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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS INVOLVING GRONWALL INEQUALITY IN BANACH SPACE

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Abstract. In this paper, we study boundary value problems for fractional integrodifferential equations involving Caputo derivative in Banach spaces. A generalized singular type Gronwall inequality is given to obtain an important priori bounds. Some sufficient conditions for the existence solutions are established by virtue of fractional calculus and fixed point method under some mild conditions

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1. Introduction

In this paper, we consider the existence of solutions of the following boundary value problems:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), (Gy)(t), (Sy)(t)), \ 0 < \alpha < 1, \ t \in J = [0, T], \\ ay(0) + by(T) = c, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f : J \times X \times X \times X \to X$ where X is a Banach spaces and a, b, c are real constants with $a + b \neq 0$. G and S are nonlinear integral operators given by

$$(Gy)(t) = \int_0^t k_1(t, s)y(s)ds,$$

and

$$(Sy)(t) = \int_0^t k_2(t, s)y(s)ds$$

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with $\gamma_0 = max\int_0^t k_1(t,s)ds : (t,s) \in [0,T] \times [0,T]$ and $\gamma_1 = max\int_0^t k_2(t,s)ds : (t,s) \in [0,T] \times [0,T]$, where $k_1, k_2 \in C\left(J \times J, \mathbb{R}^+\right)$.

The initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc (see[6,11,15]). Therefore,they have received much attention. For the most recent works for the existence and uniqueness of solution of the initial and boundary value problems for nonlinear fractional differential equations, we mention [1-5,7,9,15-23]. But in the obtained results, for the existence, the nonlinear term f needs to satisfy the condition: there exist functions $p, r \in C([0,1],[0,\infty))$ such that for $1 \geq t \geq 0$ and each $u \in R$,

$$|f(t,u)| \leq p(t)|u| + r(t)$$

and for the uniqueness,the nonlinear term f needs to satisfy the condition: there exist functions $p, r \in C([0,1],[0,\infty))$ such that for each $1 \geq t \geq 0$ and any $u, v \in R$,

$$|f(t,u) - f(t,v)| \leq p(t)|u - v|$$

such that by using these result, we cannot discuss the existence and uniqueness of solution. Particulary, Agarwal et al. [1] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative in finite diminsional spaces. Recently, some fractional differential eqquations and optimal controals in Banach spaces are studied by Balachandran et al.[5], El-Borai [7], Henderson and Ourhab [9], Hernandez et.al [11],K.Karthikeyan and J.J.Trujillo [11] Mophou and N, Guerekata [16], Wang et al.[20-22].

The rest of this paper is organized as follows. In Sect. 2, we give some notations and recall some concepts and preparation results. In Sect. 3, we give a generalized singular type Gronwall inequaltity which can be uses to establish the estimate of fixed point set. In Sect. 4, we give two main results, the first results based on Banach contraction principale, the second result based on Schaefer's fixed point theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote $C(J,X)$ the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}$. For measurable functions $m : J \to R$, define the norm $\|m\|_{L^p(J,R)} = \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}}$, $1 \leq p < \infty$. We denote $L^p(J,R)$ the Banach space of all Lebesgue measurable functions m with $\|m\|_{L^p(J,R)} < \infty$. We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [11].

Definition 1. The fractional order integral of the function $h \in L^1([a,b],R)$ of order $\alpha \in R_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the Gamma function.

Definition 2. For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 3. For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Lemma 1. *Let $\alpha > 0$, then the differential equation ${}^cD^\alpha h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0,1,2,\cdots,n, n = [\alpha]+1$.

Lemma 2. *Let $\alpha > 0$, then*

$$I^\alpha({}^cD^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in R, i = 0,1,2,\cdots,n$, $n = [\alpha]+1$.

Now, let us introduce the definition of a solution of the fractional BVP (1).

Definition 4. A function $y \in C^1(J,X)$ is said to be a solution of the fractional BVP (1) if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t), (Gy)(t), (Sy)(t))$ a.e. on J, and the condition $ay(0) + by(T) = c$.

For the existence of solutions for the fractional BVP (1), we need the following auxiliary lemma.

Lemma 3 ([1]). *A function $y \in C(J,X)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \overline{f}(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \overline{f}(s)ds - c\right],$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} {}^cD^\alpha y(t) = \overline{f}(t), \ 0 < \alpha < 1, \ t \in J, \\ ay(0) + by(T) = c. \end{cases} \tag{2}$$

As a consequence of Lemmas 3, we have the following result which is useful in what follows.

Lemma 4. *A function $y \in C(J, X)$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s)) ds \\ &\quad - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s)) ds - c \right], \end{aligned}$$

if and only if y is a solution of the fractional BVP (1).

Lemma 5 (Bochner theorem, [2])**.** *A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesbuge integrable.*

Lemma 6 (Mazur lemma, [2])**.** *If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 7 (Ascoli-Arzela theorem, [18])**.** *Let $\mathcal{S} = \{s(t)\}$ is a function family of continuous mappings $s : [a, b] \to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}(n = 1, 2, \cdots, t \in [a, b])$ in $\mathcal{S}$.*

Theorem 1 (Schaefer's fixed point theorem, [18])**.** *Let $F : X \to X$ completely continuous operator. If the set*

$$E(F) = \{x \in X : x = \lambda F x \ for \ some \ \lambda \in [0, 1]\}$$

is bounded, then F has fixed points.

3. A generalized singular type Gronwall's inequality

In order to apply the Schaefer fixed point theorem to show the existence of solutions, we need a new generalized singular type Gronwall inequality with mixed type singular integral operator. It will play an essential role in the study of BVP for fractional differential equations.

Lemma 8 (Lemma3.2, [13])**.** *Let $y \in C(J, X)$ satisfy the following inequality:*

$$\|y(t)\| \le a + b\int_0^t \|y(\theta)\|^{\lambda_1} d\theta + c\int_0^T \|y(\theta)\|^{\lambda_2} d\theta + d\int_0^t \|y_\theta\|_B^{\lambda_3} d\theta + e\int_0^T \|y_\theta\|_B^{\lambda_4} d\theta, t \in J,$$

where $\lambda_1, \lambda_3 \in [0, 1], \lambda_2, \lambda_4 \in [0, 1), a, b, c, d, e \ge 0$ are constants and $\|y_\theta\|_B = \sup_{0 \le s \le \theta} \|y(s)\|$. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \le M^*.$$

Using the above generalized Gronwall inequality, we can obtain the following new generalized singular type Gronwall inequality.

Lemma 9 (Lemma 3.2, [23]). *Let $y \in C(J, X)$ satisfy the following inequality:*

$$\|y(t)\| \leq a + b\int_0^t (t-s)^{\alpha-1}\|y(s)\|^{\lambda}ds + c\int_0^T (T-s)^{\alpha-1}\|y(s)\|^{\lambda}ds, \tag{3}$$

where $\alpha \in (0,1)$, $\lambda \in [0, 1-\frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$, $a, b, c \geq 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \leq M^*.$$

Proof. Let

$$x(t) = \begin{cases} 1, & \|y(t)\| \leq 1, \\ y(t), & \|y(t)\| > 1. \end{cases}$$

It follows from condition (3) and Hölder inequality that

$$\begin{aligned}
\|x(t)\|^{\lambda} &\leq \|x(t)\| \\
&\leq (a+1) + b\int_0^t (t-s)^{\alpha-1}\|x(s)\|^{\lambda}ds + c\int_0^T (T-s)^{\alpha-1}\|x(s)\|^{\lambda}ds \\
&\leq (a+1) + b\left(\int_0^t (t-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} \\
&\quad + c\left(\int_0^T (T-s)^{p(\alpha-1)}ds\right)^{\frac{1}{p}}\left(\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}}ds\right)^{\frac{p-1}{p}} \\
&\leq (a+1) + b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}}ds \\
&\quad + c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}}ds.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x(t)\| \leq (a+1) &+ b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^t \|x(s)\|^{\frac{\lambda p}{p-1}}ds \\
&+ c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}\int_0^T \|x(s)\|^{\frac{\lambda p}{p-1}}ds,
\end{aligned}$$

where $0 < \frac{\lambda p}{p-1} < 1$. By Lemma (3.1), one can complete the rest proof immediately. $\square$

4. Existence result

Before stating and proving the main results, we introduce the following hypotheses.

- (H1) The function $f : J \times X \times X \times X \to X$ is strongly measurable with respect to t on J.
- (H2) There exists a constant $\alpha_1 \in (0, \alpha)$ and real-valued functions $m_1(t), m_2(t), m_3(t) \in L^{\frac{1}{\alpha_1}}(J, X)$ such that

$$\begin{aligned} \|f(t, x(t), (Gx)(t), (Sx)(t)) - f(t, y(t), (Gy)(t), (Sy)(t))\| \quad &\leq \\ m_1(t)\|x - y\| + m_2(t)\|Gx - Gy\| + m_3(t)\|Sx - Sy\|, \end{aligned} \tag{4}$$

$for\ each\ t \in J,\ and\ all\ x, y \in X.$

- (H3) There exists a constant $\alpha_2 \in (0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, X)$ such that

$$\|f(t, y, (Gy), (Sy)\| \leq h(t),\ for\ each\ t \in J,\ and\ all\ y \in X.$$

For brevity, let $M = \|m_1 + \gamma_0 m_2 + \gamma_1 m_3\|_{L^{\frac{1}{\alpha_1}}(J,X)}$, $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,X)}$.
- (H4) The function $f : J \times X \times X \to X$ is continuous.
- (H5) There exist constants $\lambda \in [0, 1 - \frac{1}{p})$ for some $1 < p < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$\|f(t, u, Gu, Su)\| \leq N(1 + \gamma_0\|u\|^{\lambda} + \gamma_1\|u\|^{\lambda})\ for\ each\ t \in J\ and\ all\ u \in X.$$

- (H6) For every $t \in J$, the set $K = \left\{(t - s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s)) : y \in C(J, X), s \in [0, t])\right\}$ is relatively compact.

Our first result is based on Banach contraction principle.

Theorem 2. *Assume that (H1)–(H3) hold. If*

$$\Omega_{\alpha,T} = \frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\left(1 + \frac{|b|}{|a + b|}\right) < 1, \tag{5}$$

then the system (1)has a unique solution on J.

Proof. For each $t \in J$, we have

$$\begin{aligned} \int_0^t \|(t - s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s))\|\, ds &\leq \left(\int_0^t (t - s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds\right)^{\alpha_2} \\ &\leq \frac{T^{\alpha-\alpha_2} H}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} \end{aligned}$$

Thus, $\|(t-s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $x \in C(J, X)$. Then $(t-s)^{\alpha-1} f(s, y(s), Gy(s), Sy(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to Lemma 5. Hence, the fractional BVP (1) is equivalent to the following fractional integral equation

$$y(t) \quad = \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s), (Gy)(s), (Sy)(s)) ds$$

$$-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s),(Gy)(s),(Sy)(s))ds - c\right], t \in J.$$

Let

$$r \geq \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}.$$

Now we define the operator F on $B_r := \{y \in C(J,X) : \|y\| \leq r\}$ as follows

$$\begin{aligned}(Fy)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s),(Gy)(s),(Sy)(s))ds \\ &- \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s),(Gy)(s),(Sy)(s))ds - c\right], t \in J.\end{aligned}$$

Therefore, the existence of a solution of the fractional BVP (1) is equivalent to that the operator F has a fixed point on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy \in B_r$ for every $y \in B_r$ For every $y \in B_r$ and any $\delta > 0$, by (H3) and Hölder inequality, we get

$$\begin{aligned}
& \|(Fy)(t+\delta) - (Fy)(t)\| \\
\leq & \left\|\frac{1}{\Gamma(\alpha)}\int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] f(s,y(s),(Gy)(s),(Sy)(s))ds\right\| \\
& + \left\|\frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} (t+\delta-s)^{\alpha-1} f(s,y(s),(Gy)(s),(Sy)(s))ds\right\| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^t [(t-s)^{\alpha-1} - (t+\delta-s)^{\alpha-1}] \|f(s,y(s),(Gy)(s),(Sy)(s))\| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \|f(s,y(s),(Gy)(s),(Sy)(s))\| ds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^t [(t-s)^{\alpha-1} - (t+\delta-s)^{\alpha-1}] h(s) ds \\
& + \frac{1}{\Gamma(\alpha)}\int_t^{t+\delta} (t+\delta-s)^{\alpha-1} h(s) ds \\
\leq & \frac{1}{\Gamma(\alpha)}\left(\int_0^t [(t-s)^{\frac{\alpha-1}{1-\alpha_2}} - (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}] ds\right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds\right)^{\alpha_2} \\
& + \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}} ds\right)^{1-\alpha_2} \left(\int_t^{t+\delta} (h(s))^{\frac{1}{\alpha_2}} ds\right)^{\alpha_2} \\
\leq & \frac{H}{\Gamma(\alpha)}\left(\frac{t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}} - \frac{(t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2} + \frac{H}{\Gamma(\alpha)}\left(\frac{\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}}{\frac{\alpha-\alpha_2}{1-\alpha_2}}\right)^{1-\alpha_2}
\end{aligned}$$

$$\leq \frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\left[(t^{\frac{\alpha-\alpha_2}{1-\alpha_2}} - (t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}} + \delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}})^{1-\alpha_2} + \delta^{\alpha-\alpha_2}\right]$$

$$\leq \frac{2H\delta^{\alpha-\alpha_2}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore, F is continuous on J, i.e., $Fy \in C(J,X)$. Moreover, for $y \in B_r$ and all $t \in J$, we get

$$\begin{aligned}
\|(Fy)(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s),(Gy)(s),(Sy)(s))\|ds \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s,y(s),(Gy)(s),(Sy)(s))\|ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds\right)^{1-\alpha_2}\left(\int_0^T (h(s))^{\frac{1}{\alpha_2}}ds\right)^{\alpha_2} + \frac{|c|}{|a+b|} \\
&\leq \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|}\times\frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|} \\
&\leq r,
\end{aligned}$$

which implies that $\|Fy\|_\infty \leq r$. Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$. i.e., $F : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r. For $x, y \in B_r$ and any $t \in J$, using (H2) and Hölder inequality, we get

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),(Gx)(s),(Sx)(s)) - f(s,y(s),(Gy)(s),(Sy)(s))\|ds \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s,x(s),(Gx)(s),(Sx)(s)) - f(s,y(s),(Gy)(s),(Sy)(s))\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m_1(s)\|x(s)-y(s)\| + m_2(s)\|Gx(s)-Gy(s)\| \\
&\quad + m_3(s)\|Sx(s)-Sy(s)\|ds \\
&\quad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}m_1(s)\|x(s)-y(s)\| + m_2(s)\|Gx(s)-Gy(s)\| \\
&\quad + m_3(s)\|Sx(s)-Sy(s)\|ds \\
&\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)]ds \\
&+ \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}[m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)]ds
\end{aligned}$$

$$\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right)^{1-\alpha_1} \left(\int_0^t ([m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)])^{\frac{1}{\alpha_1}} ds\right)^{\alpha_1}$$
$$+ \frac{|b|\ \|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}} ds\right)^{1-\alpha_1} \left(\int_0^T ([m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)])^{\frac{1}{\alpha_1}} ds\right)^{\alpha_1}$$
$$\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|[m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)]\|_{L^{\frac{1}{\alpha_1}}(J,R^+)}$$
$$+ \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|[m_1(s)+\gamma_0 m_2(s)+\gamma_1 m_3(s)]\|_{L^{\frac{1}{\alpha_1}}(J,R^+)}$$
$$\leq \left[\frac{MT^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(1+\frac{|b|}{|a+b|}\right)\right] \|x-y\|_\infty.$$

So we obtain

$$\|Fx-Fy\|_\infty \leq \Omega_{\alpha,T}\|x-y\|_\infty.$$

Thus, F is a contraction due to the condition (5).

By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the fractional BVP (1). □

Our second result is based on the well known Schaefer's fixed point theorem.

Theorem 3. *Assume that (H4)–(H6) hold. Then the fractional BVP (1) has at least one solution on J.*

Proof. Transform the fractional BVP (1) into a fixed point problem. Consider the operator $F : C(J,X) \to C(J,X)$ defined as (6). It is obvious that F is well defined due to (H4). For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,X)$. Then for each $t \in J$, we have

$$\|(Fy_n)(t)-(Fy)(t)\|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_n(s),(Gy)_n(s),(Sy)_n(s)) - f(s,y(s),(Gy)(s),(Sy)(s))\|ds$$
$$+ \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|f(s,y_n(s),(Gy)_n(s),(Sy)_n(s)) - f(s,y(s),(Gy)(s),(Sy)(s))\|ds$$
$$\leq \frac{\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|}{|a+b|}\int_0^T (T-s)^{\alpha-1}ds\right]$$
$$\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_\infty.$$

Since f is continuous, we have

$$\|Fy_n - Fy\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \times \|f(\cdot,y_n(\cdot),(Gy)_n(\cdot),(Sy)_n(\cdot)) - f(\cdot,y(\cdot),(Gy)(\cdot),(Sy)(\cdot))\|_\infty \to 0 \quad \text{as } n\to\infty.$$

Step 2. F maps bounded sets into bounded sets in $C(J,X)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C(J, X) : \|y\|_\infty \leq \eta^*\}$, we have $\|Fy\|_\infty \leq \ell$.

For each $t \in J$, we get

$$\begin{aligned}
\|(Fy)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, y(s), (Gy)(s), (Sy)(s))\| ds \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|f(s, y(s), (Gy)(s), (Sy)(s))\| ds + \frac{|c|}{|a+b|} \\
&\leq \Big[\frac{N(1 + \gamma_0 \|y\|^\lambda + \gamma_1 \|y\|^\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \Big] \\
&+ \Big[\frac{|b| N(1 + \gamma_0 \|y\|^\lambda + \gamma_1 \|y\|^\lambda)}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \Big] + \frac{|c|}{|a+b|} \\
&\leq \Big[\frac{N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \Big] \\
&+ \Big[\frac{|b| N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \Big] + \frac{|c|}{|a+b|} \\
&+ \Big[\frac{N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \Big] \\
&+ \Big[\frac{|b| N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \Big] \\
&\leq \Big[\frac{T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha+1)} + \frac{|b| T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{|a+b|\Gamma(\alpha+1)} \Big] \\
&+ \frac{|c|}{|a+b|} + \Big[\frac{T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha+1)} \Big] \\
&+ \Big[\frac{|b| T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{|a+b|\Gamma(\alpha+1)} \Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|Fy\|_\infty \quad \leq \quad & \Big[\frac{T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \Big] + \frac{|c|}{|a+b|} \\
& + \Big[\frac{T^\alpha N(1 + \gamma_0 (\eta^*)^\lambda + \gamma_1 (\eta^*)^\lambda)}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \Big] := \ell.
\end{aligned}$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J, X)$.

Let $0 \leq t_1 < t_2 \leq T$, $y \in B_{\eta^*}$. Using (H5), we have

$$\begin{aligned}
& \|(Fy)(t_2) - (Fy)(t_1)\| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \|f(s, y(s), (Gy)(s), (Sy)(s))\| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,y(s),(Gy)(s),(Sy)(s))\|ds\\
&\le\Big[\frac{N(1+\gamma_0\|y\|^{\lambda}+\gamma_1\|y\|^{\lambda})}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}](1+\|y(s)\|^{\lambda})ds\Big]\\
&+(\gamma_0+\gamma_1)\Big[\frac{N}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+\|y(s)\|^{\lambda})ds\Big]\\
&\le\Big[\frac{N(1+\gamma_0(\eta^*)^{\lambda}+\gamma_1(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]ds\Big]\\
&+\Big[\frac{N(1+\gamma_0(\eta^*)^{\lambda}+\gamma_1(\eta^*)^{\lambda})}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\Big]\\
&\le\Big[\frac{N(1+\gamma_0(\eta^*)^{\lambda}+\gamma_1(\eta^*)^{\lambda})}{\Gamma(\alpha+1)}\left(|t_1^{\alpha}-t_2^{\alpha}|+2(t_2-t_1)^{\alpha}\right)\Big]\\
&\le(1+\gamma_0(\eta^*)^{\lambda}+\gamma_1(\eta^*)^{\lambda})\Big[\frac{3N(1+\gamma_0(\eta^*)^{\lambda}+\gamma_1(\eta^*)^{\lambda})(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}\Big].
\end{aligned}
$$

As $t_2\to t_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous.

Now, let $\{y_n\}$, $n=1,2,\cdots$ be a sequence on B_{η^*}, and

$$(Fy_n)(t)=(F_1y_n)(t)+(F_2y_n)(T),\ t\in J.$$

where

$$
\begin{aligned}
(F_1y_n)(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y_n(s),(Gy)_n(s),(Sy)_n(s))ds,\ t\in J,\\
(F_2y_n)(T)&=-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s,y_n(s),(Gy)_n(s),(Sy)_n(s))ds-c\right].
\end{aligned}
$$

In view of the condition (H6) and Lemma 6, we know that $\overline{conv}K$ is compact. For any $t^*\in J$,

$$
\begin{aligned}
(F_1y_n)(t^*)&=\frac{1}{\Gamma(\alpha)}\int_0^{t^*}(t^*-s)^{\alpha-1}f(s,y_n(s),(Gy)_n(s),(Sy)_n(s))ds\\
&=\frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}\left(t^*-\frac{it^*}{k}\right)^{\alpha-1}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k}),(Gy_n)(\frac{it^*}{k}),(Sy_n)(\frac{it^*}{k})\right)\\
&=\frac{t^*}{\Gamma(\alpha)}\widetilde{\xi_n},
\end{aligned}
$$

where

$$\widetilde{\xi_n}=\lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^*-\frac{it^*}{k})^{\alpha-1}f\left(\frac{it^*}{k},y_n(\frac{it^*}{k}),(Gy_n)(\frac{it^*}{k}),(Sy_n)(\frac{it^*}{k})\right).$$

Since $\overline{conv}K$ is convex and compact, we know that $\widetilde{\xi_n}\in\overline{conv}K$. Hence, for any $t^*\in J$, the set $\{(F_1y_n)(t^*)\}$ is relatively compact. From Lemma 7, every $\{(F_1y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1y_{n_k})(t)\}$, $k = 1, 2, \cdots$ on J. Thus, the set $\{F_1y : y \in B_{\eta^*}\}$ is relatively compact. Similarly, one can obtain $\{(F_2y_n)(T)\}$ contains a uniformly convergent subsequence $\{(F_2y_{n_k})(T)\}$, $k = 1, 2, \cdots$. Thus, the set $\{F_2y : y \in B_{\eta^*}\}$ is relatively compact. As a result, the set $\{Fy, y \in B_{\eta^*}\}$ is relatively compact.

As a consequence of Step 1–3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{y \in C(J, X) : y = \lambda Fy, \text{ for some } \lambda \in (0, 1)\}$$

is bounded.

Let $y \in E(F)$, then $y = \lambda Fy$ for some $\lambda \in (0, 1)$. Thus, for each $t \in J$, we have

$$\begin{aligned} y(t) =& \lambda \Bigg(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), (Sy)(s)) ds \\ & - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s), (Sy)(s)) ds - c \right] \Bigg). \end{aligned}$$

For each $t \in J$, we have

$$\begin{aligned} \|y(t)\| \quad \le \quad & \frac{NT^\alpha}{\Gamma(\alpha+1)} + \Big[\frac{|b| T^\alpha N(1 + \gamma_0(\eta^*)^\lambda + \gamma_1(\eta^*)^\lambda)}{|a+b|\Gamma(\alpha+1)} \Big] + \frac{|c|}{|a+b|} \\ & + \Big[\frac{N(1 + \gamma_0(\eta^*)^\lambda + \gamma_1(\eta^*)^\lambda)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \Big] \\ & + \Big[\frac{|b| N(1 + \gamma_0(\eta^*)^\lambda + \gamma_1(\eta^*)^\lambda)}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \Big]. \end{aligned}$$

By Lemma 9, there exists a $M^* > 0$ such that

$$\|y(t)\| \le M^*, \ t \in J.$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \le M^*.$$

This shows that the set $E(F)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional BVP (1). □

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