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IDEALS IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}_\infty$ †

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ABSTRACT. We find examples of Ideals in a tridiagonal algebra Alg$\mathcal{L}_\infty$ and study some properties of Ideals in Alg$\mathcal{L}_\infty$. We prove the following theorems:

Let k and j be fixed natural numbers. Let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ and let $\mathcal{A}_{2,\{k\}} \subset \mathcal{A} \subset \{ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0 \}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{A}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{2,\{k\}}$, where $\mathcal{A}_{2,\{k\}} = \{ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0, T_{(2k-1,2k-1)} = T_{(2k,2k)} = 0 \}$.

Let $\mathcal{B}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{2,\{j\}} \subset \mathcal{B} \subset \{ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2j+1,2j)} = 0 \}$. Then $\mathcal{B}$ is an ideal of Alg$\mathcal{A}_\infty$ if and only if $\mathcal{B} = \mathcal{B}_{2,\{j\}}$, where $\mathcal{B}_{2,\{j\}} = \{ T \in \text{Alg}\mathcal{L}_\infty | T_{(2j+1,2j)} = 0, T_{(2j,2j)} = T_{(2j+1,2j+1)} = 0 \}$.

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1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \cdots\}$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. If $x_1, x_2, \cdots, x_k$ are vectors in $\mathcal{H}$, we denote by $[x_1, x_2, \cdots, x_k]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_k$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$. We denote by $\mathcal{L}_\infty$ the subspace lattice generated by the subspaces $[e_1], [e_3], \cdots, [e_{2n-1}], \cdots, [e_1, e_2, e_3], [e_3, e_4, e_5], \cdots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], \cdots$.

By Alg$\mathcal{L}_\infty$, we mean the algebra of bounded operators which leave invariant all of the subspaces in $\mathcal{L}_\infty$. It is easy to see that all such operators have the

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matrix form

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & * & * & \\ & & & & & & \ddots \\ & & & & & & \end{pmatrix}$$

where all non-starred entries are zero.

Let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$. We say that $\mathcal{A}$ is *a left ideal* of Alg$\mathcal{L}_\infty$ if $AT \in \mathcal{A}$ for all A in Alg$\mathcal{L}_\infty$ and T in $\mathcal{A}$. $\mathcal{A}$ is called *a right ideal* of Alg$\mathcal{L}_\infty$ if $TA \in \mathcal{A}$ for all A in Alg$\mathcal{L}_\infty$ and T in $\mathcal{A}$. $\mathcal{A}$ is said to be *an ideal* of Alg$\mathcal{L}_\infty$ if $\mathcal{A}$ is a left ideal of Alg$\mathcal{L}_\infty$ and a right ideal of Alg$\mathcal{L}_\infty$.

Let $\mathcal{R}$ be a ring and let $\mathcal{A}$ be an ideal in $\mathcal{R}$. $\mathcal{A}$ is called *a maximal ideal* if there exists no proper ideal $\mathcal{B}$ such that $\mathcal{A} \subsetneqq \mathcal{B} \subsetneqq$ Alg$\mathcal{L}_\infty$. Let $\mathcal{R}$ be a (commutative)ring and let $\mathcal{P}$ be an ideal in $\mathcal{R}$. $\mathcal{P}$ is *prime* if $\mathcal{P} \neq \mathcal{R}$ and if $ab \in \mathcal{P}$ for $a, b \in \mathcal{R}$ implies either $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

In this paper, let I be the identity operator on $\mathcal{H}$. Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{N} = \{1, 2, \cdots\}$.

2. Examples of ideals in Alg$\mathcal{L}_\infty$

We can easily prove the following examples by simple calculation. We denote $T_{(i,j)}$ or t_{ij} by the (i,j)-component of an operator T in Alg$\mathcal{L}_\infty$.

Example 1. Let $\mathcal{A}_0 = \{ \ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(k,k)} = 0, \ k \in \mathbb{N} \ \}$. Then $\mathcal{A}_0$ is a left ideal in Alg$\mathcal{L}_\infty$ and a right ideal in Alg$\mathcal{L}_\infty$. Hence $\mathcal{A}_0$ is an ideal in Alg$\mathcal{L}_\infty$.

Example 2. Let J be a nonempty subset of $\mathbb{N}$. Let $\mathcal{A}_J = \{ \ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(i,i)} = 0, \ i \in J \ \}$. Then $\mathcal{A}_J$ is an ideal of Alg$\mathcal{L}_\infty$.

Example 3.

(1) Let I be the identity operator on $\mathcal{H}$ and let $\mathcal{A}_I = \{ \ \alpha I + T \mid \alpha \in \mathbb{C}, T \in \mathcal{A}_0 \ \}$. Then $\mathcal{A}_I$ is not an ideal in Alg$\mathcal{L}_\infty$.

(2) Let k and $i (i \geqslant 2)$ be fixed natural numbers. Let $\mathcal{A}_{k,i} = \{ \ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(k,k)} = T_{(k+1,k+1)} = \cdots = T_{(k+i,k+i)} \ \}$. Then $\mathcal{A}_{k,i}$ is not an ideal.

(3) Let $\mathcal{D} = \{ \ A \in \text{Alg}\mathcal{L}_\infty \mid$ A is a diagonal operator. $\}$. Then $\mathcal{D}$ is not an ideal of Alg$\mathcal{L}_\infty$.

Example 4. Let Ω be a nonempty subset of $\mathbb{N}$ and let $\mathcal{A}_\Omega = \{ \ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(i,i)} = T_{(j,j)}$ for $i, j \in \Omega$ and for all $T \in \mathcal{A} \ \}$. Then $\mathcal{A}_\Omega$ is a Lie ideal but not an ideal in Alg$\mathcal{L}_\infty$.

Example 5.

(1) Let k be a fixed natural number and let $\mathcal{A}_{0, \ \{k\}} = \{ \ T \in \mathcal{A}_0 \mid T_{(2k-1,2k)} = 0 \ \}$. Then $\mathcal{A}_{0, \ \{k\}}$ is an ideal in Alg$\mathcal{L}_\infty$.

(2) Let n be a fixed natural number and $\Gamma = \{k_1, k_2, \cdots, k_n\}$ $(n > 1)$. $\mathcal{A}_{0,\Gamma} = \{\ T \in \mathcal{A}_0 \mid T_{(2k_i-1,2k_i)} = 0, k_i \in \Gamma\ \}$. Then $\mathcal{A}_{0,\Gamma}$ is an ideal in Alg$\mathcal{L}_\infty$.

(3) Let $k_1, k_2, \cdots$ be natural numbers and $\Omega = \{k_1, k_2, \cdots\}$. Let $\mathcal{A}_{0,\Omega} = \{\ T \in \mathcal{A}_0 \mid T_{(2k_i-1,2k_i)} = 0, k_i \in \Omega\ \}$. Then $\mathcal{A}_{0,\Omega}$ is an ideal in Alg$\mathcal{L}_\infty$.

Example 6.

(1) Let j be a fixed natural number and let $\mathcal{B}_{0,\ \{j\}} = \{\ T \in \mathcal{A}_0 \mid T_{(2j+1,2j)} = 0\ \}$. Then $\mathcal{B}_{0,\ \{j\}}$ is an ideal in Alg$\mathcal{L}_\infty$.

(2) Let m be a fixed natural number and let $\Gamma = \{j_1, j_2, \cdots, j_m\}$ $(m > 1)$. Let $\mathcal{B}_{0,\Gamma} = \{\ T \in \mathcal{A}_0 \mid T_{(2j_i+1,2j_i)} = 0, j_i \in \Gamma\ \}$. Then $\mathcal{B}_{0,\Gamma}$ is an ideal in Alg$\mathcal{L}_\infty$.

(3) Let $j_1, j_2, \cdots$ be natural numbers and let $\Omega = \{j_1, j_2, \cdots\}$. Let $\mathcal{B}_{0,\Omega} = \{\ T \in \mathcal{A}_0 \mid T_{(2j_i+1,2j_i)} = 0, j_i \in \Omega\ \}$. Then $\mathcal{A}_{0,\Omega}$ is an ideal in Alg$\mathcal{L}_\infty$.

Example 7.

(1) Let k and j be fixed natural numbers. Then $\mathcal{C}_{0,\{k,j\}} = \{\ T \in \mathcal{A}_0 \mid T_{(2k-1,2k)} = 0,\ T_{(2j+1,2j)} = 0\ \}$ is an ideal in Alg$\mathcal{L}_\infty$.

(2) Let $\{k_1, k_2, \cdots, k_n\}$ and $\{j_1, j_2, \cdots, j_m\}$ be subsets of $\mathbb{N}$ and let $\Gamma = \{k_i, j_l \mid i = 1, 2, \cdots, n, l = 1, 2, \cdots, m\}$. Then $\mathcal{C}_{0,\Gamma} = \{\ T \in \mathcal{A}_0 \mid T_{(2k_i-1,2k_i)} = 0,\ T_{(2j_l+1,2j_l)} = 0, k_i, j_l \in \Gamma\ \}$ is an ideal in Alg$\mathcal{L}_\infty$.

(3) Let $\{k_1, k_2, \cdots\}$ and $\{j_1, j_2, \cdots\}$ be subsets of $\mathbb{N}$ and let $\Omega = \{k_i, j_l \mid i = 1, 2, \cdots, l = 1, 2, \cdots\}$. Then $\mathcal{C}_{0,\Omega} = \{\ T \in \mathcal{A}_0 \mid T_{(2k_i-1,2k_i)} = 0,\ T_{(2j_l+1,2j_l)} = 0, k_i, j_l \in \Omega\ \}$ is an ideal in Alg$\mathcal{L}_\infty$.

Example 8.

(1) Let k be a fixed natural number. Let $\mathcal{A} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0\ \}$. Then $\mathcal{A}$ is not an ideal in Alg$\mathcal{L}_\infty$.

(2) Let k be a fixed natural number. Let $\mathcal{A}_{2,\{k\}} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0, T_{(2k,2k)} = T_{(2k-1,2k-1)} = 0\ \}$. Then $\mathcal{A}_{2,\{k\}}$ is an ideal in Alg$\mathcal{L}_\infty$.

(3) Let n be a fixed natural number and let $\Gamma = \{k_1, k_2, \cdots, k_n\}$ $(n > 1)$. Let $\mathcal{A}_{2,\Gamma} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k_i-1,2k_i)} = 0, T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} = 0, k_i \in \Gamma\ \}$. Then $\mathcal{A}_{2,\Gamma}$ is an ideal in Alg$\mathcal{L}_\infty$.

(4) Let $k_1, k_2, \cdots$ be a subset of $\mathbb{N}$ and let $\Omega = \{k_1, k_2, \cdots\}$. Let $\mathcal{A}_{2,\Omega} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k_i-1,2k_i)} = 0, T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} = 0, k_i \in \Omega\ \}$. Then $\mathcal{A}_{2,\Omega}$ is an ideal in Alg$\mathcal{L}_\infty$.

Example 9.

(1) Let j be a fixed natural number. Let $\mathcal{B} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2j+1,2j)} = 0\ \}$. Then $\mathcal{B}$ is not an ideal in Alg$\mathcal{L}_\infty$.

(2) Let j be a fixed natural number. Let $\mathcal{B}_{2,\{j\}} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2j+1,2j)} = 0, T_{(2j,2j)} = T_{(2j+1,2j+1)} = 0\ \}$. Then $\mathcal{B}_{2,\{j\}}$ is an ideal in Alg$\mathcal{L}_\infty$.

(3) Let m be a fixed natural number and let $\Gamma = \{j_1, j_2, \cdots, j_m\}$ $(m > 1)$ be a subset of $\mathbb{N}$. Let $\mathcal{B}_{2,\Gamma} = \{\ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2j_l+1,2j_l)} = 0, T_{(2j_l,2j_l)} = T_{(2j_l+1,2j_l+1)} = 0, j_l \in \Gamma\ \}$. Then $\mathcal{B}_{2,\Gamma}$ is an ideal in Alg$\mathcal{L}_\infty$.

(4) Let $j_1, j_2, \cdots$ be a subsets of $\mathbb{N}$ and let$\Omega = \{j_1, j_2, \cdots\}$. Let $\mathcal{B}_{2,\Omega} = \{ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2j_l+1,2j_l)} = 0, T_{(2j_l,2j_l)} = T_{(2j_l+1,2j_l+1)} = 0, j_l \in \Omega \}$. Then $\mathcal{B}_{2,\Omega}$ is an ideal in $\text{Alg}\mathcal{L}_\infty$.

Example 10.

(1) Let k and j be fixed natural numbers. Then $\mathcal{C}_{2,\ \{k,j\}} = \{ T \in \text{Alg}\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0,\ T_{(2j+1,2j)} = 0,\ T_{(2k-1,2k-1)} = T_{(2k,2k)} = 0,\ T_{(2j,2j)} = T_{(2j+1,2j+1)} = 0\}$ is an ideal of $\text{Alg}\mathcal{L}_\infty$.

(2) Let $\{k_1, k_2, \cdots, k_n\}$ and $\{j_1, j_2, \cdots, j_m\}$ be subsets of $\mathbb{N}$ and let $\Gamma = \{k_i, j_l \mid i = 1, 2, \cdots, n,\ l = 1, 2, \cdots, m\}$. Then $\mathcal{C}_{2,\Gamma} = \{ T \in Alg\mathcal{L}_\infty \mid T_{(2k_i-1,2k_i)} = 0,\quad T_{(2j_l+1,2j_l)} = 0,\quad T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} = 0,\ T_{(2j_l,2j_l)} = T_{(2j_l+1,2j_l+1)} = 0, k_i, j_l \in \Gamma \}$ is an ideal of $\text{Alg}\mathcal{L}_\infty$.

(3) Let $\{k_1, k_2, \cdots\}$ and $\{j_1, j_2, \cdots\}$ be subsets of $\mathbb{N}$ and let $\Omega = \{k_i, j_l \mid i = 1, 2, \cdots, l = 1, 2, \cdots\}$. Then $\mathcal{C}_{2,\Omega} = \{ T \in Alg\mathcal{L}_\infty \mid T_{(2k_i-1,2k_i)} = 0,\quad T_{(2j_l+1,2j_l)} = 0,\quad T_{(2k_i-1,2k_i-1)} = T_{(2k_i,2k_i)} = 0,\quad T_{(2j_l,2j_l)} = T_{(2j_l+1,2j_l+1)} = 0, k_i, j_l \in \Omega \}$ is an ideal of $\text{Alg}\mathcal{L}_\infty$.

3. Properties of ideals in Alg$\mathcal{L}_\infty$

In this section we investigate some properties of ideals of $\text{Alg}\mathcal{L}_\infty$. If $J = \{ k \}$ for a fixed natural number k, we denote $\mathcal{A}_J$ by $\mathcal{A}_{\{k\}}$.

Theorem 1. *Let k be a fixed natural number. Then* (1) $\mathcal{A}_{\{k\}}$ *is prime and* (2) $\mathcal{A}_{\{k\}}$ *is maximal.*

Proof. (1) Let $A = (a_{ij})$ and let $T = (t_{ij})$ be operators in $\text{Alg}\mathcal{L}_\infty$. Let AT be in $\mathcal{A}_{\{k\}}$. Since $(AT)_{kk} = a_{kk}t_{kk} = 0$, $a_{kk} = 0$ or $t_{kk} = 0$. So A is in $\mathcal{A}_{\{k\}}$ or T is in $\mathcal{A}_{\{k\}}$.

(2) It is sufficient to show that the case k=1. Let $\mathcal{A}$ be an ideal of $\text{Alg}\mathcal{L}_\infty$ such that $\mathcal{A}_{\{1\}} \subset \mathcal{A} \subset \text{Alg}\mathcal{A}_\infty$. Suppose that there exists $T = (t_{ij})$ in $\mathcal{A}$ such that $t_{11} \neq 0$. Let $S = (s_{ij})$ be an element of $\text{Alg}\mathcal{A}_\infty$. If $s_{11} = 0$, then $S \in \mathcal{A}_{\{1\}}$. Let $s_{11} \neq 0$. Let $A = (a_{ij})$ be an operator defined by

$$\begin{cases} a_{11} = 0 \\ a_{ij} = -t_{ij} & \text{otherwise.} \end{cases}$$

Then $A + T$ is an operator in $\mathcal{A}$. Put $A_1 = A + T$. Then $A_{1(1,1)} = t_{11}$ and $A_{1(i,j)} = 0$ if $(i, j) \neq (1, 1)$. Let $B = (b_{ij})$ be an operator defined by

$$\begin{cases} b_{11} = 0 \\ b_{ij} = s_{ij} & \text{otherwise.} \end{cases}$$

Then B is an element of $\mathcal{A}_{\{1\}}$. Put $x = \frac{s_{11}}{t_{11}}$. Then $B + xA_1 = S$ is an operator of $\mathcal{A}$. Hence $\mathcal{A} = \text{Alg}\mathcal{L}_\infty$. □

Theorem 2. *Let $\mathcal{A}$ be a subalgebra of $Alg\mathcal{L}_\infty$. Let k be a fixed natural number and let $\mathcal{A}_{2,\{k\}} \subset \mathcal{A} \subset \{ T \in Alg\mathcal{L}_\infty \mid T_{(2k-1,2k)} = 0 \}$. Then $\mathcal{A}$ is an ideal of $Alg\mathcal{A}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{2,\{k\}}$.*

Proof. Let $A = (a_{ij})$ be an operator in Alg$\mathcal{L}_\infty$ and let $T = (t_{ij})$ be an operator in $\mathcal{A}$. Suppose that $\mathcal{A}$ is an ideal of Alg$\mathcal{A}_\infty$. Then the $(2k-1, 2k)$-component of AT is

$$a_{2k-1\ 2k-1}t_{2k-1\ 2k} + a_{2k-1\ 2k}t_{2k\ 2k} = a_{2k-1\ 2k}t_{2k\ 2k} = 0 \qquad (*)$$

Since (*) holds for all A in Alg$\mathcal{L}_\infty$, $t_{2k\ 2k} = 0$. Also the $(2k-1, 2k)$-component of TA is

$$t_{2k-1\ 2k-1}a_{2k-1\ 2k} + t_{2k-1\ 2k}a_{2k\ 2k} = t_{2k-1\ 2k-1}a_{2k-1\ 2k} = 0 \qquad (**)$$

Since (**) holds for all A in Alg$\mathcal{L}_\infty$, $t_{2k-1\ 2k-1} = 0$. Hence $\mathcal{A} \subset \mathcal{A}_{2,\ \{k\}}$ and $\mathcal{A} = \mathcal{A}_{2,\ \{k\}}$. Suppose that $\mathcal{A} = \mathcal{A}_{2,\{k\}}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{L}_\infty$ by Example 8-(2). □

Theorem 3. *Let j be a fixed natural number and let $\mathcal{B}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{2,\{j\}} \subset \mathcal{B} \subset \{\ T \in Alg\mathcal{L}_\infty \mid T_{(2j+1,2j)} = 0\ \}$. Then $\mathcal{B}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{B} = \mathcal{B}_{2,\{j\}}$.*

Proof. Let $A = (a_{ij})$ be an operator in Alg$\mathcal{L}_\infty$ and let $T = (t_{ij})$ be an operator in $\mathcal{B}$. Suppose that $\mathcal{B}$ is an ideal of Alg$\mathcal{A}_\infty$. Then the $(2j+1, 2j)$-component of AT is

$$a_{2j+1\ 2j+1}t_{2j+1\ 2j} + a_{2j+1\ 2j}t_{2j\ 2j} = a_{2j+1\ 2j}t_{2j\ 2j} = 0 \qquad (*)$$

Since $(*)$ holds for all A in Alg$\mathcal{L}_\infty$, $t_{2j\ 2j} = 0$. Also the $(2j+1, 2j)$-component of TA is

$$t_{2j+1\ 2j+1}a_{2j+1\ 2j} + t_{2j+1\ 2j}a_{2j\ 2j} = t_{2j+1\ 2j+1}a_{2j+1\ 2j} = 0 \qquad (**)$$

Since $(**)$ holds for all A in Alg$\mathcal{L}_\infty$, $t_{2j+1\ 2j+1} = 0$. Hence $T \in \mathcal{B}_{2,\{j\}}$ and so $\mathcal{B} \subset \mathcal{B}_{2,\{j\}}$. Thus $\mathcal{B} = \mathcal{B}_{2,\{j\}}$. Assume that $\mathcal{B} = \mathcal{B}_{2,\{j\}}$. Then $\mathcal{B}$ is an ideal of Alg$\mathcal{L}_\infty$ by Example 9-(2). □

Theorem 4. *Let k be a fixed natural number and let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{A}_{\{k,k+1\}} \subset \mathcal{A} \subset \mathcal{A}_{\{k\}}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{\{k,k+1\}}$ or $\mathcal{A} = \mathcal{A}_{\{k\}}$.*

Proof. It is sufficient to show that the case $k = 1$. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$. Suppose that $\mathcal{A} \neq \mathcal{A}_{\{1,2\}}$. Then there exists an element $T = (t_{ij})$ in $\mathcal{A}$ and $T \notin \mathcal{A}_{\{1,2\}}$. Then $t_{22} \neq 0$. Let $A = (a_{ij})$ be an element of $\mathcal{A}_{\{1\}}$. If $a_{22} = 0$, then $A \in \mathcal{A}_{\{1,2\}}$ and so $A \in \mathcal{A}$. Let $a_{22} \neq 0$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{11} = 0, \ \ b_{22} = 0 \\ b_{ij} = -t_{ij} \qquad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}$. Put $B + T = D$. Then $D \in \mathcal{A}$. Put $x = \frac{a_{22}}{t_{22}}$. Then $xD \in \mathcal{A}$. Let $S = (s_{ij})$ be an operator defined by

$$\begin{cases} s_{22} = 0 \\ s_{ij} = a_{ij} \qquad \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{A}_{\{1,2\}}$ and hence $A = S + xD \in \mathcal{A}$. □

Theorem 5. *Let k be a fixed natural number and let $\mathcal{A}$ be a subalgebra of $Alg\mathcal{L}_\infty$ such that $\{ S \in Alg\mathcal{L}_\infty \mid S_{(2k-1,2k)} = 0 \} \subset \mathcal{A} \subset Alg\mathcal{L}_\infty$. Then $\mathcal{A}$ is an ideal of $Alg\mathcal{L}_\infty$ if and only if $\mathcal{A} = Alg\mathcal{L}_\infty$.*

Proof. It is sufficient to show that the case $k = 2$. Let $\mathcal{A}$ is an ideal of $\mathrm{Alg}\mathcal{L}_\infty$. Since $\{ S \in \mathrm{Alg}\mathcal{L}_\infty \mid S_{(3,4)} = 0 \}$ is not an ideal of $\mathrm{Alg}\mathcal{L}_\infty$, there exists $T = (t_{ij}) \in \mathcal{A}$ such that $t_{34} \neq 0$. Let $A = (a_{ij})$ be an operator in $\mathrm{Alg}\mathcal{L}_\infty$. If $a_{34} = 0$, then $A \in \mathcal{A}$. Let $a_{34} \neq 0$. Let $S = (s_{ij})$ be an operator defined by

$$\begin{cases} s_{34} = 0 \\ s_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{A}$. So $S + T \in \mathcal{A}$. Put $D = S + T$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{34} = 0 \\ b_{ij} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}$. Put $x = \frac{a_{34}}{t_{34}}$. Then $B + xD = A \in \mathcal{A}$. □

Theorem 6. *Let j be a fixed natural number and let $\mathcal{B}$ be a subalgebra of $Alg\mathcal{L}_\infty$ such that $\{T \in Alg\mathcal{L}_\infty \mid T_{(2j+1,2j)} = 0\} \subset \mathcal{B} \subset Alg\mathcal{L}_\infty$. Then $\mathcal{B}$ is an ideal of $Alg\mathcal{L}_\infty$ if and only if $\mathcal{B} = Alg\mathcal{L}_\infty$.*

Proof. It is sufficient to show that $j = 1$. Let $\mathcal{B}$ be an ideal of $\mathrm{Alg}\mathcal{L}_\infty$. Since $\{ T \in \mathrm{Alg}\mathcal{L}_\infty \mid T_{(3,2)} = 0 \}$ is not an ideal of $\mathrm{Alg}\mathcal{L}_\infty$, there exists $T = (t_{ij})$ in $\mathcal{B}$ such that $t_{32} \neq 0$. Let $A = (a_{ij})$ be an operator in $\mathrm{Alg}\mathcal{L}_\infty$. If $a_{32} = 0$, then $A \in \mathcal{B}$. Let $a_{32} \neq 0$. Let $S = (s_{ij})$ be an operator defined by

$$\begin{cases} s_{32} = 0 \\ s_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{B}$. So $S + T \in \mathcal{B}$. Put $S + T = D$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{32} = 0 \\ b_{ij} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{B}$. Put $x = \frac{a_{32}}{t_{32}}$. Then $B + xD = A \in \mathcal{B}$. □

Theorem 7. *Let $\Gamma = \{k_1, k_2\}$ be a subset of $\mathbb{N}$ such that $k_1 \lneq k_2$. Let $\mathcal{A}$ be a subalgebra of $Alg\mathcal{L}_\infty$ such that $\mathcal{A}_{2,\Gamma} \subset \mathcal{A} \subset \{ T \in Alg\mathcal{L}_\infty \mid T_{(2k_i-1,2k_i)} = 0, i = 1, 2 \}$. Then $\mathcal{A}$ is an ideal of $Alg\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{2,\Gamma}$.*

Proof. Let $\mathcal{A}$ is an ideal of $\mathrm{Alg}\mathcal{L}_\infty$ and let $T = (t_{ij})$ be an operator in $\mathcal{A}$. Then $T_{(2k_i-1,2k_i)} = 0, i = 1, 2$. Since $\mathcal{A}$ is an ideal of $\mathrm{Alg}\mathcal{A}_\infty$, $T_{(2k_i-1,2k_i-1)} = 0 = T_{(2k_i,2k_i)} (i = 1, 2)$ by Theorem 2. So $\mathcal{A} \subset \mathcal{A}_{2,\Gamma}$. Hence $\mathcal{A} = \mathcal{A}_{2,\Gamma}$. □

Theorem 8. *Let $\Gamma = \{k_1, k_2\}$ be a subset of $\mathbb{N}$ such that $k_1 \lneq k_2$. Let $\mathcal{A}$ be a subalgebra of $Alg\mathcal{L}_\infty$ such that $\mathcal{A}_{2,\Gamma} \subset \mathcal{A} \subset \mathcal{B}$, where $\mathcal{B} = \mathcal{A}_{2,\{k_1\}} \cap \mathcal{A}_{\{2k_2-1,2k_2\}}$. Then $\mathcal{A}$ is an ideal of $Alg\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{2,\Gamma}$ or $\mathcal{A} = \mathcal{B}$.*

Proof. It is sufficient to show that $\Gamma = \{k_1, k_2\} = \{2, 3\}$. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$ and $\mathcal{A} \neq \mathcal{A}_{2,\Gamma}$. Then there exists an element $T \in \mathcal{A}$ such that $T \notin \mathcal{A}_{2,\Gamma}$, i.e. $T_{(5,6)} \neq 0$. Let $A = (a_{ij}) \in \mathcal{B}$. If $a_{56} = 0$, then $\mathcal{A} \in \mathcal{A}_{2,\Gamma}$ and so $\mathcal{A} \in \mathcal{A}$. Let $a_{56} \neq 0$. Let $S = (s_{ij})$ be an operator defined by

$$\begin{cases} s_{56} = 0 \\ s_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{A}_{2,\Gamma}$ and so $S \in \mathcal{A}$. Put $S + T = A_1$. Then A_1 is an operator of $\mathcal{A}$. Let $B = (b_{ij})$ be an operator defined by

$$\begin{cases} b_{56} = 0 \\ b_{ij} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}_{2,\Gamma}$. Put $x = \frac{a_{56}}{t_{56}}$. Then $A = B + xA_1 \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{B}$. □

Theorem 9. *Let n be a fixed natural number$(n > 1)$ and let $\Gamma = \{j_1, j_2\}$ be a subset of $\mathbb{N}$ such that $j_1 \lneq j_2$. Let $\mathcal{B}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{2,\Gamma} \subset \mathcal{B}$ $\subset \{\ T \in Alg\mathcal{L}_\infty \ \mid T_{(2j_i+1, 2j_i)} = 0, i = 1, 2\ \}$. Then $\mathcal{B}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{B} = \mathcal{B}_{2,\Gamma}$.*

Proof. Let $\mathcal{B}$ be an ideal of Alg$\mathcal{L}_\infty$ and let $A = (a_{ij}) \in \mathcal{B}$. Then $a_{2j_i+1\ 2j_i} = 0 (i = 1, 2)$. Since $\mathcal{B}$ is an ideal, $a_{2j_i\ 2j_i} = 0 = a_{2j_i+1\ 2j_i+1}$ for all $j_i \in \Gamma$ by Theorem 2. Hence $A \in \mathcal{B}_{2,\Gamma}$. □

Theorem 10. *Let $\Gamma = \{j_1, j_2\}$ be a subset of natural numbers. Let $\mathcal{B} = \mathcal{B}_{2,\{j_1\}} \cap \mathcal{A}_{\{2j_2, 2j_2+1\}}$. Let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{2,\Gamma} \subset \mathcal{A} \subset \mathcal{B}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{B}_{2,\Gamma}$ or $\mathcal{A} = \mathcal{B}$.*

Proof. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$ and let $\mathcal{A} \neq \mathcal{B}_{2,\Gamma}$. Then there exists an operator $T = (t_{ij})$ such that $T \in \mathcal{A}$ and $T \notin \mathcal{B}_{2,\Gamma}$, *i.e.* $t_{(2j_2+1\ 2j_2)} \neq 0$. Let $A = (a_{ij}) \in \mathcal{B}$. If $a_{2j_2+1\ 2j_2} = 0$, then $A \in \mathcal{A}$. Let $a_{2j_2+1\ 2j_2} \neq 0$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{2j_2+1\ 2j_2} = 0 \\ b_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}$ and $B + T \in \mathcal{A}$. Put $D = B + T$ and $\alpha = \frac{a_{2j_2+1\ 2j_2}}{t_{2j_2+1\ 2j_2}}$. Then $\alpha D \in \mathcal{A}$. Define an operator T_1 by

$$\begin{cases} T_{1(2j_2+1, 2j_2)} = 0 \\ T_{1(i,j)} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}$. So $\alpha D + T_1 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{B}$. □

Theorem 11. *Let k and j be fixed natural numbers. Let $\mathcal{C}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{C}_{2,\ \{k,j\ \}} \subset \mathcal{C} \subset \mathcal{B} = \{\ T \in Alg\mathcal{L}_\infty \mid T_{(2k-1, 2k)} = 0, T_{(2j+1, 2j)} = 0\ \}$. Then $\mathcal{C}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{C} = \mathcal{C}_{2,\{k,j\}}$.*

Proof. Let $\mathcal{C}$ be an ideal of Alg$\mathcal{L}_\infty$ and let $A \in \mathcal{C}$. Since $\mathcal{C} \subset \mathcal{B}$, $A \in \mathcal{B}$, i.e. $A_{(2k-1,2k)} = 0, A_{(2j+1,2j)} = 0$. Since $\mathcal{C}$ is an ideal, $A_{(2k-1,2k-1)} = 0, A_{(2k,2k)} = 0, A_{(2j,2j)} = 0, A_{(2j+1,2j+1)} = 0$ by Theorem 3. So $A \in \mathcal{C}_{2,\{k,j\}}$. Hence $\mathcal{C} = \mathcal{C}_{2,\{k,j\}}$. □

Theorem 12. *Let k be a fixed natural number and let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{A}_{0,\{k\}} \subset \mathcal{A} \subset \mathcal{A}_0$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{A}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{0,\{k\}}$ or $\mathcal{A} = \mathcal{A}_0$.*

Proof. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$. It is sufficient to show that the case $k = 1$, i.e. if $\mathcal{A}_{0,\{1\}} \subset \mathcal{A} \subset \mathcal{A}_0$, then $\mathcal{A} = \mathcal{A}_{0,\{1\}}$ or $\mathcal{A} = \mathcal{A}_0$. Assume that $\mathcal{A} \neq \mathcal{A}_{0,\{1\}}$. Then there exists $T = (t_{ij})$ in $\mathcal{A}$ such that $T \notin \mathcal{A}_{0,\{1\}}$. Then $t_{12} \neq 0$ and $t_{ii} = 0$ for all $i \in \mathbb{N}$. Let $A = (a_{ij})$ be an element of $\mathcal{A}_0$. If $a_{12} = 0$, $A \in \mathcal{A}_{0,\{1\}} \subset \mathcal{A}$. If $a_{12} \neq 0$, let A_1 be an operator defined by

$$\begin{cases} A_{1(1,2)} = 0 \\ A_{1(i,j)} = a_{ij} & \text{otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{0,\{1\}} \subset \mathcal{A}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(1,2)} = 0 \\ T_{1(i,j)} = -t_{ij} & \text{otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{0,\{1\}} \subset \mathcal{A}$. Let $T_2 = T + T_1 \in \mathcal{A}$. Then $T_{2(1,2)} = t_{12}$ and $T_{2(i,j)} = 0$ for $(i,j) \neq (1,2)$. Let $x = \frac{a_{12}}{t_{12}}$. Then $xT_2 + A_1 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_0$. □

Theorem 13. *Let $\Gamma = \{k_1, k_2\}$ and let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{A}_{0,\Gamma} \subset \mathcal{A} \subset \mathcal{A}_{0,\{k_i\}}$ $i = 1$ or $i = 2$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{A}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{0,\Gamma}$ or $\mathcal{A} = \mathcal{A}_{0,\{k_i\}}$.*

Proof. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$. It is sufficient to show that the case $k_1 = 1$, $k_2 = 2$, $k_i = 1$. Suppose that $\mathcal{A} \neq \mathcal{A}_{0,\{1,2\}}$. Then there exists an element $T = (t_{ij})$ in $\mathcal{A}$ and $T \notin \mathcal{A}_{0,\{1,2\}}$. Then $t_{34} \neq 0$. Let $A = (a_{ij})$ be an element of $\mathcal{A}_{0,\{1\}}$. If $a_{34} = 0$, then $A \in \mathcal{A}_{0,\{1,2\}}$ and so $A \in \mathcal{A}$. Let $a_{34} \neq 0$. Define an operator $S = (s_{ij})$ by

$$\begin{cases} s_{12} = 0, \quad s_{34} = 0 \\ s_{ij} = -t_{ij} & \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{A}_{0,\{1,2\}}$. Put $S + T = A_1$. Then $A_1 \in \mathcal{A}$. Put $x = \frac{a_{34}}{t_{34}}$. Then $xA_1 \in \mathcal{A}$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{12} = 0, \quad b_{34} = 0, \quad b_{ii} = 0 \\ b_{ij} = a_{ij} & \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}_{0,\{1,2\}}$ and hence $B + xA_1 = A \in \mathcal{A}$. So $\mathcal{A} = \mathcal{A}_{0,\{1\}}$. □

Theorem 14. *Let k be a fixed natural number and $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{A}_{2,\{k\}} \subset \mathcal{A} \subset \mathcal{A}_{\{2k-1,2k\}}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{A}_{2,\{k\}}$ or $\mathcal{A} = \mathcal{A}_{\{2k-1,2k\}}$.*

Proof. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$. It is sufficient to show that the case $k = 1$. i.e. if $\mathcal{A}_{2,\{1\}} \subset \mathcal{A} \subset \mathcal{A}_{\{1,2\}}$, then $\mathcal{A} = \mathcal{A}_{2,\{1\}}$ or $\mathcal{A} = \mathcal{A}_{\{1,2\}}$. Assume that $\mathcal{A} \neq \mathcal{A}_{2,\{1\}}$. Then there exists $T = (t_{ij})$ in $\mathcal{A}$ such that $T \notin \mathcal{A}_{2,\{1\}}$. Then $t_{12} \neq 0$ and $t_{11} = 0$ and $t_{22} = 0$. Let $A = (a_{ij})$ be an element of $\mathcal{A}_{\{1,2\}}$. If $a_{12} = 0$, $A \in \mathcal{A}_{2,\{1\}} \subset \mathcal{A}$. If $a_{12} \neq 0$, let A_1 be an operator defined by

$$\begin{cases} A_{1(1,2)} = 0 \\ A_{1(i,j)} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{2,\{1\}} \subset \mathcal{A}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(1,2)} = 0 \\ T_{1(i,j)} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{2,\{1\}} \subset \mathcal{A}$. Let $T_2 = T + T_1 \in \mathcal{A}$. Then $T_{2(1,2)} = t_{12}$ and $T_{2(i,j)} = 0$ for $(i, j) \neq (1, 2)$. Let $x = \frac{a_{12}}{t_{12}}$. Then $xT_2 + A_1 = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_{\{1,2\}}$. □

If we modify the proof of Theorem 14, then we can prove the following Theorem.

Theorem 15. *Let j be a fixed natural number and let $\mathcal{A}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{2,\{j\}} \subset \mathcal{A} \subset \mathcal{A}_{\{2j,2j+1\}}$. Then $\mathcal{A}$ is an ideal of Alg$\mathcal{L}_\infty$ if and only if $\mathcal{A} = \mathcal{B}_{2,\{j\}}$ or $\mathcal{A} = \mathcal{A}_{\{2j,2j+1\}}$.*

Theorem 16. *Let $\Gamma = \{j_1, j_2\}$ and let $\mathcal{B}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{B}_{0,\Gamma} \subset \mathcal{B} \subset \mathcal{B}_{0,\{j_i\}}$ $i = 1$ or $i = 2$. Then $\mathcal{B}$ is an ideal of Alg$\mathcal{A}_\infty$ if and only if $\mathcal{B} = \mathcal{B}_{0,\Gamma}$ or $\mathcal{B} = \mathcal{B}_{0,\{j_i\}}$.*

Proof. Let $\mathcal{B}$ be an ideal of Alg$\mathcal{L}_\infty$. It is sufficient to show that the case $j_1 = 1$, $j_2 = 2$ and $j_i = 2$. Let $\mathcal{B}_{0,\Gamma} \subset \mathcal{B} \subset \mathcal{B}_{0,\{2\}}$. Suppose that $\mathcal{B} \neq \mathcal{B}_{0,\Gamma}$. Then there exists an element $T = (t_{ij})$ in $\mathcal{B}$ and $T \notin \mathcal{B}_{0,\Gamma}$. Then $t_{32} \neq 0$. Let $A = (a_{ij})$ be an element of $\mathcal{B}_{0,\{2\}}$. If $a_{32} = 0$, then $A \in \mathcal{B}_{0,\Gamma}$ and so $A \in \mathcal{B}$. Let $a_{32} \neq 0$. Define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{32} = 0, \ \ b_{54} = 0 \\ b_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{B}_{0,\Gamma}$. Put $B + T = A_1$. Then $A_1 \in \mathcal{B}$. Put $x = \frac{a_{32}}{t_{32}}$. Then $xA_1 \in \mathcal{B}$. Let $S = (s_{ij})$ be an operator defined by

$$\begin{cases} s_{32} = 0, \ \ s_{54} = 0, \ \ s_{ii} = 0 \\ s_{ij} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $S \in \mathcal{B}_{0,\Gamma}$ and hence $S + xA_1 = A \in \mathcal{B}$. So $\mathcal{B} = \mathcal{B}_{0,\{2\}}$. □

If we repeat the proof of the Theorem 12 and the Theorem 13, then we can get the following theorem.

Theorem 17. *Let k, j be natural numbers and $\Omega_{1,1} = \{\ k, j\ \}$.*

i) Let $\mathcal{C}$ be a subalgebra of Alg$\mathcal{L}_\infty$ such that $\mathcal{C}_{0,\Omega_{1,1}} \subset \mathcal{C} \subset \mathcal{C}_{0,\Omega_{1,}}$, where $\mathcal{C}_{0,\Omega_{1,1}} = \{\ T \in \mathcal{A}_0 \mid T_{(2k-1,2k)} = 0 = T_{(2j+1,2j)}\ \}$ and $\mathcal{C}_{0,\Omega_{1,}} = \{\ T \in$

$\mathcal{A}_0 \mid T_{(2k-1,2k)} = 0$ }. *Then* $\mathcal{C}$ *is an ideal of* $Alg\mathcal{L}_\infty$ *if and only if* $\mathcal{C} = \mathcal{C}_{0,\Omega_{1,1}}$ *or* $\mathcal{C} = \mathcal{C}_{0,\Omega_{1,}}$.

ii) Let $\mathcal{C}$ *be a subalgebra of* $Alg\mathcal{L}_\infty$ *such that* $\mathcal{C}_{0,\Omega_{1,1}} \subset \mathcal{C} \subset \mathcal{C}_{0,\Omega_{,1}}$. *Then* $\mathcal{C}$ *is an ideal of* $Alg\mathcal{L}_\infty$ *if and only if* $\mathcal{C} = \mathcal{C}_{0,\Omega_{1,1}}$ *or* $\mathcal{C} = \mathcal{C}_{0,\Omega_{,1}}$.

Let Λ and Γ be nonempty subsets of N and $\mathcal{A}_{2,\Gamma} \cap \mathcal{A}_\Lambda$ will be denoted by $\mathcal{A}_{2,\Gamma,\Lambda}$. And we will prove only one case of relationships between ideals of $\mathcal{A}_{2,\Gamma,\Lambda}$. The other relations will be proved by the same way.

Theorem 18. *Let* $\Gamma = \{k_1, k_2\} = \{1, 2\}$ *and* $\Lambda = \{3, 4\}$. *Let* $\mathcal{A}$ *be a subalgebra of* $Alg\mathcal{L}_\infty$ *such that* $\mathcal{A}_{2,\Gamma} \subset \mathcal{A} \subset \mathcal{A}_{2,\{1\},\{3,4\}}$. *Then* $\mathcal{A}$ *is an ideal of* $Alg\mathcal{L}_\infty$ *if and only if* $\mathcal{A} = \mathcal{A}_{2,\Gamma}$ *or* $\mathcal{A} = \mathcal{A}_{2,\{1\},\{3,4\}}$.

Proof. Let $\mathcal{A}$ be an ideal of Alg$\mathcal{L}_\infty$ and $\mathcal{A} \neq \mathcal{A}_{2,\Gamma}$. Then there exists an element $T = (t_{ij}) \in \mathcal{A}$ and $T = (t_{ij}) \notin \mathcal{A}_{2,\Gamma}$, i.e. $t_{34} \neq 0$. Let $A = (a_{ij})$ be an element of $\mathcal{A}_{2,\{1\},\{3,4\}}$. If $a_{34} = 0$, then $A \in \mathcal{A}_{2,\Gamma}$ and so $A \in \mathcal{A}$. If $a_{34} \neq 0$, we let define an operator $S = (s_{ij})$ by

$$\begin{cases} s_{11} = 0,\ s_{12} = 0,\ s_{22} = 0,\ s_{33} = 0,\ s_{34} = 0,\ s_{44} = 0 \\ s_{ij} = -t_{ij} \quad \text{otherwise.} \end{cases}$$

Then $\mathcal{S} \in \mathcal{A}_{2,\Gamma}$ and so $\mathcal{S} \in \mathcal{A}$. We define an operator $B = (b_{ij})$ by

$$\begin{cases} b_{11} = 0,\ b_{12} = 0,\ b_{22} = 0,\ b_{33} = 0,\ b_{34} = 0,\ b_{44} = 0 \\ b_{ij} = a_{ij} \quad \text{otherwise.} \end{cases}$$

Then $B \in \mathcal{A}$. Put $D = S + T$ and $x = \frac{a_{34}}{t_{34}}$. Then $A = B + xD \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_{2,\{1\},\{1,4\}}$. □

If we denote $\mathcal{B}_{2,\Gamma} \cap \mathcal{B}_\Lambda$ by $\mathcal{B}_{2,\Gamma,\Lambda}$ and $\mathcal{C}_{2,\Gamma} \cap \mathcal{C}_\Lambda$ by $\mathcal{C}_{2,\Gamma,\Lambda}$, then we will prove relationships between ideals $\mathcal{B}_{2,\Gamma,\Lambda}$ and $\mathcal{C}_{2,\Gamma,\Lambda}$ by modifying the method of the proof of Theorem 15.

Theorem 19. *Let* k_i *be natural numbers such that* $k_i \lneq k_{i+1}$, $i = 1, 2, \cdots$. *Let* $\Gamma_1 = \{k_1\}$, $\Gamma_2 = \{k_1, k_2\}$, $\cdots$,$\Gamma_n = \{k_1, k_2, \cdots, k_n\}$ *and* $\Gamma = \{k_1, k_2, \cdots\}$. *Then*

$\mathcal{A}_{0,\Gamma} \subset \cdots \subset \mathcal{A}_{0,\Gamma_n} \subset \mathcal{A}_{0,\Gamma_{n-1}} \subset \cdots \subset \mathcal{A}_{0,\Gamma_2} \subset \mathcal{A}_{0,\Gamma_1} = \mathcal{A}_{0,\{k_1\}}$
$\mathcal{A}_{2,\Gamma} \subset \cdots \subset \mathcal{A}_{2,\Gamma_n} \subset \mathcal{A}_{2,\Gamma_{n-1}} \subset \cdots \subset \mathcal{A}_{2,\Gamma_2} \subset \mathcal{A}_{2,\Gamma_1} = \mathcal{A}_{2,\{k_1\}}$
$\mathcal{B}_{0,\Gamma} \subset \cdots \subset \mathcal{B}_{0,\Gamma_n} \subset \mathcal{B}_{0,\Gamma_{n-1}} \subset \cdots \subset \mathcal{B}_{0,\Gamma_2} \subset \mathcal{B}_{0,\Gamma_1} = \mathcal{B}_{0,\{k_1\}}$
$\mathcal{B}_{2,\Gamma} \subset \cdots \subset \mathcal{B}_{2,\Gamma_n} \subset \mathcal{B}_{2,\Gamma_{n-1}} \subset \cdots \subset \mathcal{B}_{2,\Gamma_2} \subset \mathcal{B}_{2,\Gamma_1} = \mathcal{B}_{2,\{k_1\}}$

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