

Some Properties of Complex Uncertain Process

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ABSTRACT

Uncertainty appears not only in real quantities but also in complex quantities. Complex uncertain process is essentially a sequence of complex uncertain variables indexed by time. In order to describe complex uncertain process, a formal definition of complex uncertain distribution is given in this paper, as well as the concepts of independence and variance. In addition, some properties of complex uncertain integral are presented.

Keywords: Complex Uncertain Process, Distribution, Independence, Variance, Integral

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1. INTRODUCTION

In order to describe these non-determinate phenomena in nature, probability theory, fuzzy mathematics and rough set theory have been produced. In 1933, Kolmogorov founded the theory of probability axiomatic theorem. The theory of fuzzy set was established by Zadeh (1965). Rough set theory was presented by Pawlak (1982) for the first time in 1982. In many cases of our real life, classical measure is widely used. Classical measure satisfies nonnegativity, but the measure has no countable additivity. In order to satisfy this demand, capacities was proposed by Choquet in 1954 and fuzzy measures introduced by Sugeno (1974). Although both capacity and fuzzy measures put emphasis on continuity, instead of on self-duality and countable subadditivity, as we all know, self-duality and countable subadditivity are of utmost importance. To define a self-dual measure, the concept of credibility measure was put forward by Liu and Liu (2002). In 2007, Liu (2004) founded uncertainty theory, then it was refined by Liu (2011) which based on an uncertain measure satisfying normality, duality, subadditivity, and product axioms. Thereafter, a lot of explorations were undertaken. For example, You (2009) researched the convergence of uncertain sequences in 2009. For studying dynamic uncertain events, the concept of uncertain process was initiated by Liu (2008).

Up to now, uncertainty theory has been applied to uncertain calculus (Liu, 2009), uncertain differential eq-

uation (Liu, 2008; Yao and Chen, 2013), uncertain logic (Liu, 2011), uncertain programming (Liu, 2009; Liu and Chen, 2015) and uncertain finance (Liu, 2009; Chen, 2011), etc.

Since the 1900s, complex stochastic variable and complex stochastic process have been widely applied in information science, electronic systems and physical fields. Inspired by this, complex uncertain variable was discussed by Peng (2013). Considering the increasing applications of complex numbers and the uncertainty involved in dynamic systems, the properties of complex uncertain process are explored in this paper, including definition and theorems of complex uncertain distribution, independence, expected value, variance and integrals of complex uncertain process.

There are four sections in this paper. In Section 2, we will recall some concepts in uncertainty theory, then some properties of complex uncertain process are discussed in Section 3. A brief summary is given in the last section.

2. PRELIMINARY

In this section, some basic knowledge in uncertainty theory are introduced, which will be used in this paper.

To measure uncertain event, uncertain measure was introduced as a set function satisfying normality axiom,

duality axiom and subadditivity axiom.

After the introduction of uncertain measure, we recall the definition of uncertain space.

Definition 2.1 (Liu, 2007) Let Γ be a nonempty set, \mathcal{L} a σ -algebra over Γ , and \mathcal{M} an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2.2 (Liu, 2007) An uncertain variable ξ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.

Definition 2.3 (Peng, 2013) A complex uncertain variable ζ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers such that $\{\zeta \in B\}$ is an event for any Borel set B of complex numbers.

Definition 2.4 (Peng, 2013) The complex uncertain distribution Φ of a complex uncertain variable ζ is defined by

$$\Phi(z) = \mathcal{M}\{\text{Re}(\zeta) \leq x, \text{Im}(\zeta) \leq y\},$$

for any complex number $z = x + iy$, $x, y \in \mathbb{R}$.

Theorem 2.1 (Peng, 2013) A function $\Phi(c): \mathbb{C} \rightarrow [0, 1]$ is a complex uncertainty distribution of complex uncertain variable if and only if it is a monotone increasing function with respect to the real part and imaginary part of c , respectively, and

- 1) $\lim_{x \rightarrow -\infty} \Phi(x + bi) \neq 1, \lim_{y \rightarrow -\infty} \Phi(a + yi) \neq 1, \forall a, b \in \mathbb{R}$.
- 2) $\lim_{x \rightarrow +\infty, y \rightarrow +\infty} \Phi(x + yi) \neq 0$.

Definition 2.5 (Liu, 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\} dx,$$

provided that at least one of the two integrals is finite.

Definition 2.6 If the real and imaginary parts of complex uncertain process Z_t exist, then the expected value of Z_t is defined by

$$E[Z_t] = E[\text{Re}[Z_t]] + iE[\text{Im}[Z_t]].$$

In order to describe dynamic uncertain systems, uncertain processes are introduced.

Definition 2.7 (Liu, 2008) Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of the real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t .

Since uncertain process is used to describe uncertain phenomena changing over time, it is a series of uncertain variables changing over time.

Remark 2.1 The above definition says X_t is an uncertain process if and only if it is an uncertain variable at each time t .

Definition 2.8 Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let T be an index set. A complex uncertain process is a function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers.

Furthermore, Z_t is a complex uncertain process if and only if there exist two uncertain processes X_t, Y_t such that $Z_t = X_t + iY_t$.

Definition 2.9 (Liu, 2009) An uncertain process C_t is said to be a canonical Liu process if

- (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
- (ii) C_t has stationary and independent increments,
- (iii) every increment $C_{t+s} - C_t$ is a normal uncertain variable with expected value 0 and variance t^2 .

Definition 2.10 If C_{1t}, C_{2t} are independent Liu processes, then $C_t = C_{1t} + iC_{2t}$ is a complex Liu process, where $i^2 = -1$. Especially, complex Liu process C_t is said to be standard if C_{1t}, C_{2t} are canonical Liu processes.

Definition 2.11 (Liu, 2009) Let X_t be an uncertain process and let C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \dots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Theorem 2.2 (Liu, 2011) If X_t is a sample-continuous uncertain process on $[a, b]$, C_t is a canonical Liu process, then X_t is integrable with respect to C_t on $[a, b]$.

Theorem 2.3 If uncertain processes X_t, Y_t are all integrable with respect to complex Liu process C_t , then the complex process Z_t is said to be integrable with respect to C_t , and

$$\int_a^b Z_t dC_t = \left(\int_a^b X_t dC_{1t} - \int_a^b Y_t dC_{2t} \right) + i \left(\int_a^b X_t dC_{2t} + \int_a^b Y_t dC_{1t} \right),$$

where $Z_t = X_t + iY_t$, $C_t = C_{1t} + iC_{2t}$.

3. THE PROPERTIES OF COMPLEX UNCERTAIN PROCESS

In this section some concepts and theorems of complex uncertain process will be discussed.

As uncertainty distribution is an important method to describe uncertain variable, the complex uncertainty distribution of a complex uncertain process is of prime importance.

Definition 3.1 Complex uncertain process Z_t is said to have a complex uncertainty distribution $\Phi_t(z)$ if at each fixed time t , complex uncertain variable Z_t has complex uncertainty distribution $\Phi_t(z)$.

Theorem 3.1 A function $\Phi_t(z): T \times C \mapsto [0, 1]$ is a complex uncertainty distribution of complex uncertain process if and only if at each time t , it is monotone increasing with respect to the real part $Re(z)$ and imaginary part $Im(z)$, respectively, and

- i) $\lim_{x \rightarrow -\infty} \Phi_t(x + bi) \neq 1, \lim_{y \rightarrow -\infty} \Phi_t(a + yi) \neq 1, \forall a, b \in R$.
- ii) $\lim_{x \rightarrow +\infty, y \rightarrow +\infty} \Phi_t(x + yi) \neq 0$.

Proof: Since Z_t is a complex uncertain process if and only if the real part and imaginary part of Z_t are complex uncertain variables at each time t , by Theorem 2.1, the theorem is proved.

Next, we introduce the definition of independence of complex uncertain processes.

Definition 3.2 Complex uncertain processes $Z_{1t}, Z_{2t}, \dots, Z_{nt}$ are said to be independent, if for any Borel sets of complex numbers B_1, B_2, \dots, B_n , we have

$$\mathcal{M} \left\{ \bigcap_{j=1}^n Z_{jt} \in B_j \right\} = \bigwedge_{j=1}^n \mathcal{M} \{ Z_{jt} \in B_j \}.$$

Theorem 3.2 Complex uncertain processes $Z_{1t}, Z_{2t}, \dots, Z_{nt}$ are independent if and only if for any Borel sets of complex numbers B_1, B_2, \dots, B_n , we have

$$\mathcal{M} \left\{ \bigcup_{j=1}^n Z_{jt} \in B_j \right\} = \bigvee_{j=1}^n \mathcal{M} \{ Z_{jt} \in B_j \}.$$

Proof: By the self-duality of uncertain measure, $Z_{1t}, Z_{2t}, \dots, Z_{nt}$ are independent if and only if

$$\mathcal{M} \left\{ \bigcup_{j=1}^n Z_{jt} \in B_j \right\} = 1 - \mathcal{M} \left\{ \bigcap_{j=1}^n Z_{jt} \in B_j^c \right\}$$

$$= 1 - \bigwedge_{j=1}^n \mathcal{M} \{ Z_{jt} \in B_j^c \} \\ = \bigvee_{j=1}^n \mathcal{M} \{ Z_{jt} \in B_j \}.$$

Theorem 3.3 Let $Z_{1t} = X_{1t} + iY_{1t}, Z_{2t} = X_{2t} + iY_{2t}$ be two independent complex uncertain processes, where $X_{1t}, Y_{1t}, X_{2t}, Y_{2t}$ are uncertain processes. Then

- i) X_{1t} is independent with X_{2t} or Y_{2t} .
- ii) X_{2t} is independent with X_{1t} or Y_{1t} .
- iii) Y_{1t} is independent with Y_{2t} .

Proof: For $B_1, B_2 \subseteq R$, we have

$$\{X_{1t} \in B_1\} \cap \{X_{2t} \in B_2\} \\ = \{X_{1t} \in B_1, Y_{1t} \in R\} \cap \{X_{2t} \in B_2, Y_{2t} \in R\} \\ = \{Z_{1t} \in \{x_t + iy_t | x_t \in B_1\}\} \cap \{Z_{2t} \in \{x_t + iy_t | x_t \in B_2\}\}.$$

Since Z_{1t}, Z_{2t} are independent, we get

$$\mathcal{M} \{ \{X_{1t} \in B_1\} \cap \{X_{2t} \in B_2\} \} \\ = \mathcal{M} \{ Z_{1t} \in \{x_t + iy_t | x_t \in B_1\} \} \wedge \mathcal{M} \{ Z_{2t} \in \{x_t + iy_t | x_t \in B_2\} \} \\ = \mathcal{M} \{ X_{1t} \in B_1, Y_{1t} \in R \} \wedge \mathcal{M} \{ X_{2t} \in B_2, Y_{2t} \in R \} \\ = \mathcal{M} \{ X_{1t} \in B_1 \} \wedge \mathcal{M} \{ X_{2t} \in B_2 \}.$$

Thus X_{1t}, X_{2t} are independent. Similarly, we can prove the other conclusions.

Theorem 3.4 Let Z_{1t}, Z_{2t} be independent complex uncertain processes with finite expected values. Then for any complex numbers a, b , we have

$$E[aZ_{1t} + bZ_{2t}] = aE[Z_{1t}] + bE[Z_{2t}],$$

where $Z_{jt} = X_{jt} + iY_{jt}, j = 1, 2$.

Proof: If Z_{1t}, Z_{2t} are independent complex uncertain processes with finite expected values, then

$$Re[Z_{1t} + Z_{2t}] = X_{1t} + X_{2t}, Im[Z_{1t} + Z_{2t}] = Y_{1t} + Y_{2t}.$$

Since X_{1t} is independent with X_{2t} , Y_{1t} is independent with Y_{2t} , we have

$$E[aZ_{1t} + bZ_{2t}] = E[aRe(Z_{1t}) + bRe(Z_{2t})] \\ + iE[aIm(Z_{1t}) + bIm(Z_{2t})] \\ = E[aX_{1t} + bX_{2t}] + iE[aY_{1t} + bY_{2t}] \\ = E[aX_{1t}] + iE[aY_{1t}] + E[bX_{2t}] + iE[bY_{2t}] \\ = aE[X_{1t}] + iaE[Y_{1t}] + bE[X_{2t}] + ibE[Y_{2t}] \\ = aE[Z_{1t}] + bE[Z_{2t}].$$

Definition 3.3 Let Z_t be a complex uncertain process with finite expected value Z . Then the variance of Z_t is defined by

$$V[Z_t] = E[(Z_t - Z)^2].$$

Some useful properties of complex uncertain processes are illustrated as follows.

Theorem 3.5 Let Z_t be a complex uncertain process, f a function from C to R . Then $f(Z_t)$ is an uncertain process.

Proof: If Z_t is a complex process, f is a function from C to R , then $f^{-1}(B)$ is a Borel set of real numbers, i.e.

$$\{Z_t \in f^{-1}(B)\} = \{\gamma \in \Gamma | Z_t \in f^{-1}(B)\}.$$

Then we have

$$\{f(Z_t) \in B\} = \{\gamma \in \Gamma | f(Z_t) \in B\}.$$

Theorem 3.6 Let $Z_{1t}, Z_{2t}, \dots, Z_{nt}$ be complex processes, f a function from C^n to C . Then $Z_t = f(Z_{1t}, Z_{2t}, \dots, Z_{nt})$ is a complex uncertain process.

Proof: If $Z_{1t}, Z_{2t}, \dots, Z_{nt}$ are complex processes, f is a function from C^n to C , then for any Borel set B of complex numbers, $f^{-1}(B)$ is a Borel set of C^n , we get

$$\{(Z_{1t}, Z_{2t}, \dots, Z_{nt}) \in f^{-1}(B)\} = \{f(Z_{1t}, Z_{2t}, \dots, Z_{nt}) \in B\},$$

then $f(Z_{1t}, Z_{2t}, \dots, Z_{nt})$ is a complex uncertain process.

Theorem 3.7 If X_t is a monotone bounded uncertain process on $[a, b]$, and C_t is a canonical Liu process, then $\int_a^b X_t dC_t$ exists.

Proof: Let uncertain process X_t be increasing on $[a, b]$ with respect to t . Thus $X_b > X_a$. For any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{X_b - X_a}$. Then partition closed interval $[a, b]$, such that the mesh

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i| < \varepsilon.$$

Since almost all sample paths of C_t are Lipschitz continuous with respect to t , we have $|\Delta C_t| < \frac{\varepsilon}{X_b - X_a}$. Let M_i , m_i be the supremum and infimum of X_t on $[t_i, t_{i+1}]$, respectively. Since X_t is increasing on $[a, b]$ with respect to t , then $M_i = X_{t_{i+1}}$, $m_i = X_{t_i}$, thus

$$\begin{aligned} \sum_{i=1}^k |\Delta C_{t_i}| \cdot (M_i - m_i) &< \frac{\varepsilon}{X_b - X_a} \cdot \sum_{i=1}^k (X_{t_{i+1}} - X_{t_i}) \\ &= \frac{\varepsilon}{X_b - X_a} (X_b - X_a) = \varepsilon. \end{aligned}$$

In a similar proof of Theorem 2.2, the theorem is proved.

Theorem 3.8 If X_t, Y_t are monotone bounded uncertain processes on $[a, b]$, then $\int_a^b X_t Y_t dC_t$ exists.

Proof: Let uncertain processes X_t, Y_t be increasing on $[a, b]$ with respect to t . For any $\varepsilon_1 > 0$, there is $\delta = \frac{\varepsilon_1}{X_b - X_a}$. Then partition $[a, b]$ into T_1 parts. Since almost all sample paths of C_t are Lipschitz continuous with respect to time t , we have $|\Delta C_t| < \frac{\varepsilon_1}{X_b - X_a}$. Let M^X be the supremum of X_t and m^Y be the infimum of Y_t on $[a, b]$. Let M_i^X, m_i^X be the supremum and infimum of X_t and M_i^Y, m_i^Y be the supremum and infimum of Y_t on $[t_i, t_{i+1}]$, respectively. Since X_t is increasing on $[a, b]$ with respect to t , then $M_i^X = X_{t_{i+1}}$, $m_i^X = X_{t_i}$, thus

$$\begin{aligned} \sum_{i=1}^{k_1} |\Delta C_{t_i}| \cdot (M_i^X - m_i^X) &< \frac{\varepsilon_1}{X_b - X_a} \cdot \sum_{i=1}^{k_1} (M_i^X - m_i^X) \\ &= \frac{\varepsilon_1}{X_b - X_a} (X_b - X_a) = \varepsilon_1. \end{aligned}$$

Similarly, for the partition T_2 of Y_t , we have

$$\begin{aligned} \sum_{i=1}^{k_2} |\Delta C_{t_i}| \cdot (M_i^Y - m_i^Y) &< \frac{\varepsilon_2}{Y_b - Y_a} \cdot \sum_{i=1}^{k_2} (M_i^Y - m_i^Y) \\ &= \frac{\varepsilon_2}{Y_b - Y_a} (Y_b - Y_a) = \varepsilon_2. \end{aligned}$$

Let $T = T_1 + T_2$, and $\varepsilon_1 = \varepsilon/2m^Y$, $\varepsilon_2 = \varepsilon/2M^X$. We have

$$\begin{aligned} &\sum_{i=1}^k |\Delta C_{t_i}| \cdot (M_i^X M_i^Y - m_i^X m_i^Y) \\ &= \sum_{i=1}^k |\Delta C_{t_i}| \cdot (M_i^X M_i^Y - M_i^X m_i^Y + M_i^X m_i^Y - m_i^X m_i^Y) \\ &= \sum_{i=1}^k |\Delta C_{t_i}| \cdot [M_i^X (M_i^Y - m_i^Y) + (M_i^X - m_i^X) m_i^Y] \\ &\leq M^X \sum_{i=1}^k |\Delta C_{t_i}| \cdot (M_i^Y - m_i^Y) + m^Y \sum_{i=1}^k |\Delta C_{t_i}| \cdot (M_i^X - m_i^X) \\ &< M^X \varepsilon_2 + m^Y \varepsilon_1 \\ &= \varepsilon, \end{aligned}$$

where $k = k_1 + k_2$.

The theorem is verified.

Theorem 3.9 If $Z_t = X_t + iY_t$ is a complex uncertain process, where X_t, Y_t are all monotone bounded uncertain processes, $C_t = C_{1t} + iC_{2t}$ is a complex uncertain Liu process, then Liu integral $\int_a^b Z_t dC_t$ exists.

Proof: Since

$$\int_a^b Z_t dC_t = \int_a^b X_t dC_{1t} + i \int_a^b X_t dC_{2t} + i \int_a^b Y_t dC_{1t} - \int_a^b Y_t dC_{2t},$$

X_t, Y_t are all monotone bounded uncertain processes. According to Theorem 3.7, $\int_a^b X_t dC_{1t}, \int_a^b X_t dC_{2t}, \int_a^b Y_t dC_{1t}$ and $\int_a^b Y_t dC_{2t}$ are finite, thus Liu integral $\int_a^b Z_t dC_t$ exists.

Theorem 3.10 If $Z_{1t} = X_{1t} + iY_{1t}$ and $Z_{2t} = X_{2t} + iY_{2t}$ are Liu integrable complex uncertain processes, where $X_{1t}, Y_{1t}, X_{2t}, Y_{2t}$ are all monotone bounded uncertain processes, $C_t = C_{1t} + iC_{2t}$ is a complex uncertain Liu process, then $\int_a^b Z_{1t} Z_{2t} dC_t$ exists.

Proof: It follows Theorem 3.8 that

$$\begin{aligned} \int_a^b Z_{1t} Z_{2t} dC_t &= \int_a^b ((X_{1t} X_{2t} - Y_{1t} Y_{2t}) + i(X_{1t} Y_{2t} + Y_{1t} X_{2t})) dC_t \\ &= \int_a^b (X_{1t} X_{2t} - Y_{1t} Y_{2t}) dC_{1t} - \int_a^b (X_{1t} Y_{2t} + Y_{1t} X_{2t}) dC_{2t} \\ &\quad + i \int_a^b (X_{1t} X_{2t} - Y_{1t} Y_{2t}) dC_{2t} + i \int_a^b (X_{1t} Y_{2t} + Y_{1t} X_{2t}) dC_{1t} \\ &= \int_a^b X_{1t} X_{2t} dC_{1t} - \int_a^b Y_{1t} Y_{2t} dC_{1t} - \int_a^b X_{1t} Y_{2t} dC_{2t} - \int_a^b Y_{1t} X_{2t} dC_{2t} \\ &\quad + i \int_a^b X_{1t} X_{2t} dC_{2t} - i \int_a^b Y_{1t} Y_{2t} dC_{2t} + i \int_a^b X_{1t} Y_{2t} dC_{1t} + \int_a^b Y_{1t} X_{2t} dC_{1t}. \end{aligned}$$

The above integrals are all exist, then $\int_a^b Z_{1t} Z_{2t} dC_t$ exists.

4. CONCLUSIONS

In this paper, distribution, independence and variance of complex uncertain process were defined which broadened the scope of uncertain process. Based on the results of uncertain theory, sufficient and necessary condition of complex uncertain distribution, some theorems about independence, expected value and integrals of complex uncertain process were deduced. These theorems provided theoretical basis for complex uncertain process and promoted the developments of uncertainty theory in complex field.

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