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EXPONENTIALLY FITTED INTERPOLATION FORMULAS DEPENDING ON TWO FREQUENCIES†

KYUNG JOONG KIM

Abstract. Our goal is to construct a two-frequency-dependent formula $\tilde{\mathcal{I}}_N$ which interpolates a product $\tilde{f}$ of two functions with different frequencies at some N points. In the beginning, it is not clear to us that the formula $\tilde{\mathcal{I}}_N$ satisfies

$$\tilde{\mathcal{I}}_N \quad = \quad \tilde{f}$$

at the points. However, it is later shown that $\tilde{\mathcal{I}}_N$ satisfies the above equation. For this theoretical development, a one-frequency-dependent formula is introduced, and some of its characteristics are explained. Finally, our newly constructed formula $\tilde{\mathcal{I}}_N$ is compared to the classical Lagrange interpolating polynomial and the one-frequency-dependent formula in order to show the advantage that is obtained by generating the formula depending on two frequencies.

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1. Introduction

The exponentially fitted technique was introduced to deal with numerical differentiation and integration tuned to oscillatory functions [4]. This technique was applied in order to approximate oscillatory functions with the information of the functions known at two points (see Chapter 4 of [5]). The formulas to approximate oscillatory functions were further studied and extended into a case using the information of the functions at two or more given points [9]. Error analysis for such exponentially-fitted-base(=EFB) formulas was also investigated [2, 7]. Recently, some characteristics of the EFB formulas using the values of first and higher-order derivatives have been dealt with more comprehensively [8, 10]. Now, we will construct EFB formulas to interpolate the product of two functions with different frequencies at some points.

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This article is organized as follows. In Section 2, a system of linear equations is derived that is satisfied by the coefficients of the formula $\mathcal{I}_N$ depending on a single frequency. In Section 3, it is shown that $\mathcal{I}_N$ is actually an interpolation formula that matches an oscillatory function at some points. In Section 4, a two-frequency-dependent interpolation formula $\tilde{\mathcal{I}}_N$ is newly constructed. A regularization process is described to solve the singular problem that occurs when one frequency approaches the other frequency. In Section 5, numerical results are given and compared.

2. Constructing $\mathcal{I}_N$ depending on a single frequency

Let us consider a formula to approximate an ω-dependent function $f(x)$ on $[a, b]$ in terms of the values of the function at a set of predetermined points on $[a, b]$ where the function f is of the form

$$f(x) \quad = \quad \epsilon_1(x)\cos(\omega x) + \epsilon_2(x)\sin(\omega x). \tag{1}$$

In the above, ϵ_1 and ϵ_2 are assumed to be smooth enough to be approximated by polynomials. The formula with the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$ is denoted by $\mathcal{I}_N$ and given by

$$f(c + ht) \quad \approx \quad \mathcal{I}_N(t) = \textstyle\sum_{k=1}^{N} \alpha_k(\omega, h, t, \mathcal{X}) f(c + hx_k) \tag{2}$$

where $c = (a + b)/2$, $h = (b - a)/2$ and $-1 \le t \le 1$ and $\mathcal{X} = (x_1, x_2, \ldots, x_N)$. In (2), x_k is given by

$$x_k \quad = \quad -1 + 2(k - 1)/(N - 1) \tag{3}$$

where $k = 1, 2, \ldots, N$. Thus, $x_1, x_2, \ldots, x_N$ are equidistant and symmetrically distributed around 0. For example, using (3) with $N = 4$ gives

$$x_1 = -1, \quad x_2 = -1/3, \quad x_3 = 1/3, \quad x_4 = 1.$$

Therefore, the points $c + hx_1,\ c + hx_2,\ \ldots,\ c + hx_N$ are equidistant on $[a, b]$ and symmetrically distributed around c. In (2), the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$ will depend on the values of ω, h, t and $\mathcal{X}$. But, for simplicity we will take the notation α_k instead of $\alpha_k(\omega, h, t, \mathcal{X})$.

Let us introduce a functional $\mathcal{M}$,

$$\mathcal{M}(f(x), h, \mathcal{A}) \quad = \quad f(x + ht) - \textstyle\sum_{k=1}^{N} \alpha_k f(x + hx_k), \tag{4}$$

where $\mathcal{A}$ is a vector of the coefficients. That is,

$$\mathcal{A} \quad = \quad (\alpha_1, \alpha_2, \ldots, \alpha_N).$$

To construct $\mathcal{I}_N$, we are concerned with determining the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$ which satisfy N conditions such as

$$\mathcal{M}(x^k e^{\pm i\omega x}, h, \mathcal{A}) \quad = \quad 0 \quad (k = 0, 1, 2, \ldots, N/2 - 1). \tag{5}$$

In the above, the functions $x^k e^{\pm i\omega x}$ will be called reference functions. Note that $\cos(\omega x)$ and $\sin(\omega x)$ are expressed by linear combinations of $e^{\pm i\omega x}$. To obtain the values of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$, we will assume two facts as follows.

(i): The value of ω is known.
(ii): The values of $f(x)$ are given at $x = c + hx_1,\ c + hx_2,\ \ldots,\ c + hx_N$.

In this article, assume that N is even. For an odd N, one more function is needed to obtain a system with the same number of equations as the number of coefficients, in addition to the reference functions $x^k e^{\pm i\omega x}$. See [4] for more details about the reference functions to be taken.

To understand the rest of this article more easily, the Ixaru's functions and their properties are stated as follows (see Section 3.4 of [3]).

(i):

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z < 0 \\ \cosh(Z^{1/2}) & \text{if } Z \geq 0, \end{cases} \tag{6}$$

(ii):

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0, \end{cases} \tag{7}$$

(iii): for $s = 1, 2, 3, \ldots,$

$$\eta_s(Z) = \begin{cases} (\eta_{s-2}(Z) - (2s-1)\eta_{s-1}(Z))/Z & \text{if } Z \neq 0 \\ 2^s s!/(2s+1)! & \text{if } Z = 0. \end{cases} \tag{8}$$

The power series and differentiation of the Ixaru's functions are given (see also Section 3.4 of [3]).

(i):

$$\eta_s(Z) = 2^s \sum_{q=0}^{\infty} g_{sq} Z^q/(2q+2s+1)! \tag{9}$$

with

$$g_{sq} = \begin{cases} 1 & \text{if } s = 0 \\ (q+1)(q+2)\ldots(q+s) & \text{if } s = 1, 2, 3, \ldots, \end{cases}$$

(ii):

$$\frac{d}{dZ}\eta_s(Z) = \frac{1}{2}\eta_{s+1}(Z), \quad s = \ -1,\ 0,\ 1, \ldots. \tag{10}$$

Under these circumstances, let us start the procedure to determine the values of $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_N$. Using (4) with $f(x) = e^{\mu x}$ gives

$$\mathcal{M}(e^{\mu x}, h, \mathcal{A}) \quad = \quad e^{\mu x}\psi(\mu h, \mathcal{A}) \tag{11}$$

where $\mu = i\omega$ and

$$\psi(u, \mathcal{A}) \quad = \quad e^{ut} - \textstyle\sum_{k=1}^{N} \alpha_k e^{u x_k}. \tag{12}$$

Likewise,

$$\mathcal{M}(e^{-\mu x}, h, \mathcal{A}) \quad = \quad e^{-\mu x}\psi(-\mu h, \mathcal{A}). \tag{13}$$

Using (12), define

$$\Psi_p(Z, \mathcal{A}) \quad = \quad \tfrac{1}{2}(\psi(u, \mathcal{A}) + \psi(-u, \mathcal{A})) \tag{14}$$

and

$$\Psi_n(Z, \mathcal{A}) \quad = \quad \tfrac{1}{2u}(\psi(u, \mathcal{A}) - \psi(-u, \mathcal{A})) \tag{15}$$

where $Z = u^2 = (\mu h)^2 = -\omega^2 h^2$. Then the Ixaru's functions give

$$\Psi_p(Z, \mathcal{A}) \quad = \quad \eta_{-1}(Zt^2) - \textstyle\sum_{k=1}^{N/2} \alpha_k^+ \eta_{-1}(Zx_k^2) \tag{16}$$

and

$$\Psi_n(Z, \mathcal{A}) \quad = \quad t\eta_0(Zt^2) - \textstyle\sum_{k=1}^{N/2} \alpha_k^- x_k \eta_0(Zx_k^2) \tag{17}$$

where, for $k = 1, 2, \ldots, N/2$,

$$\alpha_k^+ = \alpha_k + \alpha_{N+1-k} \quad \text{and} \quad \alpha_k^- = \alpha_k - \alpha_{N+1-k}. \tag{18}$$

Now, it is clear that if one of the following two properties is satisfied, the other is also satisfied:

$$\begin{array}{ll} (i) & \mathcal{M}(e^{\pm\mu x}, h, \mathcal{A}) = 0, \\ (ii) & \Psi_p(Z, \mathcal{A}) = 0, \Psi_n(Z, \mathcal{A}) = 0. \end{array} \tag{19}$$

The equivalence of (i) and (ii) is obtained from (11) and (13)-(15). Next we see that, for $m = 0, 1, 2, \ldots,$

$$\tfrac{d^m}{d\mu^m}\mathcal{M}(e^{\mu x}, h, \mathcal{A}) = \mathcal{M}(x^m e^{\mu x}, h, \mathcal{A})$$

and

$$\tfrac{d^m}{d\mu^m}\mathcal{M}(e^{-\mu x}, h, \mathcal{A}) = (-1)^m \mathcal{M}(x^m e^{-\mu x}, h, \mathcal{A}).$$

Thus, if one of the following two properties is satisfied, the other is also satisfied:

$$\begin{array}{ll} (a) & \mathcal{M}(x^m e^{\pm\mu x}, h, \mathcal{A}) = 0, \quad m = 0, 1, \ldots, N/2 - 1, \\ (b) & \tfrac{d^m}{dZ^m}\Psi_p(Z, \mathcal{A}) = 0, \tfrac{d^m}{dZ^m}\Psi_n(Z, \mathcal{A}) = 0, \\ & m = 0, 1, \ldots, N/2 - 1. \end{array} \tag{20}$$

To obtain the equivalence of (a) and (b), the chain rule is applied:

$$\tfrac{d}{d\mu}\Psi_p(Z, \mathcal{A}) = \tfrac{d}{dZ}\Psi_p(Z, \mathcal{A})\tfrac{dZ}{d\mu} = \tfrac{d}{dZ}\Psi_p(Z, \mathcal{A})2\mu h^2.$$

The details of (b) in (20) are given by

$$\tfrac{d^m}{dZ^m}\Psi_p(Z, \mathcal{A}) \quad = \quad \tfrac{1}{2^m}\left(t^{2m}\eta_{m-1}(Zt^2) - \textstyle\sum_{k=1}^{N/2} \alpha_k^+ x_k^{2m} \eta_{m-1}(Zx_k^2)\right) \tag{21}$$

and

$$\tfrac{d^m}{dZ^m}\Psi_n(Z, \mathcal{A}) \quad = \quad \tfrac{1}{2^m}\left(t^{2m+1}\eta_m(Zt^2) - \textstyle\sum_{k=1}^{N/2} \alpha_k^- x_k^{2m+1} \eta_m(Zx_k^2)\right). \tag{22}$$

Let us denote $N^* = N/2$. As seen in (21), the first system of (b) in (20),

$$\tfrac{d^m}{dZ^m}\Psi_p(Z, \mathcal{A}) = 0, \quad m = 0, 1, \ldots, N^* - 1,$$

is linear in α_k^+. Thus, it is possible to arrange it into the matrix equation

$$M^+X^+ = Y^+ \tag{23}$$

where (i) M^+ is an $N^* \times N^*$ matrix, (ii) X^+ and Y^+ are all column vectors with N^* components. The details of the matrix equation in (23) are given: for $m = 0, 1, \ldots, N^* - 1$ and $k = 1, 2, \ldots, N^*$,

$$\begin{aligned} M^+(m+1, k) &= x_k^{2m}\eta_{m-1}(Zx_k^2), \\ X^+(m+1) &= \alpha_{m+1}^+, \\ Y^+(m+1) &= t^{2m}\eta_{m-1}(Zt^2). \end{aligned} \tag{24}$$

We denote $M(j,k)$ and $V(j)$ by the (j,k) entry of a matrix M and the jth entry of a vector V, respectively.

Similarly, the second system of (b) in (20),

$$\frac{d^m}{dZ^m}\Psi_n(Z, \mathcal{A}) = 0, \quad m = 0, 1, \ldots, N^* - 1,$$

is expressed by the matrix equation

$$M^-X^- = Y^- \tag{25}$$

because it is linear in α_k^-. In the above, M^- is an $N^* \times N^*$ matrix whose entries are given by

$$M^-(m+1, k) = x_k^{2m+1}\eta_m(Zx_k^2) \tag{26}$$

for $m = 0, 1, \ldots, N^* - 1$ and $k = 1, 2, \ldots, N^*$. Also, X^- and Y^- are column vectors with N^* components, respectively, such that

$$X^-(m+1) = \alpha_{m+1}^- \quad \text{and} \quad Y^-(m+1) = t^{2m+1}\eta_m(Zt^2) \tag{27}$$

for $m = 0, 1, \ldots, N^* - 1$.

Now, α_k^+ and α_k^- are determined by solving the two matrix equations (23) and (25) where $k = 1, 2, \ldots, N^*$. Therefore, α_1, α_2, α_3, $\ldots$, α_N are obtained from (18). In the following section, some properties of α_k are investigated.

3. Properties of α_k

At the moment, we do not know yet the fact that

$$[\mathcal{I}_N(t)]_{x=c+hx_k} \quad = \quad [f(x)]_{x=c+hx_k} \tag{28}$$

where $k = 1, 2, \ldots, N$. This is because $\mathcal{I}_N$ was only constructed in a way to satisfy (5). We did not impose (28) on the construction of $\mathcal{I}_N$. But, it will be proved in Corollary 3.3 that $\mathcal{I}_N$ satisfies (28). Consequently, $\mathcal{I}_N$ represents an interpolation formula.

To begin with, let us investigate the relation between $Y^\pm$ and $M^\pm$. The first and last equations of (24) give

$$[Y^+(\cdot)]_{t=\pm x_k} = M^+(\,\cdot\,, k) \tag{29}$$

where $k = 1, 2, \ldots, N^*$. Also, (26) and the second equation of (27) give

$$[Y^-(\cdot)]_{t=x_k} = M^-(\,\cdot\,, k) \quad \text{and} \quad [Y^-(\cdot)]_{t=-x_k} = -M^-(\,\cdot\,, k) \tag{30}$$

where $k = 1, 2, \ldots, N^*$. With these findings, apply the Cramer's Rule to solve the two linear systems (23) and (25), respectively. Thus, Lemma 3.1 is obtained.

Lemma 3.1. *For $j, k = 1, 2, \ldots, N^*$,*

$$\begin{aligned} &\left[\alpha_j^+\right]_{t=\pm x_k} = \begin{cases} 1, & \textit{if } j = k, \\ 0, & \textit{otherwise}, \end{cases} \\ &\left[\alpha_j^-\right]_{t=x_k} = \begin{cases} 1, & \textit{if } j = k, \\ 0, & \textit{otherwise}, \end{cases} \quad \left[\alpha_j^-\right]_{t=-x_k} = \begin{cases} -1, & \textit{if } j = k, \\ 0, & \textit{otherwise}. \end{cases} \end{aligned} \tag{31}$$

Proof. The determinant $\det(M)$ of a square matrix M is equal to 0 if two columns (or rows) of the matrix M are equal. This property is used to get (31) when the Cramer's Rule is applied to the matrix equations (23) and (25). □

From (18), we have

$$\alpha_k = \tfrac{1}{2}(\alpha_k^+ + \alpha_k^-), \quad \alpha_{N+1-k} = \tfrac{1}{2}(\alpha_k^+ - \alpha_k^-) \tag{32}$$

where $k = 1, 2 \ldots, N^*$. Since $x_1, x_2, \ldots, x_N$ are symmetrically distributed around 0, the equation

$$x_{N+1-k} = -x_k \tag{33}$$

holds for $k = 1, 2 \ldots, N^*$. From Lemma 3.1, some properties of the coefficients of $\mathcal{I}_N$ are obtained and stated in Theorem 3.2.

Theorem 3.2. *For $j, k = 1, 2, \ldots, N$,*

$$\left[\alpha_j\right]_{t=x_k} = \begin{cases} 1, & \textit{if } j = k, \\ 0, & \textit{otherwise}. \end{cases} \tag{34}$$

Proof. Note that $N = 2N^*$. By Lemma 3.1 and (32), the following results are obtained. For $q, r = 1, 2, \ldots, N^*$,

$$\begin{aligned} (i) \quad \left[\alpha_q\right]_{t=x_r} &= \left(\left[\alpha_q^+\right]_{t=x_r} + \left[\alpha_q^-\right]_{t=x_r}\right)/2 \\ &= \begin{cases} (1+1)/2 = 1, & \text{if } q = r, \\ (0+0)/2 = 0, & \text{otherwise}, \end{cases} \\ (ii) \quad \left[\alpha_q\right]_{t=x_{N+1-r}} &= \left[\alpha_q\right]_{t=-x_r} \\ &= \left(\left[\alpha_q^+\right]_{t=-x_r} + \left[\alpha_q^-\right]_{t=-x_r}\right)/2 \\ &= \begin{cases} (1+(-1))/2 = 0, & \text{if } q = r, \\ (0+0)/2 = 0, & \text{otherwise}, \end{cases} \end{aligned}$$

$$
\begin{aligned}
(iii)\quad [\alpha_{N+1-q}]_{t=x_r} &= \left([\alpha_q^+]_{t=x_r} - [\alpha_q^-]_{t=x_r}\right)/2 \\
&= \begin{cases} (1-1)/2 = 0, & \text{if } q = r, \\ (0-0)/2 = 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(iv)\quad [\alpha_{N+1-q}]_{t=x_{N+1-r}} &= [\alpha_{N+1-q}]_{t=-x_r} \\
&= \left([\alpha_q^+]_{t=-x_r} - [\alpha_q^-]_{t=-x_r}\right)/2 \\
&= \begin{cases} (1-(-1))/2 = 1, & \text{if } q = r, \\ (0-0)/2 = 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

The above results prove (34). □

Corollary 3.3. *For $k = 1, 2, \ldots, N$,*

$$
[\mathcal{I}_N(t)]_{x=c+hx_k} \quad = \quad [f(x)]_{x=c+hx_k} \tag{35}
$$

where $x = c + ht$.

Proof. Theorem 3.2 says that, for $k = 1, 2, \ldots, N$,

$$
\begin{aligned}
[\mathcal{I}_N(t)]_{x=c+hx_k} &= [\mathcal{I}_N(t)]_{t=x_k} = \textstyle\sum_{j=1}^N [\alpha_j]_{t=x_k} f(c+hx_j) \\
&= [\alpha_k]_{t=x_k} f(c+hx_k) \\
&= 1 \cdot f(c+hx_k) = f(c+hx_k)
\end{aligned}
$$

□

As seen in (5), we did not impose (35) on $\mathcal{I}_N$ at the beginning so that $\mathcal{I}_N$ did not necessarily satisfy (35). However, Corollary 3.3 shows that $\mathcal{I}_N$ matches f at the given points. In particular, the result of Corollary 3.3 can be accessed by the theoretical developments which were studied in [8].

4. Constructing $\tilde{\mathcal{I}}_N$ depending on two frequencies

This section we consider a formula to approximate a product of two oscillatory functions f_1 and f_2 with different frequencies ω_1 and ω_2 where

$$
f_j(x) \quad = \quad f_{j,1}(x)\cos(\omega_j x) + f_{j,2}(x)\sin(\omega_j x), \quad j = 1, 2. \tag{36}
$$

In (36), $f_{j,1}$ and $f_{j,1}$ are assumed to be smooth enough to be approximated by polynomials. The product of $f_1(x)$ and $f_2(x)$, denoted by $\tilde{f}$, follows

$$
\begin{aligned}
\tilde{f}(x) &= f_1(x) \times f_2(x) \\
&= p_1(x)\cos(\tau_1 x) + p_2(x)\sin(\tau_1 x) \\
&\quad + p_3(x)\cos(\tau_2 x) + p_4(x)\sin(\tau_2 x)
\end{aligned} \tag{37}
$$

where $\tau_1 = \omega_1 - \omega_2$, $\tau_2 = \omega_1 + \omega_2$ and

$$
\begin{aligned}
p_1 &= (f_{1,1}f_{2,1} + f_{1,2}f_{2,2})/2, \quad p_2 = (f_{1,2}f_{2,1} - f_{1,1}f_{2,2})/2, \\
p_3 &= (f_{1,1}f_{2,1} - f_{1,2}f_{2,2})/2, \quad p_4 = (f_{1,1}f_{2,2} + f_{1,2}f_{2,1})/2.
\end{aligned}
$$

This shows that the product is a sum of two oscillatory functions with different frequencies τ_1 and τ_2. Thus, the formula $\mathcal{I}_N$ which was introduced in Section 2 can be used to approximate the product $\tilde{f}$. That is, we are led to the problem of determining the coefficients α_k of $\mathcal{I}_N$ with respect to $\tilde{f}$ such that

$$\mathcal{M}(x^n e^{\pm i\omega x}, h, \mathcal{A}) \quad = \quad 0 \quad (\omega = \tau_1, \tau_2 \text{ and } n = 0, 1, 2, \ldots). \tag{38}$$

If such coefficients to satisfy (38) are obtained, they obviously depend on the values of two frequencies. To indicate such facts explicitly, we will take the notations of $\tilde{\alpha}_k$ and $\tilde{\mathcal{I}}_N$ instead of α_k and $\mathcal{I}_N$, respectively. Thus, $\mathcal{I}_N$ in (2) is re-expressed by

$$\tilde{f}(c+ht) \quad \approx \quad \tilde{\mathcal{I}}_N(t) = \textstyle\sum_{k=1}^{N} \tilde{\alpha}_k \tilde{f}(c+hx_k) \tag{39}$$

where $\tilde{\alpha}_k = \tilde{\alpha}_k(\tau_1, \tau_2, h, t, \mathcal{X})$ (equivalently, $\tilde{\alpha}_k = \tilde{\alpha}_k(\omega_1, \omega_2, h, t, \mathcal{X})$). Note that c, h, t and x_k in (39) were stated in the fore part of Section 2.

By the way, some of the equations given by (38) will be associated to τ_1, while the others to τ_2. If the number of such equations is N_1^* and N_2^*, respectively, our system to solve is:

(i): for $m = 0, 1, 2, \ldots, N_1^* - 1$,

$$\begin{cases} t^{2m}\eta_{m-1}(Z_1 t^2) & = & \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ x_k^{2m} \eta_{m-1}(Z_1 x_k^2) \\ t^{2m+1}\eta_m(Z_1 t^2) & = & \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k^{2m+1} \eta_m(Z_1 x_k^2), \end{cases} \tag{40}$$

(ii): for $m = 0, 1, 2, \ldots, N_2^* - 1$,

$$\begin{cases} t^{2m}\eta_{m-1}(Z_2 t^2) & = & \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ x_k^{2m} \eta_{m-1}(Z_2 x_k^2) \\ t^{2m+1}\eta_m(Z_2 t^2) & = & \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k^{2m+1} \eta_m(Z_2 x_k^2) \end{cases} \tag{41}$$

where $N_1^* + N_2^* = N^*$, $Z_j = -\tau_j^2 h^2$ $(j = 1, 2)$ and

$$\tilde{\alpha}_k^+ = \tilde{\alpha}_k + \tilde{\alpha}_{N+1-k} \quad \text{and} \quad \tilde{\alpha}_k^- = \tilde{\alpha}_k - \tilde{\alpha}_{N+1-k}.$$

These results come from (21) and (22).

When N_1^* and N_2^* are chosen, there is no any restriction on the choice except that $N_1^* + N_2^* = N^*$. However, we need to be careful about the choice of N_1^* and N_2^*. The detailed form of $\tilde{f}$ given by (37) leads to a suitable choice of N_1^* and N_2^*. If the behaviors of p_1 and p_2 relatively smoother than those of p_3 and p_4, it is certainly acceptable to take $N_1^* < N_2^*$. For example, suppose p_1 and p_2 behave like polynomials of degree one, and suppose p_3 and p_4 do like polynomials of degree two. Then taking $N_1^* = 2$ and $N_2^* = 3$ is a good choice.

Next, let us rearrange (40) and (41) as follows:

$$\begin{cases} t^{2m}\eta_{m-1}(Z_1 t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ x_k^{2m} \eta_{m-1}(Z_1 x_k^2), m = 0, 1, \ldots, N_1^* - 1, \\ t^{2m}\eta_{m-1}(Z_2 t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ x_k^{2m} \eta_{m-1}(Z_2 x_k^2), m = 0, 1, \ldots, N_2^* - 1 \end{cases} \tag{42}$$

and

$$\begin{cases} t^{2m+1}\eta_m(Z_1t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k^{2m+1}\eta_m(Z_1x_k^2), m = 0, 1, \ldots, N_1^* - 1, \\ t^{2m+1}\eta_m(Z_2t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k^{2m+1}\eta_m(Z_2x_k^2), m = 0, 1, \ldots, N_2^* - 1. \end{cases} \quad (43)$$

As a result, (42) is linear in $\tilde{\alpha}_k^+$. Therefore it is written in the form of the matrix equation

$$\tilde{M}^+\tilde{X}^+ \quad = \quad \tilde{Y}^+ \quad (44)$$

where (i) $\tilde{M}^+$ is an $N^* \times N^*$ matrix, (ii) $\tilde{X}^+$ and $\tilde{Y}^+$ are column vectors with N^* components, respectively. By solving (44), the values of $\tilde{\alpha}_k^+$ (equivalently, $\tilde{X}^+$) are obtained. However, this only occurs when the matrix $\tilde{M}^+$ is nonsingular. If $Z_2 \to Z_1$ (equivalently $\tau_2 \to \tau_1$), the system is not stable. But, the problem of this type can be removed by a proper regularization of the system (see [6] for more details). The essence of the regularization may be understood by properly treating the following two equations:

$$\mathcal{M}(e^{\mu_1 x}, h, \mathcal{A}) = 0, \quad \mathcal{M}(e^{\mu_2 x}, h, \mathcal{A}) = 0 \quad (45)$$

where $\mu_1 = i\tau_1$ and $\mu_2 = i\tau_2$. As $\mu_2 \to \mu_1$, the two equations of (45) become more and more identical, so that the involved system becomes singular. To remove the singular problem, we write the two equations of (45) as

$$\mathcal{M}(e^{\mu_1 x}, h, \mathcal{A}) = 0 \quad (46)$$

and

$$\frac{\mathcal{M}(e^{\mu_2 x},h,\mathcal{A}) - \mathcal{M}(e^{\mu_1 x},h,\mathcal{A})}{\mu_2 - \mu_1} = 0. \quad (47)$$

Letting $\mu_2 \to \mu_1$, we note that (47) tends to

$$\mathcal{M}(xe^{\mu_1 x}, h, \mathcal{A}) = 0. \quad (48)$$

Hence, as $\mu_2 \to \mu_1$, the original two equations of (45) become the same system as we need in order to obtain one-frequency-dependent (or μ_1-dependent) interpolation formula which is exact for $f(x) = e^{\mu_1 x}$, $xe^{\mu_1 x}$.

As done in the above process, let us apply the regularization technique to (42). Thus, we have that, for $m = 0$,

$$\eta_{-1}(Z_1t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ \eta_{-1}(Z_1x_k^2) \quad (49)$$

and

$$\eta_{-1}(Z_2t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ \eta_{-1}(Z_2x_k^2). \quad (50)$$

The above two equations are written as

$$\eta_{-1}(Z_1t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ \eta_{-1}(Z_1x_k^2) \quad (51)$$

and

$$\frac{\eta_{-1}(Z_2t^2) - \eta_{-1}(Z_1t^2)}{Z_2 - Z_1} = \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ \frac{\eta_{-1}(Z_2x_k^2) - \eta_{-1}(Z_1x_k^2)}{Z_2 - Z_1}. \quad (52)$$

In particular, (52) is expressed by the series as follows:

$$\begin{aligned} &t^2 \sum_{n=1}^{\infty} \tfrac{1}{n!2^n} \eta_{n-1}(Z_1 t^2) \left(t^2(Z_2 - Z_1)\right)^{n-1} \\ &= \sum_{k=1}^{N^*} \tilde{\alpha}_k^+ x_k^2 \sum_{n=1}^{\infty} \tfrac{1}{n!2^n} \eta_{n-1}(Z_1 x_k^2) \left(x_k^2(Z_2 - Z_1)\right)^{n-1}. \end{aligned} \tag{53}$$

This is done by using the Taylor series for η_{-1} and its differentiation property given by (10).

So far in this section, the regularization has been applied to some of the equations given by (42) involved with $\tilde{\alpha}_k^+$. But, our arguments about the regularization can also be applied to (43) involved with $\tilde{\alpha}_k^-$. First, (43) can be viewed as the matrix equation

$$\tilde{M}^- \tilde{X}^- \quad = \quad \tilde{Y}^- \tag{54}$$

where (i) $\tilde{M}^-$ is an $N^* \times N^*$ matrix, (ii) $\tilde{X}^-$ and $\tilde{Y}^-$ are column vectors with N^* components. By solving (54), the values of $\tilde{\alpha}_k^-$ (equivalently $\tilde{X}^-$) are obtained. Secondly, the regularization is reflected on the following two equations,

$$t\eta_0(Z_1 t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k \eta_0(Z_1 x_k^2) \tag{55}$$

and

$$t\eta_0(Z_2 t^2) = \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k \eta_0(Z_2 x_k^2), \tag{56}$$

which come from the case of $m = 0$ in (43). Finally, from the above two equations we have an analogue of (53) represented by $\tilde{\alpha}_k^-$ as follows:

$$\begin{aligned} &t^3 \sum_{n=1}^{\infty} \tfrac{1}{n!2^n} \eta_n(Z_1 t^2) \left(t^2(Z_2 - Z_1)\right)^{n-1} \\ &= \sum_{k=1}^{N^*} \tilde{\alpha}_k^- x_k^3 \sum_{n=1}^{\infty} \tfrac{1}{n!2^n} \eta_n(Z_1 x_k^2) \left(x_k^2(Z_2 - Z_1)\right)^{n-1}. \end{aligned} \tag{57}$$

If more equations are associated and they face the singular problem, the general regularization procedure developed in [6] is applied to our systems (42) and (43), respectively, to avoid the singularity of each system. Thus, our two systems are rearranged in such a way using both the Taylor series for η_s and the differentiation property of η_s as (53) (or (57)) has been derived from (49) and (50) (or (55) and (56)). Let's approach more closely to actual circumstances. When the regularization procedure is practically carried out in the computer program, a threshold value δ can be used to calculate the fractional forms of (52). That is, (52) is calculated by its own form when $|Z_2 t^2 - Z_1 t^2| \geq \delta$ (or $|Z_2 x_k^2 - Z_1 x_k^2| \geq \delta$) whereas it is calculated by truncated Taylor series of (53) when $|Z_2 t^2 - Z_1 t^2| < \delta$ (or $|Z_2 x_k^2 - Z_1 x_k^2| < \delta$).

Now, let's consider the value of $\tilde{\mathcal{I}}_N$ at some points. As shown in (35), the same values of $\mathcal{I}_N$ and f are obtained at the given points. The equality is maintained by $\tilde{\mathcal{I}}_N$ and $\tilde{f}$. This fact is stated in the following.

Theorem 4.1. *For* $k = 1, 2, \ldots, N$,

$$\left[\tilde{\mathcal{I}}_N(t)\right]_{x=c+hx_k} \quad = \quad \left[\tilde{f}(x)\right]_{x=c+hx_k} \tag{58}$$

where $x = c + ht$.

Proof. The linear system given in (44) has the following details:

$$\begin{aligned}\tilde{M}^+(\cdot, k) = \big[&\eta_{-1}(Z_1x_k^2), x_k^2\eta_0(Z_1x_k^2), x_k^4\eta_1(Z_1x_k^2), \ldots,\\ &x_k^{2(N_1^*-1)}\eta_{N_1^*-2}(Z_1x_k^2), \eta_{-1}(Z_2x_k^2), x_k^2\eta_0(Z_2x_k^2), \ldots,\\ &x_k^{2(N_2^*-1)}\eta_{N_2^*-2}(Z_2x_k^2)\big]^T,\end{aligned} \tag{59}$$

$$\tilde{X}^+ = \left[\tilde{\alpha}_1^+, \tilde{\alpha}_2^+, \tilde{\alpha}_3^+, \ldots, \tilde{\alpha}_{N_1^*}^+, \tilde{\alpha}_{N_1^*+1}^+, \tilde{\alpha}_{N_1^*+2}^+, \ldots, \tilde{\alpha}_{N^*}^+\right]^T,$$

$$\begin{aligned}\tilde{Y}^+ = \big[&\eta_{-1}(Z_1t^2), t^2\eta_0(Z_1t^2), t^4\eta_1(Z_1t^2), \ldots,\\ &t^{2(N_1^*-1)}\eta_{N_1^*-2}(Z_1t^2), \eta_{-1}(Z_2t^2), t^2\eta_0(Z_2t^2), \ldots,\\ &t^{2(N_2^*-1)}\eta_{N_2^*-2}(Z_2t^2)\big]^T.\end{aligned} \tag{60}$$

Likewise, all components of the other linear system (54) are given below:

$$\begin{aligned}\tilde{M}^-(\cdot, k) = \big[&x_k\eta_0(Z_1x_k^2), x_k^3\eta_1(Z_1x_k^2), x_k^5\eta_2(Z_1x_k^2), \ldots,\\ &x_k^{2N_1^*-1}\eta_{N_1^*-1}(Z_1x_k^2), x_k\eta_0(Z_2x_k^2), x_k^3\eta_1(Z_2x_k^2), \ldots,\\ &x_k^{2N_2^*-1}\eta_{N_2^*-1}(Z_2x_k^2)\big]^T,\end{aligned} \tag{61}$$

$$\tilde{X}^- = \left[\tilde{\alpha}_1^-, \tilde{\alpha}_2^-, \tilde{\alpha}_3^-, \ldots, \tilde{\alpha}_{N_1^*}^-, \tilde{\alpha}_{N_1^*+1}^-, \tilde{\alpha}_{N_1^*+2}^-, \ldots, \tilde{\alpha}_{N^*}^-\right]^T,$$

$$\begin{aligned}\tilde{Y}^- = \big[&t\eta_0(Z_1t^2), t^3\eta_1(Z_1t^2), t^5\eta_2(Z_1t^2), \ldots,\\ &t^{2N_1^*-1}\eta_{N_1^*-1}(Z_1t^2), t\eta_0(Z_2t^2), t^3\eta_1(Z_2t^2), \ldots,\\ &t^{2N_2^*-1}\eta_{N_2^*-1}(Z_2t^2)\big]^T.\end{aligned} \tag{62}$$

On the one hand, (59) and (60) give

$$\left[\tilde{Y}^+(\cdot)\right]_{t=\pm x_k} = \tilde{M}^+(\,\cdot\,, k) \tag{63}$$

where $k = 1, 2, \ldots, N^*$. On the other hand, (61) and (62) give

$$\left[\tilde{Y}^-(\cdot)\right]_{t=x_k} = \tilde{M}^-(\,\cdot\,, k) \quad \text{and} \quad \left[\tilde{Y}^-(\cdot)\right]_{t=-x_k} = -\tilde{M}^-(\,\cdot\,, k) \tag{64}$$

where $k = 1, 2, \ldots, N^*$. As (29) and (30) are led to the conclusion of Corollary 3.3, the findings of (63) and (64) lead to the conclusion of Theorem 4.1. Note that the results of (29) and (30) were the starting point for achieving Corollary 3.3. Based on the equation given in (53) (or (57)), it is also expected that the results of (63) and (64) will be followed after the regularization process is performed. Thus, Theorem 4.1 is proved. □

5. Discussion

As far as one-frequency-dependent (or ω-dependent) interpolation formula $\mathcal{I}_N$ is concerned, the absolute values of $\alpha_1^{\pm}$, $\alpha_2^{\pm}$, $\ldots$, $\alpha_{N^*}^{\pm}$ (consequently, the absolute value of the error of $\mathcal{I}_N$) may be very large around some particular values of ω when h, t and x_1, x_2, $\ldots, x_N$ are given (see Section 4.3 of [5] for $N = 2$). If the error of $\mathcal{I}_N$ shows such extreme values, it will be said to exhibit what we call pole-like behaviors around the particular values of ω. This phenomenon occurs because the value of the determinant of the associated matrix approaches zero in the vicinity of ω where the pole-like behaviors appear. In fact, such pole-like behaviors were also witnessed when numerical differentiation and integration were investigated by exponentially fitted techniques (see [4] for the details). Therefore, to obtain the benefit, $\mathcal{I}_N$ should be applied to the values of ω which are placed between the pole-like behaviors. This treatment is echoed in the two-frequency-dependent case in the following. Such pole-like behaviors have also been detected for $\tilde{\mathcal{I}}_N$. If our two frequencies of interest are located between the pole-like behaviors, our formula $\tilde{\mathcal{I}}_N$ will provide a more accurate approximation with respect to the function $\tilde{f}$ that depends on the two frequencies than $\mathcal{I}_N$. Technically, to find out the proper values for ω_1 and ω_2, the error of $\tilde{\mathcal{I}}_N$ can be observed while changing the value of ω_2 (or ω_1) after fixing the value of ω_1 (or ω_2) of interest. Then, ω_2 (or ω_1) can be selected in the range in which the pole-like behavior does not appear.

To show the relative superiority of $\tilde{\mathcal{I}}_N$, numerical results will be illustrated. For this purpose, let us consider an example function given by

$$\tilde{f}(x) \quad = \quad f_1(x) \times f_2(x) \tag{65}$$

where

$$f_1(x) \quad = \quad \cos(x)\cos(\omega_1 x) - \sin(x)\sin(\omega_1 x)$$

and

$$f_2(x) \quad = \quad \cos(x)\cos(\omega_2 x) - \sin(x)\sin(\omega_2 x).$$

To compare the numerical results, we introduce the classical Lagrange interpolating polynomial (see Chap. 3 of [1]) for the function $\tilde{f}(x)$, denoted by $\mathcal{P}_N(x)$, which is constructed at $x = c + hx_k$ for $k = 1, 2, \ldots, N$ (see (3) for x_k). As might be expected, the $\mathcal{P}_N$ is the polynomial of degree $N-1$, and it satisfies that for $k = 1, 2, \ldots, N$,

$$\mathcal{P}_N(c + hx_k) \quad = \quad \tilde{f}(c + hx_k). \tag{66}$$

Assume that $c = 1$ and $h = 0.1$ in (39) (and (2), (66)). Then, for $N = 4, 8$ with $N_1^* = N_2^*$ in both cases, we have investigated $\tilde{\mathcal{I}}_N$. As a result, $\tilde{\mathcal{I}}_N$ is free of the pole-like behavior when $N = 4$, $0 \le \omega_1 \le 20$ and $0 \le \omega_2 \le \omega_1$ and when $N = 8$, $0 \le \omega_1 \le 50$ and $0 \le \omega_2 \le \omega_1$. To approximate the example function $\tilde{f}$ with $\omega_1 = 17$ and $\omega_2 = 15$, we consider three versions:

(a) classical Lagrange interpolating polynomial $\mathcal{P}_N$.
(b) one-frequency-dependent interpolation formula $\mathcal{I}_N$ when $\omega = 17$.

(c) our newly constructed two-frequency-dependent interpolation formula $\tilde{\mathcal{I}}_N$ when $\omega_1 = 17$ and $\omega_2 = 15$ (equivalently, $\tau_1 = 2$ and $\tau_2 = 32$).

In Figs. 1 and 2, the error of each version is given and compared depending on $N = 4$ and 8. As seen in Figs. 1 and 2, our formula $\tilde{\mathcal{I}}_N$ is more accurate for the function $\tilde{f}$ than $\mathcal{P}_N$ and $\mathcal{I}_N$. In fact, these benefits with greater accuracy are predictable. This is because our formula $\tilde{\mathcal{I}}_N$ is determined depending on two frequencies. The numerical results in Figs. 1 and 2 are obtained from Matlab [11].

Judging by the integration of the numerical results and theoretical developments demonstrated in this article, we consider that the formula $\tilde{\mathcal{I}}_N$ can be used as an efficient and useful tool. The error analysis for the interpolation formula handled in the article, which depends on two frequencies, may be the subject of a following study. Furthermore, the information obtained in this article may be developed into a region for generating two-frequency-dependent interpolation formulas involving first and higher-order derivatives.

Figure Captions.

(i): The notations in Figs. 1 and 2 are defined by

$$\mathrm{a} = \tilde{f} - \mathcal{P}_N, \quad \mathrm{b} = \tilde{f} - \mathcal{I}_N, \quad \mathrm{c} = \tilde{f} - \tilde{\mathcal{I}}_N$$

where $\tilde{f}$ is given by (65) with $\omega_1 = 17$ and $\omega_2 = 15$. In the above, $\mathcal{P}_N$ is the classical Lagrange interpolating polynomial and $\mathcal{I}_N$ is given for $\omega = 17$, while $\tilde{\mathcal{I}}_N$ is given for $\omega_1 = 17$ and $\omega_2 = 15$ (equivalently, for $\tau_1 = 2$ and $\tau_2 = 32$).

(ii):

Figure 1: Error comparison for $N = 4$.

Figure 2: Error comparison for $N = 8$.

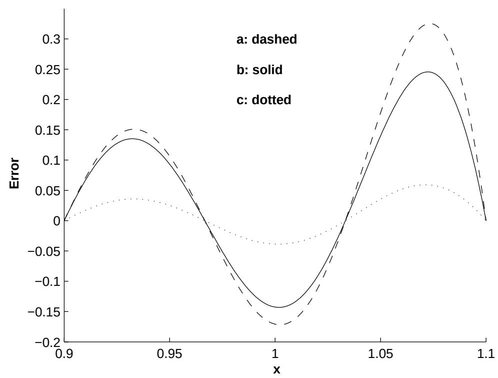

FIGURE 1. N = 4

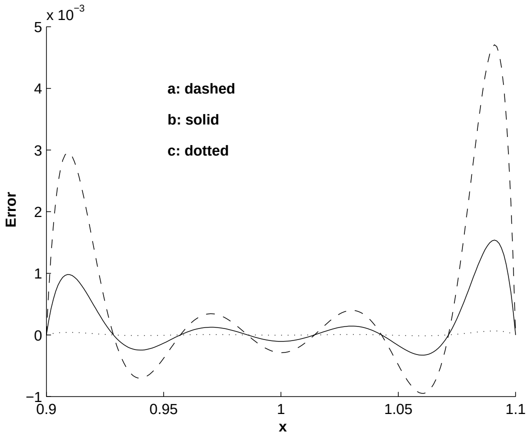

FIGURE 2. N = 8

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K.J. Kim received a Ph.D. from Yonsei University in 1998. Since 2004 he is a professor at Korea Aerospace University. His research interests include interpolation and cubature rules for multivariate integrals.

Department of General Studies, School of Liberal Arts and Sciences, Korea Aerospace University, Goyang 10540, Korea.
e-mail: kj_kim@kau.ac.kr