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A NOTE ON BLACKWELL'S THEOREM FOR FUZZY VARIABLES

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Abstract. Recently, some results of Blackwell's Theorem in which inter-arrival times are characterized as fuzzy variables under $t-$norm-based fuzzy operations are discussed by Hong. However, these results are invalid. In this note, we give counter examples of these results.

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1. Introduction

Recently, Hong [1] discussed Blackwell's Theorem in which inter-arrival times are characterized as fuzzy variables under $t-$norm-based fuzzy operations. He obtained Blackwell's Theorem for T-related fuzzy variables with respect to necessity measure holds true where T is an Archimedean t–norm and derived fuzzy Blackwell's Theorem based on the expected value of fuzzy variables. However, these results are invalid. Hong [2] provided a corrected version of a result. In this note, we give some counter examples of these results.

2. Counterexamples

Hong [1] obtained the following results of Blackwell's Theorem for fuzzy variables.

Theorem 2.1 (Hong [1]). *Let T be a continuous Archimedean t-norm. Let $\xi_i = (a, \alpha, \beta)_{LR}$, $i = 1, 2, \cdots$, be a sequence of T-related L-R fuzzy variables with $0 \leq \alpha < a$ and $N(t)$ be a fuzzy renewal variable. If c/a is a natural number then for any $\epsilon > 0$,*

$$lim_{t\to\infty} Nes\left(\left|(N(t+c) - N(t)) - \frac{c}{a}\right| < \epsilon\right) = 1.$$

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Theorem 2.2 (Hong [1])**.** *Let T be a continuous Archimedean t-norm. Let $\xi_i = (a, \alpha, \beta)_{LR}$, $i = 1, 2, \cdots$, denote a sequence of T-related L-R fuzzy variables with $0 \le \alpha < a$ and $N(t)$ be a fuzzy renewal variable. If c/a is a natural number then*

$$lim_{t\to\infty} E[N(t+c) - N(t)] = \frac{c}{a}.$$

However, these results are invalid. The following example shows that Theorem 2.1 is wrong.

Example 2.3. Let $\xi = \xi_i = (1, 1/2, 1/2)_{LR}$, $i = 1, 2, \cdots$ with $L(x) = R(x) = 1-x$. Let T be an Archimedean t-norm. Let $c = 1$. We then have for, $0 < \epsilon < 1$,

$$\begin{aligned} & Nes(1-\epsilon < N(t+1) - N(t) < 1+\epsilon) \\ = \ & 1 - Pos(N(t+1) - N(t) \notin (1-\epsilon, 1+\epsilon)) \\ = \ & 1 - sup_{n\neq 1} Pos(N(t+1) - N(t) = n) \\ \le \ & 1 - Pos(N(t+1) - N(t) = 2) \\ = \ & 1 - sup_{k\ge 1} Pos(N(t+1) = k+2, N(t) = k) \\ \le \ & 1 - Pos(N(t+1) = k+2, N(t) = k). \end{aligned} \tag{1}$$

Let $t_n = n + (1-\delta)$ for small $\delta > 0$ and $k = n$. Let

$$\xi_1 = \xi_2 = \cdots = \xi_{n+1} = 1, \xi_{n+2} = 1-\delta, \xi_{n+3} = 1.$$

Then we have

$$S_n = n, S_{n+1} = n+1, S_{n+2} = n+1+(1-\delta), S_{n+3} = n+2+(1-\delta)$$

and

$$S_n \le t_n < S_{n+1}, S_{n+2} \le t_n + 1 < S_{n+3}.$$

Then

$$\begin{aligned} & Pos(N(t_n+1) - N(t_n) = 2) \\ \ge \ & Pos(N(t_n+1) = n+2, N(t_n) = n) \\ = \ & Pos(S_{n+2} \le t_n + 1 < S_{n+3}, S_n \le t_n < S_{n+1}) \\ \ge \ & Pos(\xi_1 = \xi_2 = \cdots = \xi_{n+1} = 1, \xi_{n+2} = 1-\delta, \xi_{n+3} = 1) \\ = \ & T(1, 1, \cdots, 1, \mu_\xi(1-\delta), 1) \\ = \ & T(\mu_\xi(1-\delta), 1) \\ = \ & \mu_\xi(1-\delta). \end{aligned} \tag{2}$$

From (1) and (2), we have

$$Nes(1-\epsilon < N(t_n+c) - N((t_n) < 1+\epsilon) \le 1 - \mu_\xi(1-\delta)$$

and since $\delta > 0$ is arbitrary and $lim_{\delta\to 0}\mu_\xi(1-\delta) = 1$,

$$lim_{n\to\infty} Nes\left(\left|(N((t_n+c) - N((t_n)) - \frac{c}{a}\right| < \epsilon\right) = 0.$$

Therefore, Theorem 2.1 is wrong.

The following example shows that Theorem 2.2 is wrong.

Example 2.4. Let $\xi = \xi_i = (1, 1/2, 0)_{LR}, i = 1, 2, \cdots$ with $L(x) = R(x) = 1-x$. Let T be an continuous Archimedean t-norm. Let $c = 1$. Let $t_n = n + (1 - \delta)$ for small $\delta > 0$ and $k = n$. Let

$$\xi_1 = \xi_2 = \cdots = \xi_{n+1} = 1, \xi_{n+2} = 1 - \delta, \xi_{n+3} = 1.$$

Then, from (1) in Example 1,

$$Pos(N(t_n + 1) - N(t_n) = 2) = 1$$

By taking $\xi_i = 1, i = 1, 2, \cdots$, $Pos(N(t + 1) - N(t) = 1) = 1, t > 0$ is easy to check. We also note that

$$\begin{aligned} Pos(N(t+1) - N(t) = 0) &= sup_{k\geq 1} Pos(N(t) = k, N(t+1) = k) \\ &= sup_{k\geq 1} Pos(N(t) = k, \xi_k > 1) \\ &\leq Pos(\xi_k > 1) = 0. \end{aligned}$$

Then we have, for $n > 1$,

$$[N(t_n + 1) - N(t_n)]'_\alpha + [N(t_n + 1) - N(t_n)]''_\alpha \geq 3$$

and hence

$$lininf_{n\to\infty} E[N(t_n + c) - N(t_n)] \geq \frac{3}{2} > \frac{c}{a} = 1.$$

Therefore, Theorem 2.2 is wrong.

References

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2. D.H. Hong, *Erratum to "Blackwell's Theorem for T-related fuzzy variables"*, Information Sciences **250** (2013), 227-228.

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