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POSITIVE SOLUTIONS FOR A THREE-POINT FRACTIONAL BOUNDARY VALUE PROBLEMS FOR P-LAPLACIAN WITH A PARAMETER†

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Abstract. In this paper, we firstly use Krasnosel'skii fixed point theorem to investigate positive solutions for the following three-point boundary value problems for p-Laplacian with a parameter

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + \lambda f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$D^{\alpha}_{0+}u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta),$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ D^{\alpha}_{0+}$ is the Caputo's derivative, $\alpha \in (2, 3],\ \eta, \gamma \in (0, 1), \lambda > 0$ is a parameter. Then we use Leggett-Williams fixed point theorem to study the existence of three positive solutions for the fractional boundary value problem

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$D^{\alpha}_{0+}u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta),$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ D^{\alpha}_{0+}$ is the Caputo's derivative, $\alpha \in (2, 3],\ \eta, \gamma \in (0, 1)$.

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1. Introduction

It is well known that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, so the differential equations with fractional-order derivative are more adequate than integer order derivative for some real world problems. Therefore, the fractional differential equations have been of great interest recently, this is

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because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc. see [11, 13, 17-19, 28, 32] for example. Some recent investigations have shown that many physical systems can be represented more accurately using fractional derivative formulations [2, 3]. Boundary value problems of fractional differential equations have been investigated in many papers (see [1, 7, 8, 14, 15, 21-25, 27, 29, 33] and references cited therein). The eigenvalue problems of integer differential equations have been studied extensively by many authors. As far as the eigenvalue problems of fractional differential equations are concerned, there are a few results (see [5, 10, 34]).

Z. Bai [5] studied the eigenvalue intervals for a class of fractional boundary value problem

$$
\begin{gathered}
{}^{C}D_{0+}^{\alpha}u(t) + \lambda h(t) f(u(t)) = 0, \ 0 < t < 1, \\
u(0) = u'(1) = u''(0) = 0,
\end{gathered}
$$

where $2 < \alpha \leq 3$, ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, $\lambda > 0$ is a parameter.

C. Zhai, L. Xu [25] considered the nonlinear fractional four-point boundary value problem with a parameter

$$
\begin{gathered}
D_{0+}^{\alpha}u(t) + \lambda f(t, u(t)) = 0, \ 0 < t < 1, \\
u'(0) - \mu_1 u(\xi) = 0, \quad u'(1) + \mu_2 u(\eta) = 0,
\end{gathered}
$$

where $1 < \alpha \leq 2, \ 0 \leq \xi \leq \eta \leq 1, \ 0 \leq \mu_1, \mu_2 \leq 1, \ \lambda > 0$ is a parameter.

X. Zhang, L. Liu and Y. Wu [31] investigated the singular eigenvalue problem for a higher order fractional differential equation

$$
\begin{gathered}
-D^{\alpha}x(t) = \lambda f(x(t), \ D^{\mu_1}x(t), \ D^{\mu_2}x(t), \ \cdots, \ D^{\mu_{n-1}}x(t)), \ \ 0 < t < 1, \\
x(0) = 0, \ D^{\mu_i}x(0) = 0, \ D^{\mu}x(1) = \sum_{j=1}^{p-2} a_j D^{\mu}x(\xi_j), \ 1 \leq i \leq n-1,
\end{gathered}
$$

where $n \geq 3, \ n \in N, \ n-1 < \alpha \leq n, \ n-l-1 < \alpha - \mu_1 < n-l, \ l = 1, 2, \cdots, n-2, \ \mu - \mu_{n-1} > 0, \ \alpha - \mu_{n-1} \leq 2, \ \alpha - \mu > 1, \ a_j \in [0, +\infty), \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1, \ 0 < \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1} < 1$, D^{α} is the Riemann-Liouville fractional derivative.

The equation with a p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws, and so on. Recently, the existence of solutions to boundary value problems for fractional differential equation with p-Laplacian operator have been studied extensively in the literatures, (see [6, 16, 20, 26]).

G. Chai [6] investigated the existence and multiplicity of positive solutions for the boundary value problem of fractional differential equation with p-Laplacian operator

$$
D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u))(t) + f(t, u(t)) = 0, \ 0 < t < 1,
$$

$$u(0) = 0,\ u(1) + \sigma D^{\gamma}_{0+}u(1) = 0,\ D^{\alpha}_{0+}u(0) = 0,$$

where $D^{\beta}_{0+},\ D^{\alpha}_{0+}$ and D^{γ}_{0+} are the standard Riemann-Liouville fractional derivative with $1 < \alpha \leq 2,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,\ 0 \leq \alpha - \gamma - 1,\ \sigma$ is a positive number.

Z. Liu and L. Lu [16] studied the boundary value problem for nonlinear fractional differential equations with p-Laplacian operator

$$D^{\beta}_{0+}(\phi_p(D^{\alpha}_{0+}u))(t) = f(t, u(t), D^{\alpha}_{0+}u(t)),\ 0 < t < 1,$$

$$u(0) = \mu \int_0^1 u(s)ds + \lambda u(\xi),\ D^{\alpha}_{0+}u(0) = kD^{\alpha}_{0+}u(\eta),$$

where $0 < \alpha, \beta \leq 1,\ 1 < \alpha + \beta \leq 2,\ \mu,\ \lambda,\ k \in R,\ \xi, \eta \in [0, 1],\ D^{\alpha}_{0+}$ denotes the Caputo fractional derivative of order α. Motivated by the above works, in section 3, we consider the positive solutions for a three-point fractional boundary value problem for p-Laplacian with a parameter

$$(\phi_p(D^{\alpha}_{0+}u(t)))' + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$D^{\alpha}_{0+}u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta), \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ D^{\alpha}_{0+}$ is the Caputo's derivative, $\alpha \in (2, 3],\ \eta, \gamma \in (0, 1),\ f \in C([0, 1] \times [0, \infty), [0, \infty)), \lambda > 0$ is a parameter.

In recent years, using Leggett-Williams fixed point theorem, some authors obtained three positive solutions for the fractional boundary value problem.

In [26], Zhang used the Leggett-Williams theorem to show the existence of triple positive solutions to the fractional boundary value problem

$$D^{\alpha}_{0+}u(t) = f(t, u(t)), \quad 0 < t < 1,$$

$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0.$$

In [12], Eric R. Kaufmann and Ebene Mboumi gave sufficient conditions for the existence of at least one and at least three positive solutions to the nonlinear fractional boundary value problem

$$D^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

$$u(0) = 0, \quad u'(1) = 0.$$

where $D^{\alpha}$is the Riemann-Liouville differential operator of order $\alpha,\ f:\ [0, \infty) \to [0, \infty)$ is a given continuous function and $a(t)$ is a positive and continuous function on $[0, 1]$.

In [30], X. Zhao, C. Chai, W. Ge considered the existence of three positive solutions of the following fractional boundary value problem

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u'(0) - \beta u(\xi) = 0, \quad u'(1) + \gamma u(\eta) = 0,$$

where α is a real number with $1 < \alpha \leq 2,\ 0 \leq \xi \leq \eta \leq 1,\ 0 \leq \beta, \gamma \leq 1,\ f \in C([0, 1] \times [0, \infty) \to [0, \infty)),\ D^{\alpha}_{0+}$ is the Caputo fractional derivative.

In [9], M. Jia, X. Liu studied at least three nonnegative solutions for the following fractional differential equation with integral boundary conditions

$$\begin{aligned} {}^{C}D^{\alpha}x(t) + f(t, x(t)) &= 0, \quad t \in (0,1),\\ x(0) &= \int_0^1 g_0(s)x(s)ds,\\ x(1) &= \int_0^1 g_1(s)x(s)ds,\\ x^{(k)}(0) &= \int_0^1 g_k(s)x(s)ds, \quad k = 2, 3, \cdots, [\alpha], \end{aligned}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo derivative, $\alpha \in R$ and $2 \le n = [\alpha] < \alpha < [\alpha]+1$, $f \in C([0,1]\times R^+, R^+)$ and $g_k \in C([0,1], R)$ $(k = 0, 1, 2, \cdots, [\alpha])$ are given functions, $[\alpha]$ denotes the integer part of the real number α and $R^+ = [0, +\infty)$. By means of Leggett-Williams fixed point theorem, some new results on the existence of at least three nonnegative solutions are obtained.

Motivated by the above works, in section 4, by means of Leggett-Williams fixed point theorem, we consider the existence of three positive solutions for the following three-point fractional boundary value problem for p-Laplacian

$$\begin{aligned} (\phi_p(D_{0+}^{\alpha}u(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1,\\ D_{0+}^{\alpha}u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta), \end{aligned}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ D_{0+}^{\alpha}$ is the Caputo's derivative, $\alpha \in (2,3],\ \eta, \gamma \in (0,1),\ f \in C([0,1] \times [0,\infty), [0,\infty))$.

As far as we know, no contribution concerns the above three-point fractional boundary value problem for p-Laplacian with a parameter and the existence of three positive solutions for the three-point fractional boundary value problem for p-Laplacian. The aim of this paper is to fill the gap in the relevant literatures. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

2. Preliminaries

Definition 2.1 ([6])**.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u(t)$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided the right side is point-wise defined on $(0, +\infty)$.

Definition 2.2 ([6])**.** The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u(t)$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n = [\alpha] + 1$, provided the right side is point-wise defined on $(0, +\infty)$.

Lemma 2.3 ([20]). *The three-point boundary value problem (1), (2) has a unique solution*

$$u(t) = \int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau, u(\tau))d\tau\right) ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f\left(\tau, u(\tau)\right) d\tau\right) ds,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1 \end{cases}$$

$$G_2(\eta,s) = \begin{cases} \dfrac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le s \le \eta \le 1, \\ \dfrac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le \eta \le s \le 1. \end{cases}$$

Lemma 2.4 ([20]). *Let $\beta \in (0,1)$ be fixed. The kernel $G_1(t,s)$ satisfies the following properties.*
(1): $0 \le G_1(t,s) \le G_1(1,s)$ *for all* $s \in (0,1)$;
(2): $\min\limits_{\beta \le t \le 1} G_1(t,s) \ge \beta G_1(1,s)$ *for all* $s \in (0,1)$.

Lemma 2.5 ([20]). *The unique solution $u(t)$ of (1), (2) is nonnegative and satisfies*

$$\min_{\beta \le t \le 1} u(t) \ge \beta \|u\|. \tag{3}$$

Theorem 2.6. *Suppose E is a real Banach space, $K \subset E$ is a cone, let Ω_1, Ω_2 be two bounded open sets of E such that $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be completely continuous. Suppose that one of two conditions hold*
(i) $\|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_1, \qquad \|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_2;$
(ii) $\|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_1, \qquad \|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_2,$
then T has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Define the cone K by

$$K = \left\{ u \in C[0,1] : \ u(t) \ge 0, \ \min_{\beta \le t \le 1} u(t) \ge \beta \|u\| \right\}, \tag{4}$$

and the operator $T : \ K \to E$ by

$$(Tu)(t) = \int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau, u(\tau))d\tau\right) ds + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau, u(\tau))d\tau\right) ds. \tag{5}$$

Lemma 2.7 ([20]). *T is completely continuous and $T(K) \subseteq K$.*

Denote

$$f_\beta = \liminf_{|u|\to\beta} \min_{0\le t\le 1} \frac{f(t,u)}{|u|^{p-1}}, \quad f^\beta = \limsup_{|u|\to\beta} \max_{0\le t\le 1} \frac{f(t,u)}{|u|^{p-1}},$$

where $\beta = 0^+, \infty$,

$$M = \beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right] + \frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right].$$

3. Main results

Theorem 3.1. *Suppose that $f_\infty > 0$, $f^0 < \infty$. Then boundary value problem (1), (2) has at least one positive solution if*

$$\frac{1}{f_\infty \beta^{p-1} M^{p-1}} \ < \ \lambda \ < \ \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{f^0}. \tag{6}$$

Proof. By (6), there exists $\varepsilon > 0$, such that

$$\frac{1}{(f_\infty - \varepsilon) \beta^{p-1} M^{p-1}} \ \le \ \lambda \ \le \ \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{(f^0+\varepsilon)}. \tag{7}$$

(i) Fixed ε. By $f^0 < \infty$, there exists $H_1 > 0$, such that for $u:\ 0 < |u| \le H_1$, we have

$$f(t,u) \le (f^0+\varepsilon)|u|^{p-1}.$$

Define

$$\Omega_1 = \{u \in K:\ \|u\| < H_1\},$$

for $u \in \partial\Omega_1$, we have

$$\begin{aligned}
&\|Tu\| = \max_{0\le t\le 1} |(Tu)(t)| \\
&= \max_{0\le t\le 1}\left\{\int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \right.\\
&\qquad\left. + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds\right\} \\
&\le \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\le \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda (f^0+\varepsilon)|u|^{p-1} d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda (f^0+\varepsilon)|u|^{p-1} d\tau\right)ds
\end{aligned}$$

$$\leq \left(\lambda(f^0+\varepsilon)\right)^{q-1}|u|\int_0^1 G_1(t,s)ds+\frac{\gamma}{1-\gamma}\left(\lambda(f^0+\varepsilon)\right)^{q-1}|u|\int_0^1 G_2(\eta,s)ds$$
$$\leq \left(\lambda(f^0+\varepsilon)\right)^{q-1}|u|\frac{1}{\Gamma(\alpha)}+\frac{\gamma}{1-\gamma}\left(\lambda(f^0+\varepsilon)\right)^{q-1}|u|\frac{1}{\Gamma(\alpha)}$$
$$\leq \frac{1}{(1-\gamma)\Gamma(\alpha)}\left(\lambda(f^0+\varepsilon)\right)^{q-1}\|u\|$$
$$\leq \|u\|.$$

Therefore, $\|Tu\|\leq\|u\|$.

(ii) By $f_\infty>0$, there exists $\overline{H_2}>0$, such that for $|u|\geq\overline{H_2}$, we have

$$f(t,u)\geq(f_\infty-\varepsilon)|u|^{p-1}.$$

Choose

$$H_2=\max\left\{\frac{H_1}{\beta},\frac{\overline{H_2}}{\beta}\right\},\ \Omega_2=\left\{u\in K:\ \|u\|<H_2\right\},$$

by Lemma 5, for $u\in\partial\Omega_2$, we have

$$\beta H_2=\beta\|u\|\leq|u|\leq\|u\|=H_2,\quad t\in[\beta,1],$$

thus

$$\begin{aligned}
Tu(t)=&\int_0^1 G_1(t,s)\phi_q\left(\int_0^s\lambda f(\tau,u(\tau))d\tau\right)ds\\
&+\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s\lambda f(\tau,u(\tau))d\tau\right)ds\\
\geq&\int_\beta^1\beta G_1(1,s)\phi_q\left(\int_0^s\lambda(f_\infty-\varepsilon)|u|^{p-1}d\tau\right)ds\\
&+\frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)\phi_q\left(\int_0^s\lambda(f_\infty-\varepsilon)|u|^{p-1}d\tau\right)ds\\
\geq&\left(\lambda(f_\infty-\varepsilon)\right)^{q-1}|u|\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&+\frac{\gamma\beta}{1-\gamma}\left(\lambda(f_\infty-\varepsilon)\right)^{q-1}|u|\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\\
=&\left(\lambda(f_\infty-\varepsilon)\right)^{q-1}|u|\left\{\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right]\right.\\
&\left.+\frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\right\}\\
\geq&\left(\lambda(f_\infty-\varepsilon)\right)^{q-1}\beta H_2\left\{\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right]\right.\\
&\left.+\frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\right\}\\
\geq&H_2=\|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$. So, by Theorem 2.6 (i), we have T has a fixed point $u \in \Omega_2 \setminus \overline{\Omega_1}$, therefore, u is a positive solution of boundary value problem (1), (2). The proof is completed. □

Corollary 3.2. *Suppose that $f_\infty > 0$, $f^0 < \infty$. Then boundary value problem (1), (2) has nonnegative solution when*

$$\left(\frac{1}{f_\infty \beta^{p-1} M^{p-1}}, \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{f^0} \right) \subset D_1,$$

where $D_1 = \{\lambda > 0\}$.

Theorem 3.3. *Suppose that $f_0 > 0$, $f^\infty < \infty$. Then boundary value problem (1), (2) has at least one positive solution if*

$$\frac{1}{f_0 \beta^{p-1} M^{p-1}} < \lambda < \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{f^\infty}. \tag{8}$$

Proof. By (8), there exists $\varepsilon > 0$, such that

$$\frac{1}{(f_0 - \varepsilon)\beta^{p-1} M^{p-1}} \leq \lambda \leq \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{(f^\infty + \varepsilon)}. \tag{9}$$

(i) Fixed ε. By $f_0 > 0$, there exists $H_1 > 0$, such that for $u:\ 0 < |u| \leq H_1$, we have

$$f(t,u) \geq (f_0 - \varepsilon)|u|^{p-1}, \quad t \in [0,1].$$

Define

$$\Omega_1 = \{u \in K:\ \|u\| < H_1\},$$

by Lemma 5, for $u \in \partial\Omega_1$, we have

$$\beta H_1 = \beta\|u\| \leq |u| \leq \|u\| = H_1, \quad t \in [\beta, 1],$$

thus,

$$\begin{aligned}
Tu(t) &= \int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\geq \int_0^1 \beta G_1(1,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\geq \int_0^1 \beta G_1(1,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma \beta}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_\beta^1 \beta G_1(1,s)\phi_q\left(\int_0^s \lambda(f_0-\varepsilon)|u|^{p-1}d\tau\right)ds \\
&\quad+\frac{\gamma\beta}{1-\gamma}\int_\beta^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda(f_0-\varepsilon)|u|^{p-1}d\tau\right)ds \\
&\geq (\lambda(f_0-\varepsilon))^{q-1}|u|\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\quad+\frac{\gamma\beta}{1-\gamma}(\lambda(f_0-\varepsilon))^{q-1}|u|\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right] \\
&= (\lambda(f_0-\varepsilon))^{q-1}|u|\Bigg\{\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\quad+\frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\Bigg\} \\
&\geq (\lambda(f_0-\varepsilon))^{q-1}\beta H_1\Bigg\{\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\quad+\frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\Bigg\} \\
&\geq H_1=\|u\|.
\end{aligned}
$$

Therefore, $\|Tu\|\geq\|u\|$.

(ii) By $f^\infty<\infty$, there exists $\overline{H_2}>0$, such that for $u:\ |u|\geq\overline{H_2}$, we have

$$f(t,u)\leq(f^\infty+\varepsilon)|u|^{p-1}.$$

We shall consider two cases, case 1, f is bounded. Case 2, f is unbounded.

Case 1. Suppose that f is bounded, there exists $L>0$, such that

$$f(t,u)\leq L^{p-1}.$$

Define

$$H_2=\max\left\{\frac{H_1}{\beta},\frac{L\lambda^{q-1}}{(1-\gamma)\Gamma(\alpha)}\right\},\quad \Omega_2=\{u\in K:\ \|u\|<H_2\},$$

for $u\in\partial\Omega_2$, we have

$$
\begin{aligned}
\|Tu\|&=\max_{0\leq t\leq 1}|(Tu)(t)| \\
&=\max_{0\leq t\leq 1}\Bigg\{\int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\qquad+\frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds\Bigg\} \\
&\leq \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad+\frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda L^{p-1}d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda L^{p-1}d\tau\right)ds \\
&\leq \lambda^{q-1}L\int_0^1 G_1(t,s)ds + \frac{\gamma}{1-\gamma}\lambda^{q-1}L\int_0^1 G_2(\eta,s)ds \\
&\leq \lambda^{q-1}L\frac{1}{\Gamma(\alpha)} + \frac{\gamma}{1-\gamma}\lambda^{q-1}L\frac{1}{\Gamma(\alpha)} \\
&\leq \frac{\lambda^{q-1}L}{(1-\gamma)\Gamma(\alpha)} \\
&\leq H_2 = \|u\|.
\end{aligned}
$$

Case 2. Choose $H_2 > \max\left\{H_1,\ \overline{H_2}\right\}$, such that when $t \in [0,1]$ and $0 < |u| \leq H_2$, we have $f(t,u) \leq f(t,H_2)$. Let

$$\Omega_2 = \{u \in K:\ \|u\| < H_2\},$$

for $u \in \partial\Omega_2$, we have

$$
\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq 1}|(Tu)(t)| \\
&= \max_{0\leq t\leq 1}\left\{\int_0^1 G_1(t,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds\right. \\
&\quad \left. + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s \lambda f(\tau,u(\tau))d\tau\right)ds\right\} \\
&\leq \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda f(\tau,u(\tau))d\tau\right)ds \\
&\leq \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda f(\tau,H_2)d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda f(\tau,H_2)d\tau\right)ds \\
&\leq \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 \lambda(f^\infty+\varepsilon)H_2^{p-1}d\tau\right)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 \lambda(f^\infty+\varepsilon)H_2^{p-1}d\tau\right)ds \\
&\leq \left(\lambda(f^\infty+\varepsilon)\right)^{q-1}H_2\int_0^1 G_1(t,s)ds \\
&\quad + \frac{\gamma}{1-\gamma}\left(\lambda(f^\infty+\varepsilon)\right)^{q-1}H_2\int_0^1 G_2(\eta,s)ds
\end{aligned}
$$

$$\begin{aligned}
&\leq (\lambda(f^\infty+\varepsilon))^{q-1} H_2 \frac{1}{\Gamma(\alpha)} + \frac{\gamma}{1-\gamma} (\lambda(f^\infty+\varepsilon))^{q-1} H_2 \frac{1}{\Gamma(\alpha)} \\
&\leq \frac{(\lambda(f^\infty+\varepsilon))^{q-1} H_2}{(1-\gamma)\Gamma(\alpha)} \\
&\leq H_2 = \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \leq \|u\|$. So, by Theorem 2.6 (ii), we have T has a fixed point $u \in \Omega_2 \setminus \overline{\Omega_1}$, therefore, u is a positive solution of boundary value problem (1), (2). The proof is completed. □

Corollary 3.4. *Suppose that $f_0 > 0$, $f^\infty < \infty$. Then boundary value problem (1), (2) has nonnegative solution when*

$$\left(\frac{1}{f_0 \beta^{p-1} M^{p-1}}, \frac{((1-\gamma)\Gamma(\alpha))^{p-1}}{f^\infty} \right) \subset D_1,$$

where $D_1 = \{\lambda > 0\}$.

4. Three positive solution of the problem (10), (11)

In this section, we will give the existence of three positive solutions of the following fractional boundary value problem

$$(\phi_p(D_{0+}^\alpha u(t)))' + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{10}$$

$$D_{0+}^\alpha u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta), \tag{11}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, D_{0+}^α is the Caputo's derivative, $\alpha \in (2,3]$, $\eta, \gamma \in (0,1)$, $f \in C([0,1] \times [0,\infty)$.

The basic space used in this section is a real Banach space $E = C[0,1]$ with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$.

Definition 4.1. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 4.2. The map γ is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\gamma : P \to [0,\infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let α be a nonnegative continuous concave functional on P. Then for positive real numbers $0 < a < b$, we define the following convex sets:

$$P_r = \{x \in P \mid \|x\| < r\}, \ \ P(\alpha,\ a,\ b) = \{x \in P \mid a \leq \alpha(x),\ \|x\| \leq b\}.$$

The following fixed point theorem is fundamental in the proofs of our main results.

Theorem 4.3 ([30])**.** *Let $A:\ \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on P such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist positive numbers $0 < a < b < d \le c$ such that*

(C_1) $\{x \in P(\alpha,\ b,\ d)\ |\alpha(x) > b\} \neq \emptyset and\ \alpha(Ax) > b\ for\ x \in P(\alpha,\ b,\ d)$;

(C_2) $\|Ax\| < a\ for\ \|x\| \le a$;

(C_3) $\alpha(Ax) > b\ for\ x \in P(\alpha,\ b,\ c)\ with\ \|Ax\| > d$.

Then A has at least three fixed points $x_1,\ x_2,\ x_3$ such that

$$\|x_1\| < a, \quad b < \alpha(x_2) \quad and \quad \|x_3\| > a\ with\ \alpha(x_3) < b.$$

Let $\beta \in (0,1)$ be fixed. Define the cone P by

$$P = \left\{ u \in C[0,1]:\ u(t) \ge 0,\ \min_{\beta \le t \le 1} u(t) \ge \beta \|u\| \right\}, \tag{12}$$

and the operator $A:\ P \to E$ by

$$\begin{aligned}(Au)(t) = &\int_0^1 G_1(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds \\ &+ \frac{\gamma t}{1-\gamma}\int_0^1 G_2\left(\eta,s)\phi_q(\int_0^s f(\tau,u(\tau))d\tau\right)ds.\end{aligned} \tag{13}$$

It is obvious that the existence of a positive solution for the problem (10), (11) is equivalent to the existence of nontrivial point of A in P.

We define the nonnegative continuous concave functional on P by

$$\alpha(u) = \min_{\beta \le t \le 1} u(t), \quad \forall u \in P.$$

It is clear that $\alpha(u) \le \|u\|$ for $u \in P$.

Let

$$\begin{aligned}R =&\beta\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &+ \frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\eta-1)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right].\end{aligned} \tag{14}$$

Theorem 4.4. *Assume that there exist nonnegative numbers a, b, c such that $0 < a < b < \frac{b}{\beta} < c$ and $f(t,u)$ satisfy the following conditions*

(A_1) $f(t,u) < \phi_p(c(1-\gamma)\Gamma(\alpha))$, *for all* $(t,u) \in [0,1] \times [0,c]$;

(A_2) $f(t,u) \le \phi_p(a(1-\gamma)\Gamma(\alpha))$, *for all* $(t,u) \in [0,1] \times [0,a]$;

(A_3) $f(t,u) > \phi_p(\frac{b}{R})$, *for all* $(t,u) \in [\beta,1] \times [b, \frac{b}{\beta}]$.

Then BVP (10), (11) *has at least three positive solutions $x_1,\ x_2,\ x_3$ such that*

$$\|x_1\| < a, \quad b < \alpha(x_2) \quad and \quad \|x_3\| > a,\ with\ \alpha(x_3) < b.$$

Proof. We complete the proof by three steps.

Step 1. Show $A: \overline{P_c} \to \overline{P_c}$ and $A: \overline{P_a} \to \overline{P_a}$.

Firstly, Lemma 2.5 guarantees $A\overline{P_c} \subset P$. Secondly, for all $u \in \overline{P_c}$, we have $0 \le u(t) \le c$ and by (A_1),

$$\begin{aligned}\|Au\| &= \max_{0\le t\le 1} |(Au)(t)| \\ &\le \int_0^1 G_1(t,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))d\tau\right)ds \\ &\quad + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))d\tau\right)ds \\ &\le \int_0^1 G_1(1,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))d\tau\right)ds \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))d\tau\right)ds \\ &\le c(1-\gamma)\Gamma(\alpha)\int_0^1 G_1(t,s)ds + \frac{\gamma}{1-\gamma}c(1-\gamma)\Gamma(\alpha)\int_0^1 G_2(\eta,s)ds \\ &\le c(1-\gamma)+\gamma c \\ &= c.\end{aligned}$$

Therefore, $\|Au\| \le c$ which implies that $A\overline{P_c} \subset \overline{P_c}$. The operator A is completely continuous by Lemma 2.7. Similarly, $Au \in P_a$ for all $u \in \overline{P_a}$.

Step 2. Show

$$\left\{u \in P\left(\alpha, b, \frac{b}{\beta}\right):\ \alpha(u) > b\right\} \ne \emptyset \tag{15}$$

$$\alpha(Au) > b \quad \text{if} \quad u \in P\left(\alpha, b, \frac{b}{\beta}\right). \tag{16}$$

Let $u = \frac{b+d}{2}$; then $u \in P,\quad \alpha(u) = \frac{b+d}{2} > b$ and $\|u\| = \frac{b+d}{2} < d$. That is, (15) holds.

For $u \in P(\alpha, b, \frac{b}{\beta})$, we have

$$b \le u(t) \le \frac{b}{\beta}, \qquad \beta \le t \le 1.$$

then by (A_3), we get

$$\begin{aligned}\alpha(Au) &= \min_{\beta\le t\le 1}(Au)(t) \\ &\ge \min_{\beta\le t\le 1}\left[\int_0^1 G_1(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds\right. \\ &\quad \left. + \frac{\gamma t}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds\right]\end{aligned}$$

$$
\begin{aligned}
&\geq \left[\int_\beta^1 \beta G_1(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds\right.\\
&\quad \left.+\frac{\gamma\beta}{1-\gamma}\int_0^1 G_2(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))d\tau\right)ds\right]\\
&\geq \frac{b\beta}{R}\left[\int_\beta^1 G_1(1,s)ds+\frac{\gamma}{1-\gamma}\int_\beta^1 G_2(\eta,s)ds\right]\\
&=\frac{b\beta}{R}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1-\beta)^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&\quad +\frac{\beta\gamma}{1-\gamma}\left[\frac{(1-\beta)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\eta-1)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\eta-\beta)^{\alpha-1}}{\Gamma(\alpha)}\right]\\
&=b.
\end{aligned}
$$

Therefore, we have $\alpha(Au) > b$. Hence, (16) holds.

Step 3. Show $\alpha(Au) > b$ for all $u \in P(\alpha, b, c)$ with $\|Au\| > \frac{b}{\beta}$.

If $u \in P(\alpha, b, c)$ with $\|Au\| > \frac{b}{\beta}$, by Lemma 2.5, we have $\alpha(Au) \geq \beta\|Au\| > b$. Hence, an application of Theorem 4.3 completes the proof. □

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