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REMARKS ON CONVERGENCE OF INDUCTIVE MEANS

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ABSTRACT. We define new inductive mean constructed by a mean on a complete metric space, and see its convergence when the intrinsic mean is given. We also give many examples of inductive matrix means and claim that the limit of inductive mean constructed by the intrinsic mean is not the Karcher mean, in general.

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1. Introduction

Among a variety of applications for positive definite matrices, the process of averaging has become attractive and widely studied. Since the two-variable geometric mean of positive definite matrices A and B

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

has been introduced by Kubo and Ando [9], its algebraic and geometric properties have been studied. On the open convex cone of positive definite matrices equipped with the Riemannian trace metric $d(X,Y) = \| \log(X^{-1/2}YX^{-1/2})\|_2$, the unique geodesic connecting from A to B is given by

$$t \in [0,1] \mapsto A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$

See [3] for more details. An extensive theory of two-variable geometric mean to the multivariable geometric mean has sprung up and has remained problematic. Ando, Li, and Mathias [1] have especially suggested a symmetrization procedure to the multivariable geometric mean of positive definite matrices including ten desirable properties for extended geometric means. Moreover, a convergence for symmetrization procedure has been recently proved by Kim and Petz [8].

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A natural and attractive average of positive definite matrices is the *least squares mean*, also called the *Cartan mean, Riemannian barycenter*, and *Karcher mean.* The Karcher mean $\Lambda(\omega; A_1, \dots, A_n)$ of positive definite matrices $A_1, \dots, A_n$ and a positive probability vector $\omega = (w_1, \dots, w_n)$ is defined as the unique minimizer (provided it exists) of the weighted sum of squares of the Riemannian trace distances to each of the point. That is,

$$\Lambda(\omega; A_1, \dots, A_n) = \underset{Z \in \mathbb{P}}{\arg\min} \sum_{i=1}^{n} w_i d^2(Z, A_i). \tag{1}$$

In general, such minimizer exists uniquely on a Hadamard space, which is a complete metric space satisfying the semiparallelogram law. While many interesting properties of the Karcher mean including Ando-Li-Mathias properties have been developed, the remarkable one is that the limit of inductive mean given by

$$T_1 = A_1, \quad T_{k+1} = T_k \#_{\lambda_{k+1}} A_{\overline{k+1}}, \quad k \geq 1,$$

where $\lambda_k = \dfrac{w_{\overline{k}}}{\sum_{j=1}^{k} w_{\overline{j}}}$ and $\overline{k}$ denotes the residual of $k \mod n$, coincides with the Karcher mean [14].

One can naturally ask what if we replace the original inductive mean by other geometric means. In this article we consider a new inductive mean constructed by a given mean generally on a complete metric space. We mainly show that the limit of the inductive mean constructed by a symmetric and multiplicative mean is the given mean, and see several examples for positive definite matrices.

2. Convergence of inductive means

Let (X, d) be a complete metric space. Let $\omega \in \Delta_n$, the simplex of positive probability vectors in $\mathbb{R}^n$ convexly spanned by the unit coordinate vectors. A weighted *n-mean* G_n on X for $n \geq 2$ is a continuous map $G_n : \Delta_n \times X^n \to X$ that is idempotent in the sense that $G_n(\omega; x, \dots, x) = x$ for all $x \in X$. A weighted n-mean G_n is *symmetric* or *permutation invariant* if $G_n(\omega_\sigma; \mathbf{x}_\sigma) = G_n(\omega; \mathbf{x})$, where $\omega_\sigma = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$ and $\mathbf{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for each permutation σ on $\{1, \dots, n\}$. A mean G on X is a sequence of means $\{G_n\}_{n \geq 2}$.

For $\omega = (w_1, \dots, w_n) \in \Delta_n$ and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, we denote by

$$\omega^k = \frac{1}{k}(w_1, \dots, w_n, w_1, \dots, w_n, \dots, w_1, \dots, w_n) \in \Delta_{nk},$$
$$\mathbf{x}^k = (x_1, \dots, x_n, x_1, \dots, x_n, \dots, x_1, \dots, x_n) \in X^{nk},$$

where the number of blocks is k. Also,

$$\omega^\infty = (w_1, \dots, w_n, \dots, w_1, \dots, w_n, \dots),$$
$$\mathbf{x}^\infty = (x_1, \dots, x_n, \dots, x_1, \dots, x_n, \dots).$$

Note that ω^∞ is an infinite-dimensional vector, not a probability vector.

Definition 2.1. Let $\omega \in \Delta_n$. A mean $G = \{G_n\}$ on X is said to be *multiplicative* if for all n and $\mathbf{x} \in X^n$,

$$G_n(\omega; \mathbf{x}) = G_{nk}(\omega^k; \mathbf{x}^k), \quad k \geq 2.$$

If G is symmetric and multiplicative, then G is called *intrinsic.*

Let $\mathbf{a} = (a_1, \dots, a_m) \in X^m$ and let $\omega = (w_1, \dots, w_m) \in \Delta_m$. For a mean $G = \{G_n\}_{n \geq 2}$, we consider the inductive mean defined as

$$S_k^G(\omega; \mathbf{a}) = G_k(\omega^{(k)}; \mathbf{a}^{(k)}), \quad k \geq m, \tag{2}$$

where $\overline{k}$ denotes the residual of $k \mod m$, and

$$\begin{aligned} \omega^\infty(k) &= w_{\overline{k}}, \quad \mathbf{a}^\infty(k) = a_{\overline{k}} \\ \omega^{(k)} &= \frac{1}{\sum_{i=1}^k \omega^\infty(i)} (\omega^\infty(1), \dots, \omega^\infty(k)) \in \Delta_k, \\ \mathbf{a}^{(k)} &= (\mathbf{a}^\infty(1), \dots, \mathbf{a}^\infty(k)) \in X^k. \end{aligned}$$

We also denote the multiple of m by m. Note that

$$\begin{aligned} S_m^G(\omega; \mathbf{a}) &= G_m(\omega; \mathbf{a}) \\ S_{m+1}^G(\omega; \mathbf{a}) &= G_{m+1}\left(\frac{1}{1+w_1}(\omega, w_1); \mathbf{a}, a_1\right) \\ &\vdots \\ S_{2m-1}^G(\omega; \mathbf{a}) &= G_{2m-1}\left(\frac{1}{2-w_m}(\omega, w_1, \dots, w_{m-1}); \mathbf{a}, a_1, \dots, a_{m-1}\right) \\ S_{2m}^G(\omega; \mathbf{a}) &= G_{2m}(\omega^2; \mathbf{a}^2) \\ &\vdots \end{aligned}$$

Proposition 2.2. *Let $G = \{G_n\}_{n \geq 2}$ be a symmetric mean satisfying that for all n, $\omega = (w_1, \dots, w_n) \in \Delta_n$ and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$*

$$S_n^G(\omega; \mathbf{x}) = S_{n-p+1}^G\left(\sum_{i=1}^p w_i, w_{p+1}, \dots, w_n; x_1, x_{p+1}, \dots, x_n\right) \tag{3}$$

if $x_1 = \cdots = x_p$ for $1 \leq p < n$. Then G is the intrinsic mean.

Proof. Let $\omega = (w_1, \dots, w_m) \in \Delta_m$ and $\mathbf{x} = (x_1, \dots, x_m) \in X^m$. It is enough to show that G is multiplicative. Using the permutation invariance and the condition (3) sequentially yield that for $k \geq 2$,

$$\begin{aligned}
&G_{mk}(\omega^k; \mathbf{x}^k) \\
&= G_{mk}\left(\frac{1}{k}(w_1, \ldots, w_1, \ldots, w_m, \ldots, w_m); x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m\right) \\
&= G_{mk-k+1}\left(\frac{1}{k}(kw_1, w_2, \ldots, w_2, \ldots, w_1, \ldots, w_m, \ldots, w_m); x_1, x_2, \ldots, x_2, \ldots, x_m, \ldots, x_m\right) \\
&\vdots \\
&= G_m(w_1, \ldots, w_m; x_1, \ldots, x_m).
\end{aligned}$$

□

We now see our main result about the convergence of inductive means.

Theorem 2.3. *Let $G = \{G_n\}_{n \geq 2}$ be a symmetric mean satisfying the property (3). Then for any $\mathbf{a} = (a_1, \ldots, a_m) \in X^m$ and $\omega = (w_1, \ldots, w_m) \in \Delta_m$,*

$$\lim_{k \to \infty} S_k^G(\omega^{(k)}, \mathbf{a}^{(k)}) = G_m(\omega; \mathbf{a}).$$

Proof. Let $k \in \mathbb{N}$. Then there are $p, r \in \mathbb{N} \cup \{0\}$ such that $k = pm + r$ and $0 \leq r < m$ by the division algorithm. By permutation invariance and the condition (3),

$$\begin{aligned}
&S_k^G(\omega^{(k)}, \mathbf{a}^{(k)}) \\
&= G_k\left(\frac{1}{s_k}(\omega^\infty(1), \ldots, \omega^\infty(k)); a_1, \ldots, a_m, \ldots, a_1, \ldots, a_m, a_1, \ldots, a_r\right) \\
&= G_m\left(\frac{1}{s_k}((p+1)w_1, \ldots, (p+1)w_r, pw_{r+1}, \ldots, pw_m); a_1, \ldots, a_r, a_{r+1}, \ldots, a_m\right),
\end{aligned}$$

where $s_k = p + \sum_{i=1}^{r} w_i$. Note that $k \to \infty$ is equivalent to $p = \dfrac{k-r}{m} \to \infty$. Then $\dfrac{(p+1)w_s}{p + \sum_{i=1}^{r} w_i} \to w_s$ for $s = 1, \ldots, r$, and $\dfrac{pw_t}{p + \sum_{i=1}^{r} w_i} \to w_t$ for $t = r+1, \ldots, m$. Since a mean G_m is continuous, we conclude

$$\lim_{k \to \infty} S_k^G(\omega^{(k)}, \mathbf{a}^{(k)}) = G_m(\omega; \mathbf{a}).$$

□

3. Multivariable means of positive definite matrices

In this section we see multivariable matrix means on the open convex cone $\mathbb{P}$ of positive definite matrices. Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$ and $\omega = (w_1, \ldots, w_n) \in \Delta_n$.

Remark 3.1. The *weighted arithmetic mean* and *weighted harmonic mean*

$$\mathcal{A}(\omega; \mathbb{A}) = \sum_{i=1}^{n} w_i A_i, \quad \mathcal{H}(\omega; \mathbb{A}) = \left[\sum_{i=1}^{n} w_i A_i^{-1} \right]^{-1} \tag{4}$$

are the intrinsic means. Moreover, the identity (3) is satisfied for both $\mathcal{A}$ and $\mathcal{H}$, and hence,

$$\lim_{k\to\infty} S_k^{\mathcal{A}}(\omega^{(k)}, \mathbb{A}^{(k)}) = \mathcal{A}(\omega; \mathbb{A}), \quad \lim_{k\to\infty} S_k^{\mathcal{H}}(\omega^{(k)}, \mathbb{A}^{(k)}) = \mathcal{H}(\omega; \mathbb{A}).$$

Remark 3.2. The *resolvent mean* for parameter $\mu \geq 0$ is defined by

$$\mathcal{R}(\omega; \mathbb{A}) = \left[\sum_{i=1}^{n} w_i (A_i + \mu I)^{-1} \right]^{-1} - \mu I. \tag{5}$$

Bauschke, Moffat, and Wang [2] introduced the resolvent mean whose origin comes from the proximal average in convex analysis and optimization. Since then, many scholars have found fascinating properties such as the monotonicity for parameters and the nonexpansiveness [7, 10].

One can see the resolvent mean alternatively as

$$\mathcal{R}(\omega; \mathbb{A}) = \mathcal{H}(\omega; \mathbb{A} + \mu \mathbb{I}) - \mu I,$$

where $\mathbb{I} = (I, \ldots, I) \in \mathbb{P}^n$ and I is the identity matrix. So it is an intrinsic mean, and satisfies the equality (3). Therefore,

$$\lim_{k\to\infty} S_k^{\mathcal{R}}(\omega^{(k)}, \mathbb{A}^{(k)}) = \mathcal{R}(\omega; \mathbb{A}).$$

We now review a natural and attractive average, the Karcher mean, among many geometric means. Also we see some recent results and the connection with the Log-Euclidean mean.

Remark 3.3. The *Karcher mean* (or the least squares mean, Riemannian centroid, Cartan barycenter) $\Lambda(\omega; \mathbb{A})$ is defined as the unique minimizer of the weighted sum of the squares of the Riemannian trace metric δ:

$$\Lambda(\omega; \mathbb{A}) = \underset{X \in \mathbb{P}}{\arg\min} \sum_{i=1}^{n} w_i \delta^2(X, A_i) \tag{6}$$

where $\delta(A, B) = \| \log(A^{-1/2} B A^{-1/2}) \|_F$ and $\| \cdot \|_F$ denotes the Frobenius norm. Using Karcher's formula in [5] for the gradient of the objective function yields that the Karcher mean $\Lambda(\omega; \mathbb{A})$ is the unique positive definite solution of the *Karcher equation*

$$\sum_{i=1}^{n} w_i \log(X^{1/2} A_i^{-1} X^{1/2}) = O. \tag{7}$$

Recently, many interesting properties of the Karcher mean have been widely studied. It has been shown in [14] that power means $P_t(\omega; \mathbb{A})$ defined by the unique positive solution of the equation

$$X = \sum_{i=1}^{n} w_i X \#_t A_i,\ t \in (0,1),$$

converge to the Karcher mean as $t \to 0^+$, where $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is the weighted geometric mean of A and B in $\mathbb{P}$. Also, it has been proved in [6] that certain sequence of the Karcher means converges to the Log-Euclidean mean:

$$\lim_{m\to\infty} \Lambda(\omega; A_1^{1/m}, \ldots, A_n^{1/m}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right). \tag{8}$$

The following shows the continuity of power means $P : (0,1] \times \Delta_n \times \mathbb{P}^n \to \mathbb{P},\ P(t,\omega,\mathbb{A}) = P_t(\omega;\mathbb{A})$ with respect to the Thompson metric d on $\mathbb{P}$ given by

$$d(A,B) = \| \log(A^{-1/2}BA^{-1/2})\|,$$

where $\|X\|$ denotes the operator norm of X.

Lemma 3.1 ([12, Proposition 3.5]). *Let $\omega, \mu \in \Delta_n$ and $\mathbb{A}, \mathbb{B} \in \mathbb{P}^n$. Then for $s, t \in (0,1]$*

$$d(P_s(\omega;\mathbb{A}), P_t(\mu;\mathbb{B})) \leq \frac{|s-t|}{\max\{s,t\}}\Delta(\mathbb{A}) + \frac{1}{\max\{s,t\}} d(\omega,\mu) + \max_{1\leq i\leq n} d(A_i, B_i),$$

where $\Delta(\mathbb{A}) = \max_{1\leq i,j\leq n}\{d(A_i,A_j)\}$ denotes the diameter of $\mathbb{A} = (A_1,\ldots,A_n)$ and

$$d((s_1,\ldots,s_n),(t_1,\ldots,t_n)) = \max_{1\leq i\leq n}\left|\log\frac{s_i}{t_i}\right|$$

on $\mathbb{R}_+^n$, where $\mathbb{R}_+ = (0,\infty)$.

Theorem 3.2. *Let $\mathbb{A} = (A_1,\ldots,A_n) \in \mathbb{P}^n$ and $\omega = (w_1,\ldots,w_n) \in \Delta_n$. Then*

$$\lim_{k\to\infty} S_k^{\Lambda}(\omega^{(k)}, \mathbb{A}^{(k)}) = \Lambda(\omega;\mathbb{A}).$$

Proof. Let $\epsilon > 0$ be given and let $\omega, \mu \in \Delta_n$. By Lemma 3.1, there is $\delta > 0$ such that $d(\omega,\mu) < \delta$ implies

$$d(P_t(\omega;\mathbb{A}), P_t(\mu;\mathbb{A})) < \frac{\epsilon}{3}.$$

For such $\omega, \mu \in \Delta_n$

$$d(\Lambda(\omega;\mathbb{A}), P_t(\omega;\mathbb{A})) < \frac{\epsilon}{3} \quad \text{and} \quad d(\Lambda(\mu;\mathbb{A}), P_t(\mu;\mathbb{A})) < \frac{\epsilon}{3}$$

for $t > 0$ small enough, since the power means P_t converges to the Karcher mean as $t \to 0$ in [12, 14]. By the triangle inequality, we have

$$\begin{aligned} &d(\Lambda(\omega;\mathbb{A}), \Lambda(\mu;\mathbb{A})) \\ &\leq d(\Lambda(\omega;\mathbb{A}), P_t(\omega;\mathbb{A})) + d(P_t(\omega;\mathbb{A}), P_t(\mu;\mathbb{A})) + d(P_t(\mu;\mathbb{A}), \Lambda(\mu;\mathbb{A})) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Moreover, by Theorem 4.4, (P5) in [12]

$$d(\Lambda(\omega;\mathbb{A}),\Lambda(\omega;\mathbb{B})) \leq \max_{1\leq i\leq n} d(A_i,B_i).$$

So the Karcher mean $\Lambda : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ is continuous, and is a symmetric mean. Indeed, the Karcher mean satisfies all of the Ando-Li-Mathias properties [11].

Let $X = \Lambda(\omega;\mathbb{A})$. If $A_1 = \cdots = A_k$ for $1 \leq k < n$, then the Karcher equation (7) reduces to

$$O = \left(\sum_{i=1}^{k} w_i\right) \log(X^{1/2}A_1^{-1}X^{1/2}) + \sum_{j=k+1}^{n} w_j \log(X^{1/2}A_j^{-1}X^{1/2}).$$

It means that $X = \Lambda\left(\sum_{i=1}^{k} w_i, w_{k+1},\ldots,w_n; A_1, A_{k+1},\ldots,A_n\right)$. So by Theorem 2.3 it is proved. □

Proposition 3.3. *Let $\mathbb{A} = (A_1,\ldots,A_n) \in \mathbb{P}^n$ and $\omega = (w_1,\ldots,w_n) \in \Delta_n$. Then the double limit $\lim_{k,m\to\infty} S_k^{\Lambda}(\omega^{(k)},[\mathbb{A}^{(k)}]^{1/m})^m$ converges, and*

$$\lim_{k,m\to\infty} S_k^{\Lambda}(\omega^{(k)},[\mathbb{A}^{(k)}]^{1/m})^m = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right),$$

where $[\mathbb{A}^{(k)}]^{1/m} = (\mathbb{A}^{\infty}(1)^{1/m},\ldots,\mathbb{A}^{\infty}(k)^{1/m})$.

Proof. By Theorem 3.2 and the fact that the map $A \in \mathbb{P} \mapsto A^m$ is continuous, we have

$$\lim_{k\to\infty} S_k^{\Lambda}(\omega^{(k)},[\mathbb{A}^{(k)}]^{1/m})^m = \Lambda(\omega;\mathbb{A}^{1/m})^m.$$

By the property (8) we get

$$\lim_{m\to\infty}\lim_{k\to\infty} S_k^{\Lambda}(\omega^{(k)},[\mathbb{A}^{(k)}]^{1/m})^m = \lim_{m\to\infty}\Lambda(\omega;\mathbb{A}^{1/m})^m = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right).$$

By the definition of inductive means S_k^{Λ} and the property (8), on the other hand, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} S_k^{\Lambda}(\omega^{(k)},[\mathbb{A}^{(k)}]^{1/m})^m \\
&= \lim_{m\to\infty}\Lambda\left(\frac{1}{s(k)}((p+1)w_1,\ldots,(p+1)w_r,pw_{r+1},\ldots,pw_n); A_1^{1/m},\ldots,A_n^{1/m}\right)^m \\
&= \exp\left(\sum_{j=1}^{r}\frac{(p+1)w_j}{s(k)}\log A_j + \sum_{j=r+1}^{n}\frac{pw_j}{s(k)}\log A_j\right),
\end{aligned}$$

where $s(k) = p + \sum_{i=1}^{r} w_i$ and $k = pn + r$ for some $p \in \mathbb{N}$ and $0 \leq r < n$. Note that $k \to \infty$ is equivalent to $p \to \infty$, and hence,

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{m\to\infty} S_k^{\Lambda}(\omega^{(k)}, [\mathbb{A}^{(k)}]^{1/m})^m \\
&= \lim_{p\to\infty} \exp\left(\sum_{j=1}^{r} \frac{(p+1)w_j}{s(k)} \log A_j + \sum_{j=r+1}^{n} \frac{pw_j}{s(k)} \log A_j\right) \\
&= \exp\left(\sum_{i=1}^{n} w_i \log A_i\right).
\end{aligned}$$

□

4. Final Remarks

Hadamard spaces are important examples of complete convex metric spaces [13, 17]. Here, a complete metric space (M, d) is called a *Hadamard space* if it satisfies the semiparallelogram law; for each $x, y \in M$, there exists an $m \in M$ satisfying

$$d^2(m, z) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y) \tag{9}$$

for all $z \in M$. Such spaces are also called (global) CAT(0)-spaces or NPC (non-positively curved) spaces. The point m appearing in (9) is the unique metric midpoint between x and y. The midpoint operation gives rise to a unique minimal geodesic $\gamma_{x,y} : [0, 1] \to M$ for given two points x and y. We denoted by $x\#_t y := \gamma_{x,y}(t)$ and call it the *weighted geometric mean* of x and y. The typical example of Hadamard space is the open convex cone $\mathbb{P}$ of positive definite matrices with Riemannian trace metric.

On a Hadamard space (M, d), the least squares mean

$$\Lambda(\omega; a_1, \ldots, a_n) = \underset{z\in M}{\arg\min} \sum_{i=1}^{n} w_i d^2(z, a_i) \tag{10}$$

exists uniquely. Motivated by Strong Law of Large Number on Hadamard spaces that established by K. Sturm [17], J. Holbrook [4], Y. Lim and M. Palfia [15] found a deterministic approximation to the least squares mean mean: For $\mathbf{a} = (a_1, \ldots, a_n) \in M^n$,

$$\Lambda(\omega; \mathbf{a}) = \lim_{k\to\infty} T_k(\omega^{(k)}; \mathbf{a}^{(k)}), \tag{11}$$

where

$$T_1 = a_1, \quad T_{k+1} = T_k \#_{\lambda_{k+1}} a_{\overline{k+1}},$$

and $\lambda_k = \dfrac{w_{\overline{k}}}{\sum_{j=1}^{k} w_{\overline{j}}}$ and $\overline{k}$ denotes the residual of $k \mod n$.

We naturally ask whether or not the inductive mean S_k^G defined in (2) for given mean $G = \{G_n\}_{n\geq 2}$ satisfies the property (11). One can see that if the mean G is symmetric and satisfies the property (3), then the property (11) does not hold. Indeed, suppose that $S_k^G(\omega^{(k)}, \mathbf{a}^{(k)})$ converges to the Karcher mean as $k \to \infty$. Then every subsequence of $S_k^G(\omega^{(k)}, \mathbf{a}^{(k)})$ should converge to the Karcher mean. However, by Proposition 2.2 we have

$$S_{pn}^G(\omega^{(pn)}, \mathbf{a}^{(pn)}) = G\left(\frac{1}{pn}, \dots, \frac{1}{pn}; a_1, \dots, a_n, \dots, a_1, \dots, a_n\right) = G(\omega; \mathbf{a})$$

for any $p \in \mathbb{N}$. This means that $\lim_{p\to\infty} S_{pn}^G(\omega^{(pn)}, \mathbf{a}^{(pn)}) = G(\omega; \mathbf{a})$, which is a contradiction to converge to the Karcher mean.

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