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k-PRIME CORDIAL GRAPHS

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Abstract. In this paper we introduce a new graph labeling called k-prime cordial labeling. Let G be a (p, q) graph and $2 \leq p \leq k$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge uv, assign the label $\gcd(f(u), f(v))$. f is called a k-prime cordial labeling of G if $\left|v_f(i) - v_f(j)\right| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ and $\left|e_f(0) - e_f(1)\right| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is called a k-prime cordial graph. In this paper we investigate the k-prime cordial labeling behavior of a star and we have proved that every graph is a subgraph of a k-prime cordial graph. Also we investigate the 3-prime cordial labeling behavior of path, cycle, complete graph, wheel, comb and some more standard graphs.

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1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to some conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. The graph labeling problem has a fast development recently. This problem was first introduced by Alex Rosa in 1967. Since Rosa's article, many different types of graph labeling problems have been defined around this. This is not only due to its mathematical importance but also because of the wide range of the applications arising from this area, for instance, x-rays, crystallography, coding theory, radar, astronomy, circuit design, and design of good radar type codes, missile guidance codes and convolution codes with optimal autocorrelation properties and communication design [2]. All graphs considered here are finite simple and undirected. The number of vertices of a graph G is called order of G, and the number of edges is called size of G. Let G_1 and G_2 be two graphs with vertex sets V_1 and

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V_2 and edge sets E_1 and E_2 respectively. Then their *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. In 1987, Cahit introduced the concept of cordial labeling of graphs [1]. Sundaram, Ponraj, Somasundaram [7] have introduced the notion of prime cordial labeling. A prime cordial labeling of a graph G with vertex set V is a bijection $f : V \to \{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. Also they discussed the prime cordial labeling behavior of various graphs. In [5, 6], Seoud and Salim gave an upper bound for the number of edges of a graph with a prime cordial labeling as a function of the number of vertices. For bipartite graphs they gave a stronger bound and also they determine all prime cordial graphs of order at most 6. Recently Ponraj et al. [4], introduced the concept of k-difference cordial labeling of graphs and studied the 3-difference cordial labeling behavior of of star, m copies of star etc. Also they discussed the 3-difference cordial labeling behavior of path, cycle, complete graph, complete bipartite graph, star, bistar, comb, double comb, quadrilateral snake, $C_4^{(t)}$, $S(K_{1,n})$, $S(B_{n,n})$. Motivated by these labelings we introduce k-prime cordial labeling of graphs. Also in this paper we investigate the 3-prime cordial labeling behavior of path, cycle, complete graph, star, wheel etc. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here follow from Harary [3].

2. k-prime cordial labeling

Definition 2.1. Let G be a (p, q) graph and $2 \leq p \leq k$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a function. For each edge uv, assign the label $\gcd(f(u), f(v))$. f is called a k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is called a k-prime cordial graph.

Example 2.2. An example of a 3-prime cordial labeling of a graph is given in Figure 1.

Remark 2.1. A 2-prime cordial labeling is a product cordial labeling [2].

Theorem 2.3. *Every graph is a subgraph of a connected k-prime cordial graph.*

Proof. Let G be a given (p, q) graph. Take k-copies of complete graph K_p. Let G_i be the i^{th} copy of K_p. Let $V(G_i) = \{u_j^i : 1 \leq j \leq p\}$. Let $m = (k-2)\binom{p}{2} - k + 1$.

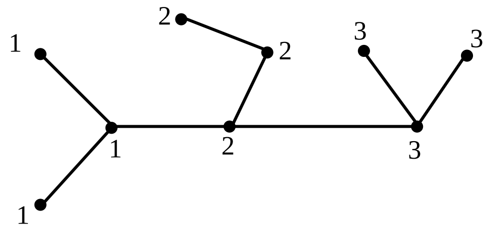

FIGURE 1

The vertex set and edge set of the super graph G^* of G is as follows: Let $V(G^*) = \bigcup_{i=1}^{k} V(G_i) \cup \{v_i : 1 \le i \le m\}$ and $E(G^*) = \bigcup_{i=1}^{k} E(G_i) \cup \{u_1^1 v_i : 1 \le i \le m\} \cup \{u_1^j u_1^{j+1} : 1 \le j \le p-1\}$. Clearly G^* has $kp + m$ vertices and $2(k-1)\binom{p}{2}$ edges. Next we assign a labeling to the vertices of G^*. Let $m = kt+r$, $0 \le r < k$. Assign the label i to all the vertices of G_i $(1 \le i \le k)$. Then assign the label 1 to $v_1, v_2, \dots, v_n$, 2 to $v_{t+1}, v_{t+2}, \dots, v_{2t}$, 3 to $v_{2t+1}, v_{2t+2}, \dots, v_{3t}$, …… k to $v_{(k-1)t+1}, v_{(k-1)t+2}, \dots, v_{kt}$. Finally assign the labels $1, 2, \dots, r$ to the vertices $v_{kt+1}, v_{kt+2}, \dots, v_{kt+r}$ respectively. The vertex and edge conditions of the above labeling is given below: $v_f(1) = v_f(2) = \ldots = v_f(r-1) = v_f(r) = p+t+1$, $v_f(r+1) = \ldots = v_f(k) = p+t$. $e_f(0) = (k-1)\binom{p}{2}$ and $e_f(1) = \binom{p}{2} + k - 1 + m = (k-1)\binom{p}{2}$. This forces f to be a k-prime cordial labeling of G^*.

Hence every graph is a subgraph of a connected k-prime cordial graph. $\square$

Theorem 2.4. *If k is even, then the path $P_n, n \neq 3$ is k-prime cordial.*

Proof. Let P_n be the path $u_1 u_2 \ldots u_n$. Let $n = kt+r$, $0 \le r < k$. Assign the label $2, 4, 6, \dots, k$ to first $\frac{k}{2}$ path vertices. Then assign again the label $2, 4, 6, \dots, k$ to the next $\frac{k}{2}$ path vertices. Continue in this way until we reach the vertex $u_{\frac{kt}{2}}$ that receive the label k.

Case 1. r is even.

Then assign the labels $2, 4, 6, \dots, r$ to the vertices $u_{\frac{kt}{2}+1}$, $u_{\frac{kt}{2}+2}$, $\dots$, $u_{\frac{kt}{2}+\frac{r}{2}}$ and $1, 3, 5, \dots, r-1$ to the vertices $u_{\frac{kt}{2}+\frac{r}{2}+1}$, $u_{\frac{kt}{2}+\frac{r}{2}+2}$, $\dots$, $u_{\frac{kt}{2}+r}$.

Case 2. r is odd.

Here assign the labels $2, 4, 6, \dots, r+1$ to the vertices $u_{\frac{kt}{2}+1}$, $u_{\frac{kt}{2}+2}$, $\dots$, $u_{\frac{kt}{2}+\frac{r-1}{2}}$ and $3, 5, \dots, r$ to the vertices $u_{\frac{kt}{2}+\frac{r-1}{2}+1}$, $u_{\frac{kt}{2}+\frac{r-1}{2}+2}$, $\dots$, $u_{\frac{kt}{2}+r}$.

Now assign the labels $1, 3, 5, \dots, k-1$ to the vertices $u_{\frac{kt}{2}+r+1}$, $u_{\frac{kt}{2}+r+2}$, $\dots$, $u_{\frac{kt}{2}+r+2t}$, then label the next t vertices of the path by $1, 3, 5, \dots, k-1$. Continue this process until we reach the vertex u_{kt+r}. Obviously this labeling is a k-prime cordial labeling. $\square$

Corollary 2.5. *The cycle C_n, $n \neq 3$ is k-prime cordial where k is even.*

Proof. Let $u_1u_2\dots u_nu_1$ be the cycle C_n. Assign the label to the vertices u_i as in theorem 2.4. Then replace the label of $u_{\frac{kt}{2}+r}$ by k. Clearly this vertex labeling is a k-prime cordial labeling of C_n. □

Theorem 2.6. *The bistar $B_{n,n}$ is k-prime cordial for all even k.*

Proof. Let u, v be the central vertices of $B_{n,n}$ and u_i and v_i $(1 \le i \le n)$ be the pendent vertices which are adjacent to u, v respectively. Let $n = kt+r$, $0 \le r < k$. Assign the label 1, 2 respectively to the vertices u, v. Then assign the label to the vertices u_i $(1 \le i \le n)$ as follows: Assign the odd integers $1, 3, 5, \dots, k-1$ to the first $\frac{k}{2}$ vertices namely $u_1, u_2, \dots, u_{\frac{k}{2}}$ respectively. Then assign the labels $1, 3, 5, \dots, k-1$ to the next $\frac{k}{2}$ vertices $u_{\frac{k}{2}+1}, u_{\frac{k}{2}+2}, \dots, u_k$ respectively. Continue this process until we reach the last '$\frac{k}{2}$' parts vertices $u_{kt-\frac{k}{2}+1}$, $u_{kt-\frac{k}{2}+2}$, $\cdots$, u_{kt}. Note that in this process the vertex u_{kt} receive the label $k-1$. If r is even, then assign the labels $1, 3, \dots, r-1$ to the vertices $u_{kt+1}, u_{kt+2}, \dots, u_{kt+\frac{r}{2}}$ and $r-1, r-3, \dots \frac{r}{2}$ to the vertices $u_{kt+r}, u_{kt+r-1}, \dots, u_{kt+\frac{r}{2}+1}$; if r is odd then assign the labels $1, 3, \dots, r$ to the vertices $u_{kt+1}, u_{kt+2}, \dots, u_{kt+\frac{r+1}{2}}$ and $r, r-2, \dots \frac{r-1}{2}$ to the vertices $u_{kt+r}, u_{kt+r-1}, \dots, u_{kt+\frac{r-1}{2}+1}$.

Next we move into the other side pendent vertices, namely, v_i $(1 \le i \le n)$. Assign the labels $2, 4, \dots, k$ to the vertices $v_1, v_2, \dots, v_{\frac{k}{2}}$ respectively. Then assign the labels $2, 4, 6, \dots, k$ to the vertices $v_{\frac{k}{2}+1}, v_{\frac{k}{2}+2}, \dots, v_k$ respectively. Proceed this way until we reach the last part $v_{kt-\frac{k}{2}+1}$, $v_{kt-\frac{k}{2}+2}$, $\cdots$, v_{kt}. Clearly the vertex v_{kt} received the label k. If r is even, then assign the labels $2, 4, \dots, r$ to the vertices $v_{kt+1}, v_{kt+2}, \dots, v_{kt+\frac{r}{2}}$ and $r, r-2, \dots \frac{r}{2}$ to the vertices $v_{kt+r}, v_{kt+r-1}, \dots, v_{kt+\frac{r}{2}+1}$; if r is odd then assign the labels $1, 3, \dots, r-1$ to the vertices $v_{kt+1}, v_{kt+2}, \dots, v_{kt+\frac{r-1}{2}}$ and $r-1, r-3, \dots \frac{r-1}{2}$ to the vertices $v_{kt+r}, v_{kt+r-1}, \dots, v_{kt+\frac{r-1}{2}+1}$. □

Next we investigate the 3-prime cordial labeling behavior of some graphs.

3. 3-prime cordial labeling

Theorem 3.1. *The path P_n is 3-prime cordial if and only if $n \ne 3$.*

Proof. For $n = 3$, it is trivial that $v_f(1) = v_f(2) = v_f(3) = 1$. But $e_f(0) = 0$. This implies $|e_f(0) - e_f(1)| > 1$. Assume $n \ne 3$. Let P_n be the path $u_1u_2\dots u_n$.
Case 1. $n \equiv 0, 1 \pmod 3$.

Assign the label 2 to the vertices $u_1, u_2, \dots, u_{\lceil \frac{n}{3} \rceil}$. Then assign the label 3 consecutively to the vertices $u_{\lceil \frac{n}{3} \rceil+1}, u_{\lceil \frac{n}{3} \rceil+2}, \dots$ until we have received the $\lceil \frac{n}{2} \rceil$ edges with the label 0. If all the $\lfloor \frac{n}{3} \rfloor$ 3's are exhausted then assign the label 1 to the remaining vertices; otherwise consider the non labeled vertex u_i such that u_{i-1} is labeled and assign the labels 1, 3 to the vertices $u_i, u_{i+1}, u_{i+2}, \dots$ alternatively until $\lfloor \frac{n}{3} \rfloor$ 3's are exhausted. Finally assign the label 1 to the remaining vertices.

Case 2. $n \equiv 2 \pmod 3$.

As in case 1, assign the labels to the vertices $u_1, u_2, \ldots, u_n$. Now, let i be the least positive integer such that the label of u_{i-1} = the label of $u_{i+1} = 3$, and the label of $u_i = 1$. Finally interchange the labels of u_i and u_{i+1}. Clearly this vertex labeling satisfies both vertex and edge conditions. □

Corollary 3.2. *The cycle C_n is 3-prime cordial if and only if $n \notin \{3, 4, 6\}$.*

Proof. One can easily verify that if $n \in \{3, 4, 6\}$, then C_n is not 3-prime cordial. Consider the case that $n \notin \{3, 4, 6\}$. It is clear that the 3-prime cordial labeling of the path given in theorem 3.1 is also a 3-prime cordial labeling of C_n. □

Theorem 3.3. *If T is a 3-prime cordial tree, then $T \odot K_1$ is also a 3-prime cordial graph.*

Proof. Let the order of T be p. Then the following three cases arise. Let f be a 3-prime cordial labeling of T.
Case 1. $p \equiv 0 \pmod 3$.
Let $p = 3t$. In this case, $v_f(1) = v_f(2) = v_f(3) = t$. We now assign the labeling g to the vertices of $T \odot K_1$ with the help of the labeling f of T. Assign the label 2 to the vertices of $T \odot K_1$ whose support received the label 2 under the labeling f. Then assign the label 3 to the $\left\lceil \frac{t}{2} \right\rceil$ pendent vertices whose supports received the label 3 and assign 1 to all the remaining pendent vertices whose support received the label 3. Now we turn to the pendent vertices whose supports received the label 1. Assign 3 to $\left\lfloor \frac{t}{2} \right\rfloor$ of these vertices. Finally assign 1 to the remaining vertices. Obviously the labeling given above is a 3-prime cordial labeling of $T \odot K_1$ for this case.
Case 2. $p \equiv 1 \pmod 3$.
Let $p = 3t + 1$. Here the following three subcases arise:

(a) $v_f(1) = t + 1$, $v_f(2) = v_f(3) = t$
(b) $v_f(1) = v_f(3) = t$, $v_f(2) = t + 1$
(c) $v_f(1) = v_f(2) = t$, $v_f(3) = t + 1$.

Take the same labeling g of case 1. In the case of (a), there is a non labeled vertex whose support received the label 1. Assign 3 to that vertex. In the case of (b), assign 3 to the non labeled vertex. In the case (c), assign the label 2 to the non labeled vertex. It is easy to verify that this vertex labeling h is a 3-prime cordial labeling of $T \odot K_1$ for this case.
Case 3. $p \equiv 2 \pmod 3$.
Let $p = 3t + 2$. In this case, the following three subcases arise:

(a) $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$
(b) $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$
(c) $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$.

In the case of (a), take the labeling h of subcase (b) of case 2. Assign 1 to the non labeled vertex. In the case of (b), take the labeling h of subcase (a) of case 2. Assign 2 to the non labeled vertex. In the case of (c), take the labeling h of subcase (a) of case 2. Assign 3 to the non labeled vertex. It is easy to verify that this labeling is a 3-prime cordial labeling for this case. $\square$

Corollary 3.4. *The comb $P_n \odot K_1$ is 3-prime cordial.*

Proof. The proof follows from theorems 3.1 and 3.3 for $n \neq 3$. For $n = 3$, the 3-prime cordial labeling of $P_3 \odot K_1$ is given in Figure 2.

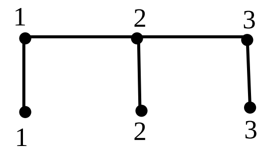

Figure 2

$\square$

Corollary 3.5. *The crown $C_n \odot K_1$ is 3-prime cordial if and only if $n \neq 3$.*

Proof. Suppose $n = 3$. Then $v_f(1) = v_f(2) = v_f(3) = 2$. Also $e_f(0) \leq 2$. It follows that, $|e_f(0) - e_f(1)| = 2$. Hence $C_3 \odot K_1$ is not 3-prime cordial. Conversely, one can easily verify that the labeling given in corollary 3.4 is also a 3-prime cordial labeling of $C_n \odot K_1$. $\square$

A rooted tree consisting of n branches, where the i^{th} branch is a path of length i is called an olive tree and it is denoted by OT_n.

Theorem 3.6. *Olive tree OT_n is 3-prime cordial.*

Proof. Assign the label 1 to the central vertex. We now consider the path of order n. Assign the label 3 to all the vertices of this path. Note that the number of edges with label 0 is $n-2$. Next we move to the next path of order $n-1$. Assign the label 3 to the vertices of the path until $\left\lceil \frac{\binom{n}{2}+1}{3} \right\rceil$ 3's are used. Now check the edges with label 0. If it is $\left\lceil \frac{\binom{n}{2}}{2} \right\rceil$ then we have half of the edges with the label 0. Otherwise we consider the label 2. Assign the label 2 to the next non labeled vertices of this path. (If the label 3 has appeared in the pendent vertex of the path then move to the next path). Proceed this way assign the label 2 to the vertices of the paths until we get the $\left\lceil \frac{\binom{n}{2}}{2} \right\rceil$ edges with the label zero. It is easy to observe that we have some remaining 2's. Now assign the labels 1 and 2 alternatively. The edges with vertex labels 1 and 2 together with the edges incident with the central vertex and the edge with the vertex label 3 and 2 contributes $\left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor$ 1's. (Possible if the last used 3 is not a label of the pendent vertex of a path). Clearly the above labeling pattern is a 3-prime cordial labeling. $\square$

The next investigation is about $K_2 + mK_1$. Let $V(K_2 + mK_1) = \{u, v, u_i : 1 \leq i \leq m\}$ and $E(K_2 + mK_1) = \{uv, uu_i, vu_i : 1 \leq i \leq m\}$.

Theorem 3.7. *$K_2 + mK_1$ is not 3-prime cordial.*

Proof. Let u and v be the vertices of K_2. Note that the order and size of $K_2 + mK_1$ are $m + 2$ and $2m + 1$ respectively.

Case 1. $f(u) = f(v) = 1$. Here, all the edges receive the label 1. This implies $e_f(1) = 2m + 1$, a contradiction.

Case 2. $f(u) = f(v) = 2$.

Subcase 2a. $m \equiv 0 \pmod 3$. Let $m = 3t$, $t \geq 1$. In this case $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$ or $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$. Suppose $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$ then $e_f(0) = 2t - 1$ and $e_f(1) = 4t + 2$. This shows that $e_f(1) - e_f(0) = 2t + 3 > 1$, a contradiction. If $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$, then $e_f(0) = 2t - 3$ and $e_f(1) = 4t + 4$. This implies $e_f(1) - e_f(0) = 2t + 7 > 1$, a contradiction.

Subcase 2b. $m \equiv 1 \pmod 3$. Let $m = 3t + 1$. In this case, $v_f(1) = v_f(2) = v_f(3) = t+1$. Then $e_f(0) = 2t-1$, $e_f(1) = 4t+4$. Thus $e_f(1)-e_f(0) = 2t+5 > 1$, a contradiction.

Subcase 2c. $m \equiv 2 \pmod 3$. Let $m = 3t + 2$. Here, $v_f(1) = t + 2$ $v_f(2) = v_f(3) = t + 1$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t + 2$ or $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t + 2$. If $v_f(1) = t + 2$, $v_f(2) = v_f(3) = t + 1$ or $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t+2$ then $e_f(0) = 2t-1$, $e_f(1) = 4t+6$. Hence $e_f(1)-e_f(0) = 2t+7 > 1$, a contradiction. For $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t + 2$, $e_f(0) = 2t + 1$, $e_f(1) = 4t + 4$. Therefore $e_f(1) - e_f(0) = 2t + 3 > 1$, a contradiction.

Case 3. $f(u) = f(v) = 3$. Similar to case 2.

Case 4. $f(u) = 1$, $f(v) = 2$.

Subcase 4a. $m \equiv 0 \pmod 3$. Let $m = 3t$, $t \geq 1$. In this case $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$ or $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$. Suppose $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$ then $e_f(0) = t$ and $e_f(1) = 5t + 1$. This shows that $e_f(1) - e_f(0) = 4t + 1 > 1$, a contradiction. If $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$, then $e_f(0) = t - 1$ and $e_f(1) = 5t + 2$. This implies $e_f(1) - e_f(0) = 4t + 3 > 1$, a contradiction.

Subcase 4b. $m \equiv 1 \pmod 3$. Let $m = 3t + 1$. In this case, $v_f(1) = v_f(2) = v_f(3) = t + 1$. Then $e_f(0) = t$, $e_f(1) = 5t + 3$. Thus $e_f(1) - e_f(0) = 4t + 3 > 1$, a contradiction.

Subcase 4c. $m \equiv 2 \pmod 3$. Let $m = 3t + 2$. Here, $v_f(1) = t + 2$ $v_f(2) = v_f(3) = t + 1$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t + 2$ or $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t + 2$. If $v_f(1) = t + 2$, $v_f(2) = v_f(3) = t + 1$ or $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t + 2$ then $e_f(0) = t$, $e_f(1) = 5t + 5$. Hence $e_f(1) - e_f(0) = 4t + 5 > 1$, a contradiction. For $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t + 2$, $e_f(0) = t + 1$, $e_f(1) = 5t + 4$. Therefore $e_f(1) - e_f(0) = 4t + 3 > 1$, a contradiction.

Case 5. $f(u) = 1$, $f(v) = 3$. Similar to case 4.
Case 6. $f(u) = 2$, $f(v) = 3$.

Subcase 6a. $m \equiv 0 \pmod 3$. Let $m = 3t$, $t \geq 1$. In this case $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$ or $v_f(1) = t$, $v_f(2) = v_f(3) = t+1$. Suppose $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$, then $e_f(0) = 2t-1$ and $e_f(1) = 4t+2$. This shows that $e_f(1) - e_f(0) = 2t+3 > 1$, a contradiction. If $v_f(1) = t$, $v_f(2) = v_f(3) = t+1$, then $e_f(0) = 2t$ and $e_f(1) = 4t+1$. This implies $e_f(1) - e_f(0) = 2t+1 > 1$, a contradiction.

Subcase 6b. $m \equiv 1 \pmod 3$. Let $m = 3t+1$. In this case, $v_f(1) = v_f(2) = v_f(3) = t+1$. Then $e_f(0) = 2t$, $e_f(1) = 4t+3$. Thus $e_f(1) - e_f(0) = 2t+3 > 1$, a contradiction.

Subcase 6c. $m \equiv 2 \pmod 3$. Let $m = 3t+2$. Here, $v_f(1) = t+2$ $v_f(2) = v_f(3) = t+1$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t+2$ or $v_f(1) = v_f(2) = t+1$, $v_f(3) = t+2$. If $v_f(1) = t+2$, $v_f(2) = v_f(3) = t+1$, then $e_f(0) = 2t$, $e_f(1) = 4t+5$. Hence $e_f(1) - e_f(0) = 2t+5 > 1$, a contradiction. Suppose $v_f(1) = v_f(2) = t+1$, $v_f(3) = t+2$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t+2$, then $e_f(0) = 2t+1$, $e_f(1) = 4t+4$. Therefore $e_f(1) - e_f(0) = 2t+3 > 1$, a contradiction.

Hence $K_2 + mK_1$ is not 3-prime cordial. □

Corollary 3.8. *$K_{2,n}$ is not 3-prime cordial.*

Proof. Since $K_{2,n}$ is obtained from $K_2 + nK_1$, by removing the edge uv where $deg(u) = deg(v) = n+1$ in K_2+nK_1. Then by theorem 3.7, the proof follows. □

Theorem 3.9. *The star $K_{1,n}$ is 3-prime cordial if and only if $n \leq 3$.*

Proof. It is easy to see that $K_{1,n}$, $n \leq 3$ is 3-prime cordial. On the other hand, if possible, there exist a 3-prime cordial labeling of $K_{1,n}$, say f. Let u be the vertex with degree n.
Case 1. $f(u) = 1$. In this case $e_f(0) = 0$ and $e_f(1) = n$. This is a contradiction.
Case 2. $f(u) = 2$.

Subcase 2a. $n \equiv 0 \pmod 3$. Let $n = 3t$. In this case, $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$ or $v_f(1) = t+1$, $v_f(2) = v_f(3) = t$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$. Suppose $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$ then $e_f(0) = t$, $e_f(1) = 2t$. Therefore $|e_f(1) - e_f(0)| = t > 1$, a contradiction. If $v_f(1) = t+1$ $v_f(2) = v_f(3) = t$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$ then $e_f(0) = t-1$, $e_f(1) = 2t+1$. Hence $|e_f(1) - e_f(0)| = t+2 > 1$, a contradiction.

Subcase 2b. $n \equiv 1 \pmod 3$. Let $n = 3t+1$. Here, $v_f(1) = t$ $v_f(2) = v_f(3) = t+1$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$ or $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$. If $v_f(1) = t$, $v_f(2) = v_f(3) = t+1$ or $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ then $e_f(0) = t$, $e_f(1) = 2t+1$. Hence $|e_f(1) - e_f(0)| = t+1 > 1$, a contradiction. For $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$, $e_f(0) = t-1$, $e_f(1) = 2t+2$. Therefore $|e_f(1) - e_f(0)| = t+3 > 1$, a contradiction.

Subcase 2c. $n \equiv 2 \pmod 3$. Let $n = 3t+2$. In this case, $v_f(1) = v_f(2) = v_f(3) = t+1$. Then $e_f(0) = t$, $e_f(1) = 2t+2$. Thus $|e_f(1) - e_f(0)| = t+2 > 1$, a contradiction.

Case 3. $f(u) = 3$. Similar to case 2.

Hence $K_{1,n}$ is 3-prime cordial if and only if $n \leq 3$. □

Theorem 3.10. *The complete graph K_n is 3-prime cordial if and only if $n < 3$.*

Proof. If $n < 3$, the proof is trivial. Assume $n \geq 3$. If possible, let f be a 3-prime cordial labeling of K_n.

Case 1. $n \equiv 0 \pmod 3$.

Let $n = 3t$. In this case, $v_f(1) = v_f(2) = v_f(3) = t$. Then $e_f(0) = \binom{t}{2} + \binom{t}{2} = t(t-1)$ and $e_f(1) = \binom{3t}{2} - t(t-1) = \frac{t(7t-1)}{2}$. This implies $|e_f(1) - e_f(0)| = \frac{t(5t+1)}{2} > 1$, a contradiction.

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t+1$. In this case, $v_f(1) = t+1$ $v_f(2) = v_f(3) = t$ or $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$. If $v_f(1) = t+1$ $v_f(2) = v_f(3) = t$ then $e_f(0) = \binom{t}{2} + \binom{t}{2} = t(t-1)$, $e_f(1) = \binom{3t+1}{2} - t(t-1) = \frac{t(7t+5)}{2}$. This forces $|e_f(1) - e_f(0)| = \frac{t(5t+7)}{2} > 1$, a contradiction. Suppose $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$ then $e_f(0) = \binom{t}{2} + \binom{t+1}{2} = t^2$, $e_f(1) = \binom{3t+1}{2} - t^2 = \frac{t(7t+3)}{2}$. Hence $|e_f(1) - e_f(0)| = \frac{t(5t+3)}{2} > 1$, a contradiction.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t+2$. In this case, $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$ or $v_f(1) = t$ $v_f(2) = v_f(3) = t+1$. If $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$ then $e_f(0) = \binom{t+1}{2} + \binom{t}{2} = t^2$, $e_f(1) = \binom{3t+2}{2} - t^2 = \frac{7t^2+9t+2}{2}$ and hence $|e_f(1) - e_f(0)| = \frac{5t^2+9t+2}{2} > 1$, a contradiction. For $v_f(2) = v_f(3) = t+1$, $v_f(1) = t$, $e_f(0) = \binom{t+1}{2} + \binom{t+1}{2} = t(t+1)$, $e_f(1) = \binom{3t+2}{2} - t(t+1) = \frac{7t^2+7t+2}{2}$. This implies $|e_f(1) - e_f(0)| = \frac{5t^2+5t+2}{2} > 1$, a contradiction.

Hence K_n is not 3-prime cordial. □

Theorem 3.11. *The wheel W_n is not 3-prime cordial.*

Proof. Suppose there exist a 3-prime cordial labeling of W_n, say f. Let u be the vertex with degree n.

Case 1. $f(u) = 1$.

Subcase 1a. $n \equiv 0 \pmod 3$. Let $n = 3t$. Here, $v_f(1) = t+1$ $v_f(2) = v_f(3) = t$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$ or $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$. Suppose $v_f(1) = t+1$, $v_f(2) = v_f(3) = t$ then $e_f(0) \leq (t-1)+(t-1) = 2t-2$ and $e_f(1) \geq 6t - (2t-2) = 4t+2$. Therefore $e_f(1) - e_f(0) \geq 2t+4$, a contradiction. If $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$ or $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$ then $e_f(0) \leq (t-1) + t = 2t-1$ and $e_f(1) \geq 6t - (2t-1) = 4t+1$. Hence $e_f(1) - e_f(0) = 2t+2$, a contradiction.

Subcase 1b. $n \equiv 1 \pmod 3$. Let $n = 3t + 1$. In this case, $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$ or $v_f(1) = t$ $v_f(2) = v_f(3) = t + 1$. If $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$ then $e_f(0) \leq t + (t-1) = 2t-1$ and $e_f(1) \geq 6t+2-(2t-1) = 4t+3$. Hence $e_f(1) - e_f(0) = 2t + 4$, a contradiction. For $v_f(2) = v_f(3) = t + 1$, $v_f(1) = t$, $e_f(0) \leq t + t = 2t$ and $e_f(1) \geq 6t + 2 - 2t = 4t + 2$. This forces $e_f(1) - e_f(0) = 2t + 2$, a contradiction.

Subcase 1c. $n \equiv 2 \pmod 3$. Let $n = 3t + 2$. In this case, $v_f(1) = v_f(2) = v_f(3) = t + 1$. Then $e_f(0) \leq t + t = 2t$, $e_f(1) \geq 6t + 4 - 2t = 4t + 4$. Thus $|e_f(1) - e_f(0)| \geq 2t + 4$, a contradiction.

Case 2. $f(u) = 2$.

Subcase 2a. $n \equiv 0 \pmod 3$. Let $n = 3t$. Here, $v_f(1) = t + 1$ $v_f(2) = v_f(3) = t$ or $v_f(1) = v_f(2) = t$, $v_f(3) = t+1$ or $v_f(1) = v_f(3) = t$, $v_f(2) = t+1$. If $v_f(2) = v_f(3) = t$, $v_f(1) = t+1$ then $e_f(0) \leq (t-2)+(t-1)+(t-1) = 3t-4$ and $e_f(1) \geq 6t - (3t-4) = 3t+4$. Hence $e_f(1) - e_f(0) \geq 8$, a contradiction. If $v_f(1) = v_f(2) = t$, $v_f(3) = t + 1$ then $e_f(0) \leq (t-2) + (t-1) + t = 3t-3$ and $e_f(1) \geq 6t - (3t-3) = 3t+3$. This forces $e_f(1) - e_f(0) \geq 6$, a contradiction. For $v_f(2) = t + 1$ $v_f(1) = v_f(3) = t$, $e_f(0) \leq (t-1) + (t-1) + t = 3t-2$ and $e_f(1) \geq 6t - (3t+2) = 3t+2$. Therefore $e_f(1) - e_f(0) \geq 4$, a contradiction.

Subcase 2b. $n \equiv 1 \pmod 3$. Let $n = 3t + 1$. Here, $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$ or $v_f(1) = v_f(3) = t+1$, $v_f(2) = t$ or $v_f(1) = t$ $v_f(2) = v_f(3) = t+1$. If $v_f(1) = v_f(2) = t+1$, $v_f(3) = t$ then $e_f(0) \leq (t-1)+(t-1)+t = 3t-2$ and $e_f(1) \geq 6t+2-(3t-2) = 3t+4$. This implies $e_f(1)-e_f(0) \geq 6$, a contradiction. If $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$ then $e_f(0) \leq (t-2) + (t-1) + t = 3t-3$ and $e_f(1) \geq 6t+2-(3t-3) = 3t+5$. Hence $e_f(1)-e_f(0) \geq 8$, a contradiction. Suppose $v_f(2) = v_f(3) = t+1$, $v_f(1) = t$ then $e_f(0) \leq (t-1)+t+t = 3t-1$ and $e_f(1) \geq 6t+2-(3t-1) = 3t+3$. This implies $e_f(1)-e_f(0) \geq 4$, a contradiction.

Subcase 2c. $n \equiv 2 \pmod 3$. Let $n = 3t + 2$. In this case, $v_f(1) = v_f(2) = v_f(3) = t+1$. Then $e_f(0) \leq t+(t-1)+t = 3t-1$, $e_f(1) \geq 6t+4-(3t-1) = 3t+5$. Hence $e_f(1) - e_f(0) \geq 6$, a contradiction.

Case 3. $f(u) = 3$. Similar to case 2.

Hence W_n is not 3-prime cordial. □

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