manifold

In this section, we study Ricci semi-symmetric lightlike hypersurfaces of indefinite Kenmotsu manifolds which have constant $\overline{\phi}$-holomorphic sectional curvature, tangent to the structure vector field ξ. We prove that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under some condition.

A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be Ricci semi-symmetric if the following condition is satisfied [5]

$$(R(W_1, W_2) \cdot Ric)(X, Y) = 0, \ \forall \ W_1, W_2, X, Y \in \Gamma(TM). \tag{6.1}$$

Where R and Ric are induced Riemannian curvature and Ricci tensor on M, respectively. This latter condition is equivalent to

$$-Ric((R(W_1, W_2)X, Y) - Ric(X, (R(W_1, W_2)Y) = 0.$$

In the following theorem we give result which shows the effect of Ricci semi-symmetry condition on the geometry of lightlike hypersurfaces of an indefinite Kenmotsu manifold $\overline{M}^c$.

Theorem 6.1. *Let M be a Ricci semi-symmetric lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in \Gamma(TM)$. Then either $c = -1$ or $Ric(E, \overline{\phi}E) = 0$. Moreover, if $c = -1$, then either M is totally geodesic or $Ric(E, A_N E) = 0$.*

Proof. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in \Gamma(TM)$. By using relation (5.13), we have, for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}(R(E,X)\cdot Ric)(E,Y) = \frac{c+1}{4}\big\{&3au(X)u(Y) + 3u(X)B(V,Y)trA_N \\ &- 3u(X)B(A_N V, Y) + u(Y)B(\phi X, A_N E) \\ &- \overline{g}(\overline{\phi}X, Y)B(V, A_N E) + 2u(X)B(\phi Y, A_N E)\big\} \\ &+ B(X,Y)B(A_N E, A_N E),\end{aligned} \tag{6.2}$$

where $a = \frac{(2n+1)(c-3)+8}{4}$. If M is Ricci semi-symmetric, then, by taking $Y = E$ into (6.2), we obtain

$$\frac{3}{4}(c+1)u(X)B(\phi E, A_N E) = 0$$

which implies, for $X = \overline{\phi}N$, $\frac{3}{4}(c-1)Ric(E, \overline{\phi}E) = 0$, since $B(\phi E, A_N E) = -Ric(E, \overline{\phi}E)$. On the other hand, suppose that $c = -1$. Using relation (6.2) and since $B(A_N E, A_N E) = -Ric(E, A_N E)$, we have, for any $X, Y \in \Gamma(TM)$, $B(X,Y)Ric(E, A_N E) = 0$ which completes the proof. □

Theorem 6.2. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E, A_N E) \neq 0$. Then M is Ricci semi-symmetric if and only if it is totally geodesic.*

Proof. In virtue of Theorem 6.1. The converse follows from (4.24), (6.1) and (5.13). □

Corollary 6.3. *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$, with $\xi \in \Gamma(TM)$ and $Ric(E, A_N E) \neq 0$. Then M is not Ricci semi-symmetric.*

Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$. If M is totally contact umbilical, then, in virtue of corollary 3.7 in [15], $c = -1$ and using the relation (6.2), we have, for any $X, Y \in \Gamma(TM)$

$$\begin{aligned}(R(E,X)\cdot Ric)(E,Y) &= B(X,Y)B(A_N E, A_N E) \\ &= \lambda^2(g(X,Y) - \eta(X)\eta(Y))(g(A_N E, A_N E) - 1),\end{aligned} \tag{6.3}$$

which lead, by taking $X = V$ and $Y = U$ to

$$(R(E,V)\cdot Ric)(E,U) = \lambda^2(g(A_N E, A_N E) - 1), \tag{6.4}$$

and we have the following result.

Proposition 6.4. *Let M be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in TM$ and $g(A_N E, A_N E) \neq 1$. Then M is Ricci semi-symmetric if and only if M is totally geodesic.*

Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$, with $\xi \in TM$. It is clear that, if $Ric(E, A_N E) \neq 0$, then $A_N E \in \Gamma(S(TM)) - \langle \xi \rangle$. Thus, from Theorem 4.5, Theorem 4.10, Theorem 5.6, Theorem 5.10 and Theorem 6.2, we obtain equivalence between parallel, semi-parallel, local symmetry, semi-symmetry and Ricci semi-symmetry notions.

Theorem 6.5. *In lightlike hypersurfaces of indefinite Kenmotsu space form $\overline{M}(c)$, tangent to the structure vector field ξ and $Ric(E, A_N E) \neq 0$, the conditions* (4.3), (4.27), (5.1), (5.16) *and* (6.1) *are equivalent.*

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