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ARITHMETIC AVERAGE ASIAN OPTIONS WITH STOCHASTIC ELASTICITY OF VARIANCE

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ABSTRACT. This article deals with the pricing of Asian options under a constant elasticity of variance (CEV) model as well as a stochastic elasticity of variance (SEV) model. The CEV and SEV models are underlying asset price models proposed to overcome shortcomings of the constant volatility model. In particular, the SEV model is attractive because it can characterize the feature of volatility in risky situation such as the global financial crisis both quantitatively and qualitatively. We use an asymptotic expansion method to approximate the no-arbitrage price of an arithmetic average Asian option under both CEV and SEV models. Subsequently, the zero and non-zero constant leverage effects as well as stochastic leverage effects are compared with each other. Lastly, we investigate the SEV correction effects to the CEV model for the price of Asian options.

1. INTRODUCTION

Since the seminal achievement (Black & Scholes [1]) of Black and Scholes on European vanilla options, pricing methods for a lot of complex exotic options also have been developed. This study particularly concerns with the pricing of Asian options among those exotic options. Asian options have a payoff frame more complex than the original European vanilla options because Asian options have a strongly path-dependent feature. The name of Asian is known to originate from the fact that two founders of the first pricing formula used to belong to Asia (Tokyo, Japan) (cf. Wilmott [2]). Due to their averaging property, Asian options could diminish the risk of financial market manipulation of underlying risky assets at the expiration day.

Like many other exotic derivatives, there is no analytic closed form formula for the price of Asian options. So, many researchers have been devoted to develop how to price these options approximately or numerically. For example, Geman and Yor [3], and Linetsky [4] used the

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approximation method. Kemna and Vorst [5] used the Monte Carlo method. Ingersoll [6], Rogers and Shi [7], and Večeř [8] used the partial differential equation (PDE) method.

On the other hand, the Black−Scholes model (Black & Scholes [1]) corresponding to constant volatility has been extended to fulfill requirements given by practical market phenomena including volatility skew/smile, fat-tailed and asymmetricity of returns distributions, mean-reversion of volatility and etc. For instance, constant elasticity of variance (CEV) model suggested by Cox [9], stochastic volatility models by Heston [10] or Fouque et al. [11], and a Levy model by Carr et al. [12] are among those popular ones. So, it is desirable to price Asian options also based upon these advanced models. In fact, there are quite a number of studies published along the lines of the extension. For example, there are Peng and Peng [13] for the CEV model, Fouque and Han [14] for a mean-reverting stochastic volatility model, and Lemmens et al. [15] for a Levy type model.

This paper chooses not only the CEV model but also the stochastic elasticity of variance (SEV) model of Kim et al. [16] and compares the models by studying the price of an arithmetic average Asian option. In the SEV model, stochastic leverage effect is a important feature of the SEV model. It's shown that the SEV model gives results of desirable correction to the price under the Black−Scholes model in regard to both dynamics and geometry of the resultant implied volatilities. The model has been utilized to study a perpetual American option (Yoon et al. [17]) and an asset allocation problem (Yang et al. [18]). Further, the model can characterize the feature of volatility during the peak period of the 2007-2008 global financial crisis both quantitatively and qualitatively as shown in Kim et al. [19]. So, it would be quite interesting to derive the value of the Asian option under this model.

This paper is organized as follows. In Section 2, we use the CEV model to characterize the price of an arithmetic average Asian option. Section 3 is devoted to use the SEV model to obtain a dynamic law of the price of the Asian option. In Section 4, we obtain the option price in expanded form. Section 5 investigates the implications of the solution. Section 6 provides concluding remarks.

2. Pricing under CEV model

In this section, we use a PDE method for the price of an Asian floating-strike option based on the CEV model. In fact, Peng and Peng [13] have already studied the pricing of arithmetic Asian options under the CEV model. They used a binomial tree method. In this paper, however, we utilize Večeř's dimension reduction skill given by Večeř [8] and some details in Fouque and Han [14] to derive the corresponding option price.

Under a risk-neutral measure P^*, the price S_t of an underlying asset at time t follows the CEV model following stochastic differential equation (SDE) given by

$$dS_t = rS_t dt + \sigma S_t^{\frac{\theta}{2}} dW_t^*, \quad t < T, \tag{2.1}$$

where r is an interest rate, σ is a volatility coefficient, θ is an elasticity parameter, W_t^* is a Brownian motion. In this paper, we truncate smoothly the value of S_t in such a way that the resultant price is bounded and bounded away from zero in the interval $[0,T]$ almost surely. However, we are going to use the same notation S_t for the resultant price still.

A payoff function for arithmetic average Asian options is defined by

$$h\left(\frac{1}{T}\int_0^T S_t dt - K_1 S_T - K_2\right) \tag{2.2}$$

for some constants K_1 and K_2. Here, h is a homogeneous function with the property

$$h(\alpha x) = \alpha h(x). \tag{2.3}$$

Here, if $K_1 = 0$, (2.2) becomes a payoff for fixed strike Asian options, while, if $K_2 = 0$, it is a payoff for floating strike Asian options.

Now, we define the Asian option price at $t = 0$ by

$$P(0,s;T,K_1,K_2) = \mathbb{E}^*\left[e^{-rT} h\left(\frac{1}{T}\int_0^T S_u du - K_1 S_T - K_2\right) | S_0 = s\right],$$

where $\mathbb{E}^*$ means the expectation under the risk-neutral measure P^*.

As a preliminary, we compose a portfolio (α_t, β_t) whose value is given by

$$X_t = \alpha_t S_t + \beta_t e^{rt} \tag{2.4}$$

to replicate an averaged process $\frac{1}{t}\int_0^t S_u du$, where α_t and β_t are numbers of the risky asset and risk-free asset at time t so that they are to be determined later. Here, α_t is assumed to be a non-random function. Applying the self-financing strategy, we obtain that from (2.1) and (2.4)

$$\begin{aligned} dX_t &= \alpha_t dS_t + \beta_t d(e^{rt}) \\ &= \alpha_t dS_t + r(X_t - \alpha_t S_t)dt \\ &= rX_t dt + \alpha_t(dS_t - rS_t dt) \\ &= rX_t dt + \alpha_t \sigma S_t^{\frac{\theta}{2}} dW_t^*. \end{aligned} \tag{2.5}$$

Since the function α_t is a non-randomness, we can obtain

$$d(e^{r(T-t)}\alpha_t S_t) = e^{r(T-t)}\alpha_t(dS_t - rS_t dt) + e^{r(T-t)} S_t d\alpha_t \tag{2.6}$$

Then, using equation (2.5) and (2.6), one can have

$$\begin{aligned} d(e^{r(T-t)}X_t) &= -re^{r(T-t)}X_t dt + e^{r(T-t)} dX_t \\ &= -re^{r(T-t)}X_t dt + e^{r(T-t)}(rX_t dt + \alpha_t(dS_t - rS_t dt)) \\ &= e^{r(T-t)}\alpha_t(dS_t - rS_t dt) \\ &= d(e^{r(T-t)}\alpha_t S_t) - e^{r(T-t)} S_t d\alpha_t \end{aligned} \tag{2.7}$$

Thus, by integrating equation (2.7), one can obtain

$$X_T = e^{rT}X_0 + \alpha_T S_T - \alpha_0 e^{rT} S_0 - \int_0^T e^{r(T-t)} S_t d\alpha_t$$

If we choose the trading strategy α_t and the portfolio value X_0 at $t=0$ as

$$\alpha_t = \frac{1-e^{-r(T-t)}}{rT}, \quad X_0 = x = \frac{1-e^{-rT}}{rT}S_0 - K_2 e^{-rT},$$

respectively, then the portfolio value X_T at $t=T$ becomes $\frac{1}{T}\int_0^T S_t dt - K_2$ and so the payoff function (2.2) becomes $h(X_T - K_1 S_T)$. Refer to Večeř [8], and Fouque and Han [14]. Thus the price at $t=0$ can be represented by

$$P(0,s;T,K_1,K_2) \quad = \quad \mathbb{E}^* \left[e^{-rT} h(X_T - K_1 S_T) \mid S_0 = s \right].$$

with the portfolio process X_t.

Now, we obtain a dynamic law of the arithmetic Asian option price in a PDE form as shown in the next theorem.

Theorem 2.1. *By the change of numeraire $\psi_t := \frac{X_t}{S_t}$, the arithmetic Asian option price at $t=0$ under the CEV model (1) is given by*

$$P(0,s;T,K_1,K_2) = su(0,\psi;T,K_1,K_2),$$

where $\psi = \frac{x}{s} = \frac{1-e^{-rT}}{rT} - \frac{K_2}{s}e^{-rT}$ and $u(t,\psi;T,K_1,K_2)$ is the solution of the PDE

$$u_t + \frac{1}{2}(\psi - \alpha_t)^2 \sigma^2 s^{\theta-2} u_{\psi\psi} = 0$$

with the final condition $u(T,\psi;T,K_1,K_2) = h(\psi - K_1)$. Further, the price at arbitrary $t>0$ satisfies

$$P(t,s;T,K_1,K_2) = \frac{T-t}{T} su(0,\psi;T,K_1,K_2).$$

Proof. From the Itô formula and the SDE (2.1), we obtain the SDEs

$$d(S_t^{-1}) = S_t^{-1}\left[(\sigma^2 S_t^{\theta-2} - r)dt - \sigma S_t^{\frac{\theta}{2}-1} dW_t^*\right], \quad dX_t = S_t\left[r\psi_t dt + \alpha_t \sigma S_t^{\frac{\theta}{2}-1} dW_t^*\right].$$

Then, using the Itô product rule, we have

$$\begin{aligned}
d\psi_t &= X_t d(S_t^{-1}) + S_t^{-1} dX_t + dX_t d(S_t^{-1}) \\
&= \psi_t \left[(\sigma^2 S^{\theta-2} - r)dt - \sigma S_t^{\frac{\theta}{2}-1} dW_t^*\right] + r\psi_t dt + \alpha_t \sigma S_t^{\frac{\theta}{2}-1} dW_t^* - \alpha_t \sigma^2 S_t^{\theta-2} dt \\
&= \psi_t \left[\sigma^2 S_t^{\theta-2} dt - \sigma S_t^{\frac{\theta}{2}-1} dW_t^*\right] - \alpha_t \left[\sigma S_t^{\theta-2} dt - \sigma S_t^{\frac{\theta}{2}-1} dW_t^*\right] \qquad (2.8) \\
&= \sigma S^{\frac{\theta}{2}-1}(\alpha_t - \psi_t)(dW_t - \sigma S_t^{\frac{\theta}{2}-1} dt) \\
&= \sigma S_t^{\frac{\theta}{2}-1}(\alpha_t - \psi_t) d\tilde{W}_t^*,
\end{aligned}$$

where $\tilde{W}_t^*$ is given by $d\tilde{W}_t^* = dW_t^* - \sigma S_t^{\frac{\theta}{2}-1}dt$. From the Girsanov theorem (cf. Oksendal [20]), we change the probability measure P^* into a measure $\tilde{P}^*$ through

$$\frac{d\tilde{P}^*}{dP^*} = e^{-rT}\frac{S_T}{S_0} = \exp\left[\int_0^T \sigma S_t^{\frac{\theta}{2}-1}d\tilde{W}_t^* - \frac{1}{2}\int_0^T \sigma^2 S_t^{\theta-2}dt\right]. \tag{2.9}$$

Then, from (2.3) and (2.9), we have

$$\begin{aligned} P(0,s;T,K_1,K_2) &= \mathbb{E}^*\left[e^{-rT}h(X_T - K_1 S_T) \mid S_0 = s\right] \\ &= s\mathbb{E}^*\left[e^{-rT}\frac{S_T}{S_0}h(\psi_T - K_1) \mid \psi_0 = \psi\right] \\ &= s\tilde{\mathbb{E}}\left[h(\psi_T - K_1) \mid \psi_0 = \psi\right], \end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes expectation with respect to the probability measure $\tilde{P}^*$.

Now, we introduce a function u defined by

$$u(t,\psi;T,K_1,K_2) = \tilde{\mathbb{E}}[h(\phi_T - K_1) \mid \psi_t = \psi].$$

Then the option price $P(0,s;T,K_1,K_2)$ at $t=0$ satisfies $P(0,s;T,K_1,K_2) = su(0,\psi;T,K_1,K_2)$. Here, from the Feynman-Kac formula (cf. Oksendal [20]) and (2.8), $u(t,\psi;T,K_1,K_2)$ satisfies the PDE

$$u_t + \frac{1}{2}(\psi - \alpha_t)^2\sigma^2 s^{\theta-2}u_{\psi\psi} = 0$$

with a final condition given by $u(T,\psi;T,K_1,K_2) = h(\psi - K_1)$. If the solution u is substituted into $P(0,s;T,K_1,K_2) = su(0,\psi;T,K_1,K_2)$, then the option price $P(t,s;T,K_1,K_2)$ at any time t can be immediately determined by the aid of the following identity whose derivation can be found at Fouque and Han [14].

$$P(t,s;T,K_1,K_2) = \frac{T-t}{T}P(0,s;T,K_1,K_2). \tag{2.10}$$

□

3. Pricing under SEV model

In this section, the constant elasticity of variance is randomized and subsequently a dynamic law of the price of Asian floating-strike option is obtained.

3.1. **SEV Formulation.** In the stochastic elasticity of variance (SEV) model of Kim et al. [19], the elasticity of variance is assumed by a stochastic process in such a way that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t^{\gamma_t}dW_t^*, \\ \gamma_t &= \frac{\theta}{2} + \sqrt{\epsilon}f(Y_t), \\ dY_t &= \left[\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t)\right]dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}d\hat{Z}_t^* \end{aligned} \tag{3.1}$$

under a risk-neutral probability measure P^*, where r, σ, θ, m and ν are positive constants, and W_t^* and $\hat{Z}_t^*$ are Brownian motions correlated by $d\langle W_t^*, \hat{Z}_t^*\rangle = \rho dt$, and Λ represents the market price of elasticity risk. When ρ is positive, Y_t may explode to infinite, and so S_t may fail to be a true martingale. Refer to Andersen & Piterbarg [24]. Also, based on the analysis of the most financial data excluding some commodity markets, it's observed that the correlation between stock price and its volatility is negative. So, ρ is assumed to be negative. Furthermore, it is considered as a constant for simplicity. For avoiding the non-existence of moments of S_t, we assume that the function f have the condition that $0 < c_1 \le f \le c_2 < \infty$ for some constants c_1 and c_2.

As the Ornstein-Uhlenbeck process(OU) Y_t is an ergodic process with an invariant distribution given by $N(m, \nu^2)$, for the later sections, we denote the average $\langle \cdot \rangle$ with respect to this invariant distribution by

$$\langle g \rangle = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^{+\infty} g(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy,$$

for any function g.

Now, we suppose that $0 < \epsilon \ll 1$ so that the process Y_t reverts fast a mean. The introduction of this process to finance by Fouque et al. [11] was motivated by an empirical analysis of financial data (example, S&P 500 index). The process gives a quite useful analytic tool to deal with the problems related to the valuation of financial derivatives. Refer to Fouque et al. [21]. Kim et al. [16] used it as a process for the elasticity of variance.

3.2. **Price Dynamics.** In this section, we extend Večeř's dimension reduction technique introduced by Fouque and Han [14] to obtain the option price in the form of two space dimensional PDE. Since the option price under the SEV model depends on the stochastic processes S_t and Y_t (differently from the CEV case), the payoff function for the option has a generalized form given by

$$P(0, s, y; T, K_1, K_2) = \mathbb{E}^* \left[e^{-r(T-t)} h\left(\frac{1}{T}\int_0^T S_u du - K_1 S_T - K_2 \right) | S_0 = s, Y_0 = y \right] \quad (3.2)$$

under a risk-neutral measure P^*, where h has the condition (2.3).

Similarly with the previous section, we want to replicate an averaged process $\frac{1}{t}\int_0^t S_u du$ with a portfolio $X_t = \alpha_t S_t + \beta_t e^{rt}$. Here, α_t is also supposed to be a non-random function. Then applying the self-financing strategy to X_t yields

$$dX_t \quad = \quad rX_t dt + \alpha_t \sigma S_t^{\frac{\theta}{2} + \sqrt{\epsilon} f(Y_t)} dW_t^*. \quad (3.3)$$

Let us choose α_t and X_0 as $\alpha_t = \frac{1 - e^{-r(T-t)}}{rT}$ and $X_0 = x = \alpha_0 S_0 + e^{-rT} K_2$, respectively. Then $X_T = \frac{1}{T}\int_0^T S_t dt - K_2$, and so the payoff function (3.2) becomes

$$P(0, s, y; T, K_1, K_2) \quad = \quad \mathbb{E}^* \left[e^{-rT} h(X_T - K_1 S_T) \mid S_0 = s, Y_0 = y \right] \quad (3.4)$$

at $t = 0$.

We define a probability measure $\tilde{P}^*$ by

$$\frac{d\tilde{P}^*}{dP^*} = e^{-rT}\frac{S_T}{S_0} = \exp\left[\int_0^T \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)} d\tilde{W}_t^* - \frac{1}{2}\int_0^T \sigma^2 S_t^{\theta-2+2\sqrt{\epsilon}f(Y_t)} dt\right]. \quad (3.5)$$

Now, we derive the option price as follows:

Theorem 3.1. *By the change of numeraire $\psi_t := \frac{X_t}{S_t}$, the arithmetic Asian option price at $t = 0$ under the SEV model is given by*

$$P(0, s, y; T, K_1, K_2) = su(0, \psi, y; T, K_1, K_2),$$

where $\psi = \frac{x}{s} = \frac{1-e^{-rT}}{rT} - \frac{K_2}{s}e^{-rT}$ and $u(t, \psi, y; T, K_1, K_2)$ sastisfies the PDE

$$u_t + \frac{1}{2}(\psi-\alpha_t)^2\sigma^2 s^{\theta-2+2\sqrt{\epsilon}f(y)}u_{\psi\psi} + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}(\alpha_t-\psi)\sigma s^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(y)}u_{\psi y}$$
$$+\left(\frac{1}{\epsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\Lambda(y) - \rho\sigma s^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(y)})\right)u_y + \frac{\nu^2}{\epsilon}u_{yy} = 0$$

with the final condition $u(T, \psi, y; T, K_1, K_2) = h(\psi - K_1)$. Furthermore, the price at arbitrary $t > 0$ sastisfies

$$P(t, s, y; T, K_1, K_2) = \frac{T-t}{T}su(0, s, y; T, K_1, K_2).$$

Proof. By the Itô formula and (3.3), we have the SDEs

$$d(S_t^{-1}) = S_t^{-1}\left[(\sigma^2 S_t^{\theta-2+2\sqrt{\epsilon}f(Y_t)} - r)dt - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^*\right],$$
$$dX_t = S_t\left[r\psi_t dt + \alpha_t\sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^*\right].$$

Then, we obtain

$$\begin{aligned}
d\psi_t &= X_t d(S_t^{-1}) + S_t^{-1}dX_t + dX_t d(S_t^{-1}) \\
&= \psi_t\left[(\sigma S^{\theta-2+2\sqrt{\epsilon}f(Y_t)} - r)dt - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^*\right] \\
&\qquad + r\psi_t dt + \alpha_t\sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^* - \alpha_t\sigma^2(Y_t)S_t^{\theta-2+2\sqrt{\epsilon}f(Y_t)}dt \\
&= \psi_t\left[\sigma S_t^{\theta-2+2\sqrt{\epsilon}f(Y_t)}dt - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^*\right] \\
&\qquad - \alpha_t\left[\sigma S_t^{\theta-2+2\sqrt{\epsilon}f(Y_t)}dt - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dW_t^*\right] \\
&= \sigma S^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}(\alpha_t-\psi_t)(dW_t^* - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dt) \\
&= \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}(\alpha_t-\psi_t)d\tilde{W}_t^*,
\end{aligned} \quad (3.6)$$

where $\tilde{W}_t^*$ is given by $d\tilde{W}_t^* = dW_t^* - \sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}dt$ and it is a Brownian motion under the measure $\tilde{P}^*$. Since $d\langle W_t^*, \hat{Z}_t^*\rangle = \rho dt$, $\hat{Z}_t^* = \rho W_t^* + \sqrt{1-\rho^2}Z_t^*$ for some independent Brownian motion Z_t^* with W_t^*. Then the process Y_t satisfies the SDE

$$\begin{aligned} dY_t = &\left[\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\left(\Lambda(Y_t) - \rho\sigma S_t^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(Y_t)}\right)\right]dt \\ &+ \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\rho d\tilde{W}_t^* + \sqrt{1-\rho^2}dZ_t^*). \end{aligned} \tag{3.7}$$

From (2.3), (3.4) and (3.5), we have

$$\begin{aligned} P(0, s, y; T, K_1, K_2) &= \mathbb{E}^*\left[e^{-rT}h(X_T - K_1 S_T) \mid S_0 = s, Y_0 = y\right] \\ &= s\mathbb{E}^*\left[e^{-rT}\frac{S_T}{S_0}h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y\right] \\ &= s\tilde{\mathbb{E}}^*\left[h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y\right]. \end{aligned}$$

If we define a function u by

$$u(t, \psi, y; T, K_1, K_2) := \tilde{\mathbb{E}}^*[h(\psi_T - K_1) \mid \psi_t = \psi, Y_t = y]. \tag{3.8}$$

Then the option price at $t = 0$ can be represented by $P(0, s, y; T, K_1, K_2) = su(0, \psi, y; T, K_1, K_2)$.

Applying the Feynman-Kac formula to (3.6) and (3.7), we obtain a PDE for u of (3.8) as given in Theorem 3.1. Furthermore, the option price at $t > 0$ satisfies

$$P(t, s, y; T, K_1, K_2) = \frac{T-t}{T}P(0, s, y; T, K_1, K_2) = \frac{T-t}{T}su(0, s, y; T, K_1, K_2)$$

by the work of Fouque and Han [14].

□

4. Asymptotic Expansion

It is difficult to solve the PDE obtained by Theorem 3.1. In this section, we use an asymptotic expansion method to obtain PDEs whose numerical solutions can be computed easily. So, we suppose the solution of the form $u = \sum_{i=0}^{\infty}\epsilon^{\frac{i}{2}}u_i$ for solveing the PDE problem in Theorem 3.1.

First, the PDE problem for u can be rewritten as

$$\begin{aligned} &u_t + \frac{1}{2}(\psi - \alpha_t)^2\sigma^2 s^{\theta-2+2\sqrt{\epsilon}f(y)}u_{\psi\psi} \\ &+ \frac{1}{\sqrt{\epsilon}}\left(\rho\nu\sqrt{2}(\alpha_t - \psi)\sigma s^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(y)}u_{\psi y} - \nu\sqrt{2}(\Lambda(y) - \rho\sigma s^{\frac{1}{2}(\theta-2)+\sqrt{\epsilon}f(y)})u_y\right) \\ &+ \frac{1}{\epsilon}\left((m-y)u_y + \nu^2 u_{yy}\right) = 0 \end{aligned} \tag{4.1}$$

with the final condition $u(T,\psi,y;T,K_1,K_2) = h(\psi - K_1)$. Applying the Taylor series expansion (cf. regular perturbation) to the PDE (4.1), we obtain

$$\left(\frac{1}{\epsilon}\mathcal{L}_{00} + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{10} + (\mathcal{L}_{11}+\mathcal{L}_{20}) + \sqrt{\epsilon}(\mathcal{L}_{12}+\mathcal{L}_{21}) + \epsilon(\mathcal{L}_{13}+\mathcal{L}_{22})\right)u(t,\psi,y) = 0, \quad (4.2)$$

where

$$\begin{aligned}
&\mathcal{L}_{00} := (m-y)\partial_y + \nu^2\partial_{yy}\\
&\mathcal{L}_{1i} := \rho\sigma\nu\sqrt{2}s^{\frac{1}{2}(\theta-2)}\frac{(\log sf(y))^i}{i!}((\alpha_t-\psi)\partial_{\psi y} - \partial_y) - \delta_i\nu\sqrt{2}\Lambda(y)\partial_y, \quad i = 0,1,2,\cdots\\
&\mathcal{L}_{20} := \partial_t + \frac{1}{2}(\psi-\alpha_t)^2\sigma^2 s^{\theta-2}\partial_{\psi\psi}\\
&\mathcal{L}_{2i} := \frac{1}{2}(\psi-\alpha_t)^2\sigma^2 s^{\theta-2}\frac{(2\log sf(y))^i}{i!}\partial_{\psi\psi},\ i = 1,2,\cdots
\end{aligned}$$

where $\delta_0 = 1$ and $\delta_i = 0$ for $i = 1,2,...$ Putting the expansion $u = \sum_{i=0}^{\infty}\epsilon^{\frac{i}{2}}u_i$ into the PDE (4.2) and the final condition, we obtain

$$\begin{aligned}
&\frac{1}{\epsilon}\mathcal{L}_{00}u_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_{00}u_1 + \mathcal{L}_{10}u_0) + (\mathcal{L}_{00}u_2 + \mathcal{L}_{10}u_1 + \mathcal{L}_{11}u_0 + \mathcal{L}_{20}u_0)\\
&\quad + \sqrt{\epsilon}(\mathcal{L}_{00}u_3 + \mathcal{L}_{10}u_2 + \mathcal{L}_{11}u_1 + \mathcal{L}_{20}u_1 + \mathcal{L}_{12}u_0 + \mathcal{L}_{21}u_0) = 0
\end{aligned} \quad (4.3)$$

with the final condition $\sum_{i=0}^{\infty}\epsilon^{\frac{i}{2}}u_i(T,\psi,y;T,K_1,K_2) = h(\psi - K_1)$.

Now, we obtain desirable PDEs for u_0 and u_1.

Theorem 4.1. *Suppose that u_i, $(i = 0,1,2,...)$, does not increase as much as $u_i \sim e^{\frac{y^2}{2}}$ as y goes to infinity. Then u_0 is a y-independent function and satisfies the PDE*

$$\begin{aligned}
\mathcal{L}_{20}u_0(t,\psi;T,K_1,K_2) &= 0, \quad t < T,\\
u_0(T,\psi;T,K_1,K_2) &= h(\psi - K_1).
\end{aligned}$$

Proof. The $\frac{1}{\epsilon}$ term of (4.3) yields the following equation

$$\mathcal{L}_{00}u_0 = 0.$$

The assumed growth condition on u_0 leads that the solution u_0 of this ODE doesn't depend on y, i.e., $u_0 = u_0(t,\psi)$. The $\frac{1}{\sqrt{\epsilon}}$ term of (4.3) gives

$$\mathcal{L}_{00}u_1 + \mathcal{L}_{10}u_0 = 0.$$

Since the operator $\mathcal{L}_{10}$ has the partial derivative with respect to y in its every terms and u_0 does not depend on the variable y, we have $\mathcal{L}_{00}u_1 = 0$ and so u_1 is independent of the y variable. By the y-independence of u_0 and u_1, the $\mathcal{O}(1)$ term of (4.3) becomes the PDE

$$\mathcal{L}_{00}u_2 + \mathcal{L}_{20}u_0 = 0.$$

From the solvability condition (cf. Ramm [22]) of this Poisson equation, we have $\mathcal{L}_{20}u_0 = 0$. Therefore, Theorem 4.1 is proved. □

Continuously, we derive a PDE for the correction term u_1 from the following theorem.

Theorem 4.2. *Suppose that* $u_i,\ (i=0,1,2,...)$, *does not increase as much as* $u_i \sim e^{\frac{y^2}{2}}$ *as* y *goes to infinity. Then* u_1 *is a* y*-independent function and satisfies the PDE*

$$\mathcal{L}_{20}u_1(t,\psi;T,K_1,K_2) = -\langle\mathcal{L}_{21}\rangle u_0(t,\psi;T,K_1,K_2), \quad t<T,$$
$$u_1(T,\psi;T,K_1,K_2) = 0,$$

where u_0 *is obtained by Theorem 4.1.*

Proof. We have already found that u_0 and u_1 are independent on the variable y from the proof of Theorem 4.1. Also, u_2 is independent of y since the proof process of Theorem 4.1 gives $\mathcal{L}_{00}u_2 = 0$. Then, from the $\sqrt{\epsilon}$ term of (4.3), we obatin. $\mathcal{L}_{00}u_3 + \mathcal{L}_{20}u_1 + \mathcal{L}_{21}u_0 = 0$ and the solvability condition of this Poisson equation for u_3 yields $\mathcal{L}_{20}u_1 + \langle\mathcal{L}_{21}\rangle u_0 = 0$. Hence, we obtain Theorem 4.2. □

5. Approximate Option Price

This section investigates the influence of the stochastic elasticity of variance on the constant elasticity of variance by using a numerical experiment.

In Section 4, we derived the first order approximation for u given by $u(t,\psi,y;T,K_1,K_2) \approx u_0(T,\psi;T,K_1,K_2)+\sqrt{\epsilon}u_1(t,\psi;T,K_1,K_2)$. We define the leading order term P_0 and the first order term P_1 by

$$P_0(t,s;T,K_1,K_2) = \frac{T-t}{T}su_0(0,\psi;T,K_1,K_2),$$
$$P_1(t,s;T,K_1,K_2) = \frac{T-t}{T}su_1(0,\psi;T,K_1,K_2),$$

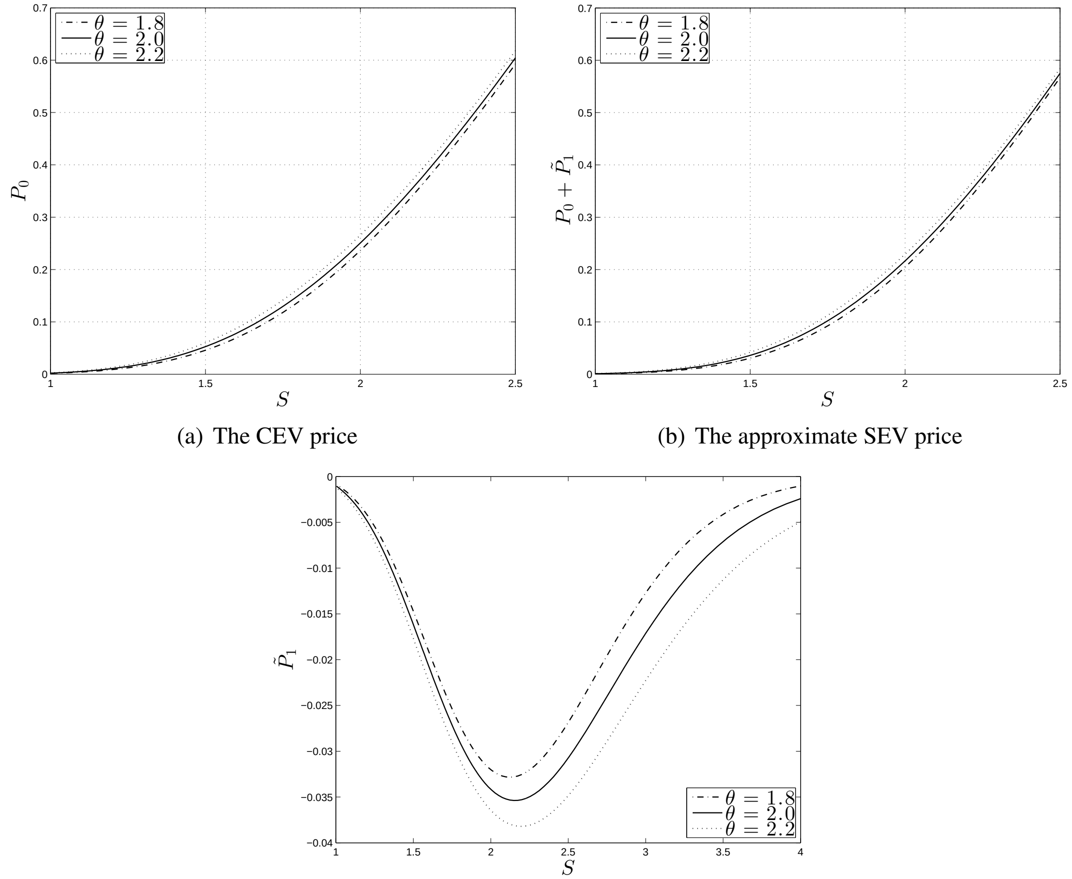
so that the option price $P(t,s,y;T,K_1,K_1)$ at time t has the approximation

$$P(t,s,y;T,K_1,K_2) \approx P_0(t,s;T,K_1,K_2) + \sqrt{\epsilon}P_1(t,s;T,K_1,K_2).$$

The accuracy of this approximation depends on the property of the payoff h. If the payoff is sufficiently smooth, it is straightforward to find an approximation error in the pointwise convergent sense. Otherwise, it needs a regularization of the payoff function as the case of European vanilla options in Fouque et al. [23]. This paper checks a numerical error instead of the theoretical proof of accuracy. See Figure 1 (c).

Now, we solve the PDEs for the leading order P_0 and the correction $\tilde{P}_1 := \sqrt{\epsilon}P_1$ by using the finite difference method (the Crank-Nicolson method). The solution has the truncation error given by $\mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta\psi)^2)$ with $\Delta = 0.005$ and $\psi = 0.0104$. Here, the parameters are given by $r = 0.06$, $\sigma = 0.5$, $T = 1$, $K_1 = 0$, $K_2 = 2$, $\epsilon = 0.01$, and $f(y) = e^y$ is chosen. Based on the observed financial data (such as S&P 500 index), we choose the value of parameter θ by three values $\theta = 1.8$, $\theta = 2$, and $\theta = 2.2$.

Figure 1 (a) shows the Asian option price P_0 underlying the CEV model and Figure 1 (b) shows the approximate price $P_0 + \tilde{P}_1$ under the SEV model. The CEV price also increases as the elasticity θ becomes larger but the stochastic elasticity of variance tends to lower the



(a) The CEV price

(b) The approximate SEV price

(c) The correction term $\tilde{P}_1$

FIGURE 1. The price under the CEV model, and the approximate price under the SEV models, and the correction term $\tilde{P}_1$ of the SEV price; $r = 0.06$, $\sigma = 0.5$, $T = 1$, $K_1 = 0$, and $K_2 = 2$

increase. In fact, Figure 1 (c) shows the correction term $\tilde{P}_1$ which is negative and has a hump shape in every case of θ. Here, the sign of $\tilde{P}_1$ is determined by the choice of f, i.e., $\tilde{P}_1$ has the opposite sign to the sign of $\langle f \rangle$. Note that the lowering effect has a maximum value near the strike price K_2 and it is more pronounced as θ becomes larger.

Figure 2 shows the sensitivity of the correction term $\tilde{P}_1$ to the asymptotic parameters $\bar{f}$. One can observe that the correction term $\tilde{P}_1$ decrease as the asymptotic parameter $\bar{f}$ increases and the slope is more large when the parameter θ is large. So one can figure out that the correction term $\tilde{P}_1$ is more sensitive to the asymptotic parameter as the parameter θ increases.

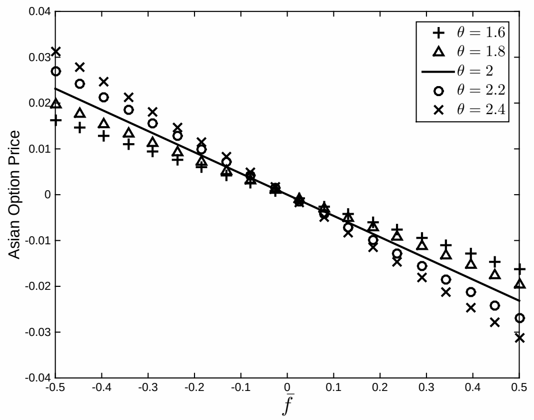


FIGURE 2. The sensitivity of $\tilde{P}_1$ to the asymptotic parameter $\bar{f}$; $S_0 = 2.5$, $r = 0.06$, $\sigma = 0.5$, $T = 1$, $K_1 = 0$, and $K_2 = 2$

6. CONCLUSION

The SEV model has been devised based upon a direct observation on the market elasticity of variance and successfully contributed to making up for the limitation of the CEV model for the European vanilla option (path-independent option) price. From the standpoint of this success, we price an Asian option (one of typical path-dependent options) under the SEV model by using a dimension reduction technique and a singular-regular perturbation method. This study finds that the CEV option price is somewhat over priced regardless of the elasticity parameter θ. The degree of over valuation has a maximum value near the strike price K_2. This may give a remarkable feature of the SEV model in that Gamma, which is important due to the fact that it corrects for the convexity of option value, is greatest approximately at-the-money.

REFERENCES

[1] F. Black and M. Scholes, *The pricing of options andcorporate liabilities*, J. Polit. Econ., **81** (1973), 637–654.
[2] P. Wilmott, *Paul Wilmott on Quantitative Finance*, John Wiley and Sons (2006).
[3] H. Geman and M. Yor, *Bessel processes, Asian option and perpetuities*, Math. Financ., **3** (1993), 349–375.
[4] V. Linetsk, *Spectral expansions for Asian (average price) options*, Oper. Res., **52**(6) (2004), 856–867.
[5] A.G.Z. Kemna and A.C.F. Vorst, *A pricing method for options based on average asset values*, J. Bank. Financ., **14**(1) (1990), 113–129.
[6] J.E. Ingersoll, *Theory of Financial Decision Making*, Rowman and Littlefield, Savage, Md (1987).
[7] L.C.G. Rogers and Z. Shi, *The value of an Asian option*, J. Appl. Prob., **32**(4) (1995), 1077–1088.
[8] J. Večeř, *Unified pricing of Asian options*, Risk, **15**(6) (2002), 113–116.
[9] J. Cox, *Notes on option pricing I: Constant elasticity of variance diffusions*, (1975) Working paper, Stanford University (Reprinted in Portf, J., 1996, manage **22**, 15–17).
[10] S.L. Heston, *Closed-form solution for options with stochastic volatility with applications to bond and currency options*, Rev. Financ. Stud., **6** (1993), 327–343.

[11] J.P. Fouque, G. Papanicolaou, and R. Sircar, *Asymptotics of a two-scale stochastic volatility model*, Equations aux derivees partielles et applications, in honour of Jacques-Louis Lions, Gauthier-Villars, Paris, (1998), 517–526.

[12] P. Carr, H. Geman, D. Madam, and M. Yor, *Stochastic volatility for Levy processes*, Math. Financ., **13**(3) (2003), 345–382.

[13] B. Peng, F. Peng, *Pricing arithmetic Asian options under the CEV process*, J. Econ. Financ. Adm. Sci., **15**(29) (2010), 7–13.

[14] J.P. Fouque and C.H. Han, *Pricing Asian options with stochastic volatility*, Quant. Financ., **3**(5) (2003), 352–362.

[15] D. Lemmens, L.Z. Liang, J. Tempere, and A. De Schepper, *Pricing bounds for discrete arithmetic Asian options under Levy models*, Physica A, **389**(22) (2010), 5193–5207.

[16] J.H. Kim, J.W. Lee, S.P. Zhu, and S.H. Yu, *A multiscale correction to the Black−Scholes formula*, Appl. Stoch. Model. Bus. **30**(6) (2014), 753–765.

[17] J.H. Yoon, J.H. Kim, and S.Y. Choi, *Multiscale analysis of a perpetual American option with the stochastic elasticity of variance*, Appl. Math. Lett., **26** (2013), 670–675.

[18] S.J. Yang, M.K. Lee, and J.H. Kim, *Portfolio optimization under the stochastic elasticity of variance*, Stoch. Dynam., **14**(03) (2014), 1350024.

[19] J.H. Kim, J.H. Yoon, J. Lee, and S.Y. Choi, *On the stochastic elasticity of variance diffusions*, Econ. Model., **51** (2015), 263–268.

[20] B. Oksendal, *Stochastic Differential Equations*. Springer, New York (2003).

[21] J.P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, *Multiscale Stochastic Volatility for Equity, Interest Rate and Credit Derivatives*, Cambridge University Press, Cambridge (2011).

[22] A.G. Ramm, *A simple proof of the Fredholm alternative and a characterization of the Fredholm operators*, Math. Assoc. America, **108** (2001), 855–860.

[23] J.P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, *Singular perturbations in option pricing*, SIAM J. Appl. Math., **62**(5) (2003), 1648–1665.

[24] L.B.G. Andersen and V.V. Piterbarg, *Moment explosions in stochastic volatility models*, Financ. Stoch., **11** (2007), 29–50.