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INTERACTIVE DYNAMICS IN A BISTABLE ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

YoonMee Ham and Sang-Gu Lee*

Abstract. We consider a bistable attraction-repulsion chemotaxis system in one dimension. The study in this paper asserts that conditions for chemotactic coefficients for attraction and repulsion to show existence of stationary solutions and Hopf bifurcation in the interfacial problem as the bifurcation parameters vary are obtained analytically.

1. Introduction

We consider an attraction-repulsion chemotaxis system ([5, 7, 9, 11, 16]):

$$(1) \qquad \begin{cases} \varepsilon\sigma\rho_t = \varepsilon^2\rho_{xx} - \varepsilon\kappa_1(\rho\, a_x)_x + \varepsilon\kappa_2(\rho\, b_x)_x + F(\rho, a), \\ a_t = a_{xx} + \mu\rho - a, \\ b_t = b_{xx} + \rho + a - b, \quad x > 0,\, t > 0, \end{cases}$$

where $\rho(x,t)$ is the cell density, $a(x,t)$ and $b(x,t)$ are the chemical concentrations of attractant and repellent, respectively. The function F is of the bistable type investigated by McKean [12], namely $F(\rho, a) = -\rho + H(\rho - a_0) - a$, where H is a Heaviside step function. The constants ε, σ, a_0 and μ are positive and the parameters κ_1 and κ_2 measure

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∗ Corresponding author.

the strength of the attraction and repulsion, respectively. The system with $F = 0$ and $\varepsilon = 1$ was proposed in [11] to describe the aggregation of microglia observed in Alzheimer's disease and in [15] to address the quorum effect in the chemotactic process. The system with $F = 0$ and $\varepsilon = 1$ in the one dimensional case is globally well-posed in the sense that $\kappa_2 - \mu\kappa_1 > 0$ in [10] and in the multi-dimensional case was essentially determined by the competition of attraction and repulsion which is characterized by the sign of $\kappa_2 - \mu\kappa_1$ in [6,16].

When ε is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place with in an x-interval of length $O(\varepsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called an internal layer when it is in the interior of the interval. An analysis of the layer solutions suggests that the layer of width $O(\varepsilon)$ converges to an interfacial curve $x = \eta(t)$ in x, t-space as $\varepsilon \downarrow 0$. In this paper, we derive an evolutional equation of an interfacial curve that is controlled by two chemicals a and b and examine the behavior of solutions in this free boundary problem.

Suppose that there is an interfacial curve $\eta(t)$, which is simply a single closed curve given in $\Omega = [0, \infty)$ in such a way that $\Omega = \Omega_1(t) \cup \eta(t) \cup \Omega_0(t)$, where $\Omega_1(t) = \{x \in \mathbb{R}^+ : \rho(x,t) > a_0\}$ and $\Omega_0(t) = \{x \in \mathbb{R}^+ : \rho(x,t) < a_0\}$. The velocity of the one-dimensional interface $\eta(t)$ is given by (see [13,17]);

$$\frac{d\eta(t)}{dt} = \frac{1}{\sigma}\Big(C(a(\eta(t))) + \kappa_1\, a_x(\eta(t),t) - \kappa_2\, b_x(\eta(t),t)\Big), \;\; x \in \eta(t), \tag{2}$$

where C is a continuously differentiable function defined on an interval $I := (-a_0, 1 - a_0)$, which is given by ([3,8])

$$C(a(\eta)) = -\frac{1 - 2a_0 - 2a(\eta)}{\sqrt{(a(\eta) + a_0)(1 - a_0 - a(\eta))}}\,. \tag{3}$$

Hence, a free boundary problem of (1) when ε is equal to zero is given by :

$$\begin{cases} a_t = a_{xx} - (\mu + 1)a + \mu, & t > 0,\; x \in \Omega_1(t) \\ a_t = a_{xx} - (\mu + 1)a, & t > 0,\; x \in \Omega_0(t) \\ a(\eta(t) - 0, t) = a(\eta(t) + 0, t) & \\ a_x(\eta(t) - 0, t) = a_x(\eta(t) + 0, t) & \end{cases} \tag{4}$$

and

$$\begin{cases} b_t = b_{xx} - b + 1, \quad t > 0, \ x \in \Omega_1(t) \\ b_t = b_{xx} - b, \quad t > 0, \ x \in \Omega_0(t) \\ b(\eta(t) - 0, t) = b(\eta(t) + 0, t) \\ b_x(\eta(t) - 0, t) = b_x(\eta(t) + 0, t). \end{cases} \tag{5}$$

The organization of the paper is as follows: In section 2, a change of variables is given which regularizes problem (4) and (5) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain a regularity of the solution which is sufficient for an analysis of the bifurcation. In section 3, we show the existence of equilibrium solutions for (4) and (5) and obtain the linearization of problem (4) and (5) under the condition $\kappa_2 - \mu\kappa_1 > 0$. In the last section, we investigate the conditions to obtain the periodic solutions and the bifurcation of the interface problem as the parameter σ varies.

2. Regularization of the interface equation

Now, we consider the existence problem of (4) and (5).

(6)
$$\begin{cases} a_t = a_{xx} - (\mu + 1)a + \mu H(x - \eta(t)), \ x > 0, t > 0 \\ b_t = b_{xx} - b + H(x - \eta(t)), \ x > 0, t > 0 \\ a_x(0, t) = 0, b_x(0, t) = 0, \ t > 0 \\ \sigma\eta'(t) = C(a(\eta)) + \kappa_1 \, a_x(\eta(t), t) - \kappa_2 \, b_x(\eta(t), t), \ t > 0\,; \eta(0) = \eta_0. \end{cases}$$

Let A be an operator defined by $A := -\frac{d^2}{dx^2} + \mu + 1$ with domain $D(A) = \{a \in H^{2,2}(\mathbb{R}) \, : \, a_x(0, t) = 0\}$. Let $A_0 := -\frac{\partial^2}{\partial x^2} + 1$ with domain $D(A_0) = \{b \in H^{2,2}((0, \infty)) \, : \, b_x(0, t) = 0\}$. In order to apply semigroup theory to (6), we choose the space $X := L_2(0, \infty)$ with norm $\| \cdot \|_2$.

To get differential dependence on initial conditions, we decompose v in (6) into two parts: u, which is a solution to a more regular problem and g, which is less regular but explicitly known in terms of the Green's function G of the operator A. Namely, we define $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$, by

$$g(x,\eta) := A^{-1}(\mu H(\cdot - \eta)(x)) = \mu \int_0^\infty G(x,y)\, H(y-\eta)\, dy,$$

where $G : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a Green's function of A satisfying the Neumann boundary conditions, and $\gamma : [0,\infty) \to \mathbb{R}$,

$$\gamma(\eta) := g(\eta,\eta).$$

If we take a transformation $u(t)(x) = a(x,t) - g(x,\eta(t))$, we have $(u_x)(t)(x) = a_x(x,t) - g_x(x,\eta(t))$. Since $G_x(x,\eta)$ is discontinuous, we cannot obtain one step more regular than that of (6).

To overcome this difficulty, let $p(x,t) = a_x(x,t)$ and define $\hat{g} : [0,\infty) \times [0,\infty) \to \mathbb{R}$,

$$\hat{g}(x,\eta) := A^{-1}(\mu\delta(\cdot - \eta)(x)) = \mu \int_0^\infty \hat{G}(x,y)\, \delta(y-\eta)\, dy,$$

where $\hat{G} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a Green's function of A satisfying the Dirichlet boundary conditions, and $\hat{\gamma} : [0,\infty) \to \mathbb{R}$,

$$\hat{\gamma}(\eta) := \hat{g}(\eta,\eta).$$

We note that

$$\tfrac{\partial}{\partial \eta}\, \hat{g}(x,\eta) = \mu \textstyle\int_0^\infty \hat{G}(x,y)\, \tfrac{\partial}{\partial \eta}\delta(y-\eta)\, dy = \tfrac{\mu}{\eta}\, \hat{G}(x,\eta)$$

and

$$\hat{\gamma}(\eta) = \mu\hat{G}(\eta,\eta) = \gamma'(\eta) + \mu G(\eta,\eta),\ \hat{\gamma}'(\eta) = -\sqrt{1+\mu}\,\gamma'(\eta)$$

are positive for all $\eta > 0$.

We define $j : [0,\infty) \times [0,\infty) \to \mathbb{R}$,

$$j(x,\eta) := A_0^{-1}(H(\cdot - \eta)(r)) = \int_0^\infty J(x,y) H(y-\eta)\, dy$$

and $\alpha : [0,\infty) \times \mathbb{C} \to \mathbb{C}$,

$$\alpha(\eta) := j(\eta,\eta).$$

Here $J : [0,\infty)^2 \to \mathbb{R}$ is a Green's function of A_0 satisfying the boundary conditions.

To overcome this difficulty, let $q(x,t) = b_x(x,t)$ and define $\hat{j} : [0,\infty) \times [0,\infty) \to \mathbb{R}$,

$$\hat{j}(x,\eta) := A_0^{-1}(\delta(\cdot - \eta)(x)) = \int_0^\infty \hat{J}(x,y)\, \delta(y-\eta)\, dy,$$

where $\hat{J} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a Green's function of A_0 satisfying the Dirichlet boundary conditions and $\hat{\alpha} : [0, \infty) \to \mathbb{R}$,

$$\hat{\alpha}(\eta) := \hat{j}(\eta, \eta).$$

Applying the transformations $u(t)(x) = a(x,t) - g(x, \eta(t))$, $v(t)(x) = p(x,t) - \hat{g}(x, \eta(t))$ and $w(t)(x) = b(x,t) - j(x, \eta(t))$, $s(t)(x) = q(x,t) - \hat{j}(x, \eta(t))$, then (6) becomes

(7)

$$\begin{cases} u_t + Au = \frac{1}{\sigma}\mu G(x,\eta)\Big(C(u(\eta)+\gamma(\eta)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))\Big) \\ v_t + Av = -\frac{1}{\sigma}\frac{\mu}{\eta}\hat{G}(x,\eta)\Big(C(u(\eta)+\gamma(\eta)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))\Big) \\ w_t + A_0 w = \frac{1}{\sigma}J(x,\eta)\Big(C(u(\eta)+\gamma(\eta)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))\Big) \\ s_t + A_0 s = -\frac{1}{\sigma\eta}\hat{J}(x,\eta)\Big(C(u(\eta)+\gamma(\eta)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))\Big) \\ \eta'(t) = \frac{1}{\sigma}\Big(C(u(\eta)+\gamma(\eta)) + \kappa_1(v(\eta)+\hat{\gamma}(\eta)) - \kappa_2(s(\eta)+\hat{\alpha}(\eta))\Big),\ t > 0. \end{cases}$$

Thus, we obtain an abstract evolution equation equivalent to (6) :

$$\text{(8)} \qquad \begin{cases} \frac{d}{dt}(u,v,w,s,\eta) + \widetilde{A}(u,v,w,s,\eta) = \frac{1}{\sigma} f(u,v,w,s,\eta), \\ (u,v,w,s,\eta)(0) = (u_0(x), v_0(x), w_0(x), s_0(x), \eta_0), \end{cases}$$

where $\widetilde{A}$ is a 5×5 matrix where (1,1) and (2,2)-entries are an operator A, (3,3) and (4,4)-entries are an operator A_0 and all the others are zero. The nonlinear forcing term f is

$$f(u,v,w,s,\eta)$$

$$= \begin{pmatrix} f_1(\eta)\cdot(f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_2(\eta)\cdot(f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_3(\eta)\cdot(f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_4(\eta)\cdot(f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta)) \\ f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta) - f_{23}(u,v,w,s,\eta) \end{pmatrix},$$

where $f_1 : (0,\infty) \to X$, $f_1(\eta)(x) := \mu G(x,\eta)$, $f_2 : (0,\infty) \to X$, $f_2(\eta)(x) := -\frac{\mu}{\eta}\hat{G}(x,\eta)$, $f_3 : (0,\infty) \to X$, $f_3(\eta)(x) := J(x,\eta)$, $f_4 : (0,\infty) \to X$, $f_4(\eta)(x) := -\frac{1}{\eta}\hat{J}(x,\eta)$, $f_{21} : W \to \mathbb{C}$, $f_{21}(u,v,w,s,\eta) := C(u(\eta)+\gamma(\eta))$, $f_{22} : W \to \mathbb{C}$, $f_{22}(u,v,w,s,\eta) := \kappa_1(v(\eta)+\hat{\gamma}(\eta))$, $f_{23}(u,v,w,s,\eta) := \kappa_2(s(\eta)+\hat{\alpha}(\eta))$ and $W := \{(u,v,w,s,\eta) \in C^1(0,\infty)\times$

$C^1(0,\infty)\times C^1(0,\infty)\times C^1(0,\infty)\times(0,\infty)\ :\ u(\eta)+\gamma(\eta)\in I,\ v(\eta)+\hat\gamma(\eta)\in I,\ w(\eta)+\alpha(\eta)\in I,\ s(\eta)+\hat\alpha(\eta)\in I\}\subset_{\text{open}} C^1(\mathbb{R})\times C^1(\mathbb{R})\times C^1(\mathbb{R})\times C^1(\mathbb{R})\times\mathbb{R}$.

The well-posedness of solutions of (8) is shown in [1, 13, 14] with the help of the semigroup theory using domains of fractional powers $\theta\in(3/4,1]$ of A, A_0 and $\widetilde{A}$. Moreover, the nonlinear term f is a continuously differentiable function from $W\cap\widetilde{X}^\theta$ to $\widetilde{X}$, where $\widetilde{X}:=D(\widetilde{A})=D(A)\times D(A)\times D(A_0)\times D(A_0)\times\mathbb{R}$, $X^\theta:=D(A^\theta)$; $X_0^\theta:=D(A_0^\theta)$ and $\widetilde{X}^\theta:=D(\widetilde{A}^\theta)=X^\theta\times X^\theta\times X_0^\theta\times X_0^\theta\times\mathbb{R}$.

The velocity of η is denoted by

$$C(\eta)=-\frac{1-2a_0-2(u(\eta)+\gamma(\eta))}{\sqrt{(u(\eta)+\gamma(\eta)+a_0)\big(1-a_0-(u(\eta)+\gamma(\eta))\big)}}\,. \tag{9}$$

The derivative of f can be obtained from the following in [4]:

LEMMA 2.1. *The functions* $G(\cdot,\eta):(0,\infty)\to X$, $\hat{G}(\cdot,\eta):(0,\infty)\to X$, $J(\cdot,\eta):(0,\infty)\to X$, $\hat{J}(\cdot,\eta):(0,\infty)\to X$, $C(\cdot):W\to\mathbb{C}$ *and* $f:W\to X\times\mathbb{R}$ *are continuously differentiable with derivatives given by*

$$\begin{aligned}
&Df_{21}(u,v,w,s,\eta)(\hat u,\hat v,\hat w,\hat s,\hat\eta)=C'(u(\eta)+\gamma(\eta))\cdot(u'(\eta)\tilde\eta+\tilde u(\eta)+\gamma'(\eta)\tilde\eta)\\
&Df_{22}(u,v,w,s,\eta)(\hat u,\hat v,\hat w,\hat s,\hat\eta)=\kappa_1\big(v'(\eta)\hat\eta+\hat v(\eta)+\hat\gamma'(\eta)\hat\eta\big)\\
&Df(u,v,w,s,\eta)(\hat u,\hat v,\hat w,\hat s,\hat\eta)=(f_{21}(u,v,w,s,\eta)+f_{22}(u,v,w,s,\eta))\\
&\qquad\cdot(f_1'(\eta),f_2'(\eta),f_3'(\eta),f_4'(\eta),0)\,\hat\eta+(Df_{21}(u,v,w,s,\eta)\\
&\qquad+Df_{22}(u,v,w,s,\eta))(\hat u,\hat v,\hat w,\hat s,\hat\eta)\cdot(f_1(\eta),f_2(\eta),f_3(\eta),f_4(\eta),1).
\end{aligned}$$

3. Equilibrium solutions and Linearization of the interface equation

In this section, we shall examine the existence of equilibrium solutions of (8). We look for $(u^*,v^*,w^*,s^*,\eta^*)\in D(\widetilde{A})\cap W$ satisfying the following equations:

(10)

$$\begin{cases} Au = \frac{1}{\sigma}\,\mu G(\cdot,\eta)\big(C(u(\eta)+\gamma(\eta))+\kappa_1(v(\eta)+\hat{\gamma}(\eta))-\kappa_2(s(\eta)+\hat{\alpha}(\eta))\big) \\ Av = -\frac{1}{\sigma\eta}\,\mu\hat{G}(\cdot,\eta)\big(C(u(\eta)+\gamma(\eta))+\kappa_1(v(\eta)+\hat{\gamma}(\eta))-\kappa_2(s(\eta)+\hat{\alpha}(\eta))\big) \\ A_0 w = \frac{1}{\sigma}\,J(\cdot,\eta^*)\big(C(u(\eta)+\gamma(\eta))+\kappa_1(v(\eta)+\hat{\gamma}(\eta))-\kappa_2(s(\eta)+\hat{\alpha}(\eta))\big) \\ A_0 s = -\frac{1}{\sigma\eta}\,\hat{J}(\cdot,\eta^*)\big(C(u(\eta)+\gamma(\eta))+\kappa_1(v(\eta)+\hat{\gamma}(\eta))-\kappa_2(s(\eta)+\hat{\alpha}(\eta))\big) \\ 0 = C(u(\eta)+\gamma(\eta))+\kappa_1(v(\eta)+\hat{\gamma}(\eta))-\kappa_2(s(\eta)+\hat{\alpha}(\eta)) \\ u'(0)=0=u'(\infty),\ v(0)=0=v(\infty),\ w'(0)=0=w'(\infty),\ s(0)=0=s(\infty). \end{cases}$$

THEOREM 3.1. *Suppose that* $\frac{\mu}{2(1+\mu)} < \frac{1}{2} - a_0 < \frac{1}{1+\mu}$ *and* $\mu\kappa_1 < \kappa_2$. *Then equation (8) has at least one equilibrium solution* $(0,0,0,0,\eta^*)$. *The linearization of* f *at the stationary solution* $(0,0,0,0,\eta^*)$ *is*

$$Df(0,0,0,0,\eta^*)(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) = \begin{pmatrix} \mu G(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ -\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ J(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ -\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)\,Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \\ Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) \end{pmatrix},$$

where $Q(\hat{u},\hat{v},\hat{w},\hat{s},\hat{\eta}) = 4(\hat{u}(\eta^*)+\gamma'(\eta^*)\hat{\eta})+\kappa_1(\hat{v}(\eta^*)+\hat{\gamma}'(\eta^*)\hat{\eta})-\kappa_2(\hat{s}(\eta^*)+\hat{\alpha}'(\eta^*)\hat{\eta})$. *The pair* $(0,0,0,0,\eta^*)$ *corresponds to a unique steady state* (a^*,p^*,b^*,q^*,η^*) *of (6) for* $\sigma \neq 0$ *with* $a^*(x) = g(x,\eta^*)$, $p^*(x) = \hat{g}(x,\eta^*)$, $b^*(x) = j(x,\eta^*)$ *and* $q^*(x) = \hat{j}(x,\eta^*)$.

Proof. From the system of equations (10), we have $u^* = 0, v^* = 0, w^* = 0$ and $s^* = 0$. In order to show existence of η^*, we define

$$\Gamma(\eta) := C(\gamma(\eta)) + \kappa_1\,\hat{\gamma}(\eta) - \kappa_2\hat{\alpha}(\eta).$$

$\Gamma(\eta) = 0$ is solvable with η^* if $\Gamma(0) > 0, \Gamma(\infty) < 0$ and $\Gamma'(\eta) < 0$ for al $\eta > 0$. Since $\Gamma(\infty) < C(\gamma(\infty)) + \frac{1}{2}(\mu\kappa_1 - \kappa_2)$ and $\mu\kappa_1 < \kappa_2$ for all η, $\Gamma(\infty)$ is negative if $C(\gamma(\infty))$ is negative. $\Gamma'(\eta) = C'(\gamma(\eta))\gamma'(\eta) + \kappa_1\hat{\gamma}'(\eta) - \kappa_2\hat{\alpha}'(\eta) < C'(\gamma(\eta))\gamma'(\eta) + (\mu\kappa_1 - \kappa_2)e^{-2\eta}$ is negative since $\mu\kappa_1 < \kappa_2$. Hence η^* exists if $\gamma(\infty) < \frac{1}{2} - a_0 < \gamma(0)$ with $\mu\kappa_1 < \kappa_2$.

The formula for $Df(0,0,0,0,\eta^*)$ follows from the relation $C'(1/2) = 4$, and the corresponding steady state (a^*,p^*,b^*,q^*,η^*) for (6) is obtained by using the transformation and Theorem 2.1 in [4]. □

4. A Hopf bifurcation

In this section, we shall show that there exists a Hopf bifurcation from the curve $\sigma \mapsto (0,0,0,0,\eta^*)$ of the equilibrium solution. First, let us introduce the following relevant definition.

DEFINITION 4.1. *Under the assumptions of Theorem 3.1, define (for $1 \geq \theta > 3/4$) the linear operator B from $\widetilde{X}^\theta$ to $\widetilde{X}$ by*

$$B := Df(0,0,0,0,\eta^*)\,.$$

We then define $(0,0,0,0,\eta^*)$ to be a Hopf point *for (8) if and only if there exists an $\epsilon_0 > 0$ and a C^1-curve*

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A} + \tau B$ with

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*) = i\beta$ *with* $\beta > 0$;
(iii) $\mathrm{Re}\,(\lambda) \neq 0$ *for all* λ *in the spectrum of* $(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ *(transversality);*

where $\tau = 1/\sigma$.

Next, we check (8) for Hopf points. For this, we solve the eigenvalue problem:

$$-\widetilde{A}(u,v,w,s,\eta) + \tau B(u,v,w,s,\eta) = \lambda I_5(u,v,w,s,\eta),$$

where I_5 is an 5×5 identity matrix. This is equivalent to:

(11)
$$\begin{cases} (A+\lambda)u = \tau\mu\, G(\cdot,\eta^*)\big(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) \\ \qquad\qquad -\kappa_2(s(\eta^*) + \hat{\alpha}'(\eta^*)\eta)\big), \\ (A+\lambda)v = -\tau\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\big(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) \\ \qquad\qquad -\kappa_2(s(\eta^*) + \hat{\alpha}'(\eta^*)\eta)\big), \\ (A_0+\lambda)w = \tau\, J(\cdot,\eta^*)\big(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) \\ \qquad\qquad -\kappa_2(s(\eta^*) + \hat{\alpha}'(\eta^*)\eta)\big), \\ (A_0+\lambda)s = -\tau\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\big(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) \\ \qquad\qquad -\kappa_2(s(\eta^*) + \hat{\alpha}'(\eta^*)\eta)\big), \\ \lambda\,\eta = \tau\big(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \hat{\gamma}'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \hat{\alpha}'(\eta^*)\eta)\big). \end{cases}$$

We shall show that an equilibrium solution is a Hopf point.

THEOREM 4.2. *Suppose that* $\frac{\mu}{2(1+\mu)} < \frac{1}{2} - a_0 < \frac{1}{1+\mu}$ *and* $\mu\kappa_1 < \kappa_2$ *for all* η. *Moreover, assume that* κ_1 *satisfies that* $4 > \frac{\kappa_1}{\eta^*}$. *Additionally, suppose the operator* $-\widetilde{A}+\tau^* B$ *has a unique pair* $\{\pm i\beta\}$, $\beta > 0$ *of purely imaginary eigenvalues for some* $\tau^* > 0$. *Then,* $(0,0,0,0,\eta^*,\tau^*)$ *is a Hopf point for (8).*

Proof. We assume without loss of generality that $\beta > 0$, and Φ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^* B$ with eigenvalue $i\beta$. We have to show that $(\Phi^*, i\beta)$ can be extended to a C^1-curve $\tau \mapsto (\Phi(\tau), \lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau B$ with $\mathrm{Re}(\lambda'(\tau^*)) \neq 0$.

For this, let $\Phi^* = (\psi_0, v_0, w_0, s_0, \eta_0) \in D(A)\times D(A)\times D(A_0)\times D(A_0)\times \mathbb{R}$. First, we note that $\eta_0 \neq 0$. Otherwise, by (11), $(A+i\beta)\psi_0 = \mu\, i\beta\, \eta_0\, G(\cdot,\eta^*) = 0$ and $(A+i\beta)v_0 = -\frac{\mu}{\eta^*}\, i\beta\, \eta_0\, \hat{G}(\cdot,\eta^*) = 0$, which is not possible given A is symmetric. So, without loss of generality, let $\eta_0 = 1$. Then $E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) = 0$ by (11), where

$$E : D(A)_{\mathbf{C}}\times D(A)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times D(A_0)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{R} \longrightarrow X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times X_{\mathbf{C}}\times\mathbb{C},$$

$$E(u,v,w,s,\lambda,\tau) := \begin{pmatrix} (A+\lambda)u - \tau\mu\, G(\cdot,\eta^*)\big(4(u(\eta^*)+\gamma'(\eta^*)) + \kappa_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s(\eta^*)+\hat{\alpha}'(\eta^*)\big) \\ (A+\lambda)v + \tau\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\big(4(u(\eta^*)+\gamma'(\eta^*)) + \kappa_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s(\eta^*)+\hat{\alpha}'(\eta^*)\big) \\ (A_0+\lambda)w - \tau J(\cdot,\eta^*)\big(4(u(\eta^*)+\gamma'(\eta^*)) + \kappa_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s(\eta^*)+\hat{\alpha}'(\eta^*)\big) \\ (A_0+\lambda)s + \tau\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\big(4(u(\eta^*)+\gamma'(\eta^*)) + \kappa_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s(\eta^*)+\hat{\alpha}'(\eta^*)\big) \\ \lambda - \tau\big(4(u(\eta^*)+\gamma'(\eta^*)) + \kappa_1(v(\eta^*)+\hat{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*)+\hat{\alpha}'(\eta^*)\big) \end{pmatrix}.$$

The equation $E(u,v,w,s,\lambda,\tau) = 0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,v,w,s,1)$. We shall apply the implicit

function theorem to E to check that E is of C^1- class in [2] and that
(12)

$$D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) \in L(D(A)_{\mathbf{C}} \times D(A)_{\mathbf{C}} \times D(A_0)_{\mathbf{C}} \times D(A_0)_{\mathbf{C}} \times \mathbb{C} \times \mathbb{R}, X^4_{\mathbf{C}} \times \mathbb{C})$$

is an isomorphism. In addition, the mapping

$$D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) = \begin{pmatrix} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 - \tau^* \mu G(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A + i\beta)\hat{v} + \hat{\lambda} v0 + \tau^* \frac{\mu}{\eta^*} \hat{G}(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A_0 + i\beta)\hat{w} + \hat{\lambda} w_0 - \tau^* J(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A_0 + i\beta)\hat{s} + \hat{\lambda} s_0 + \tau^* \frac{1}{\eta^*} \hat{J}(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ \hat{\lambda} - \tau^* \big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) \longmapsto \big((A + i\beta)\hat{u}, (A + i\beta)\hat{v}\,, (A_0 + i\beta)\hat{w}, (A_0 + i\beta)\hat{s}, \hat{\lambda}\big)$$

which is invertible. Thus, $D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)$ is a Fredholm operator of index 0. Therefore, in order to verify (12), it suffices to show that the system of equations

$$D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) = 0$$

which is equivalent to
(13)

$$\begin{cases} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \tau^* \mu G(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A + i\beta)\hat{v} + \hat{\lambda}\xi_0 = -\tau^* \frac{\mu}{\eta^*} \hat{G}(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A_0 + i\beta)\hat{w} + \hat{\lambda} q_0 = \tau^* J(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ (A_0 + i\beta)\hat{s} + \hat{\lambda} s_0 = -\tau^* \frac{1}{\eta^*} \hat{J}(\cdot, \eta^*)\big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \\ \hat{\lambda} = \tau^* \big(4\hat{u}(\eta^*) + \kappa_1 \hat{v}(\eta^*) - \kappa_2 \hat{s}(\eta^*)\big) \end{cases}$$

necessarily implies that $\hat{u} = 0$, $\hat{v} = 0$, $\hat{w} = 0$, $\hat{s} = 0$ and $\hat{\lambda} = 0$. If we define $\phi := \psi_0 - \mu G(\cdot, \eta^*)$, $\xi := v_0 + \frac{\mu}{\eta^*} \hat{G}(\cdot, \eta^*)$, $\rho = w_0 - J(\cdot, \eta^*)$ and

$\zeta = s_0 + \frac{1}{\eta^*}\hat{J}(\cdot, \eta^*)$, then (13) becomes

$$(A + i\beta)\hat{u} + \hat{\lambda}\phi = 0, \tag{14}$$

$$(A + i\beta)\hat{v} + \hat{\lambda}\xi = 0, \tag{15}$$

$$(A_0 + i\beta)\hat{w} + \hat{\lambda}\rho = 0, \tag{16}$$

$$(A_0 + i\beta)\hat{s} + \hat{\lambda}\zeta = 0, \tag{17}$$

$$\frac{\hat{\lambda}}{\tau^*} = 4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*). \tag{18}$$

Since $E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) = 0$, ϕ, ξ, ρ and ζ are solutions to the equations, we have:

$$(A + i\,\beta)\phi = -\mu\,\delta_{\eta^*}, \tag{19}$$

$$(A + i\beta)\xi = \frac{\mu}{\eta^*}\delta_{\eta^*}, \tag{20}$$

$$(A_0 + i\,\beta)\rho = -\delta_{\eta^*}, \tag{21}$$

$$(A_0 + i\,\beta)\zeta = \frac{1}{\eta^*}\delta_{\eta^*}, \tag{22}$$

(23)

$$\frac{i\beta}{\tau^*} = 4(\phi(\eta^*) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) + \kappa_1(\xi(\eta^*) - \frac{\mu}{\eta^*}\hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)) -\kappa_2(\zeta(\eta^*) - \frac{1}{\eta^*}\hat{J}(\eta^*, \eta^*) + \hat{\alpha}'(\eta^*)).$$

Multiplying (15) and (20) by ϕ, and (14) and (19) by ξ and subtracting one from the other, we now obtain

$$\hat{u}(\eta^*) = -\eta^*\,\hat{v}(\eta^*),\ \hat{w}(\eta^*) = -\eta^*\,\hat{s}(\eta^*) \tag{24}$$

$$\phi(\eta^*) = -\eta^*\,\xi(\eta^*),\ \rho(\eta^*) = -\eta^*\,\zeta(\eta^*). \tag{25}$$

Multiplying (14) by $4\overline{\phi}$, (15) by $-\eta^*\kappa_1\overline{\xi}$, and (17) by $\mu\eta^*\kappa_2\overline{\zeta}$ and adding the resultants to each, we obtain

(26)

$$-4\mu\hat{u}(\eta^*) - \kappa_1\mu\,\hat{v}(\eta^*) + \mu\kappa_2\hat{s}(\eta^*) + \hat{\lambda}(4||\phi||^2 - \eta^*\kappa_1||\xi||^2 + \mu\eta^*\kappa_2||\zeta||^2) +2i\beta\int\left(4\hat{u}\overline{\phi} - \eta^*\kappa_1\hat{v}\overline{\xi} + \mu\eta^*\kappa_2\hat{s}\overline{\zeta}\right) = 0.$$

Multiplying (19) by $4\overline{\phi}$, (20) by $-\eta^*\kappa_1\overline{\xi}$ and (17) by $\mu\eta^*\kappa_2\overline{\zeta}$ and adding the resultants to each, we obtain

$$\begin{aligned}4||A^{1/2}\phi||^2 - \eta^*\kappa_1||A^{1/2}\xi||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\zeta||^2 \\ +i\beta\Big(4||\phi||^2 - \eta^*\kappa_1||\xi||^2 + \mu\eta^*\kappa_2||\zeta||^2\Big) \\ = -4\,\mu\overline{\phi(\eta^*)} - \mu\kappa_1\overline{\xi(\eta^*)} + \mu\eta^*\kappa_2\overline{\zeta(\eta^*)},\end{aligned}$$

and from (23), we get

$$\frac{\mu}{\tau^*} = 4||\phi||^2 - \eta^*\kappa_1||\xi||^2 + \mu\eta^*\kappa_2||\zeta||^2. \tag{27}$$

Thus (26) implies that

$$\int \Big(4\hat{u}\overline{\phi} - \eta^*\kappa_1\hat{v}\overline{\xi} + \mu\eta^*\kappa_2\hat{s}\overline{\zeta}\Big) = 0. \tag{28}$$

Now, multiplying (14) by $4\overline{\hat{u}}$, (19) by $-\eta^*\kappa_1\overline{\hat{v}}$ and (17) by $\mu\eta^*\kappa_2\overline{\hat{s}}$ and adding the resultants to each, we obtain

$$\begin{aligned}\Big(4||A^{1/2}\hat{u}||^2 - \eta^*\kappa_1||A^{1/2}\hat{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\hat{s}||^2\Big) \\ +i\beta\Big(4||\hat{u}||^2 - \eta^*\kappa_1||\hat{v}||^2 + \mu\eta^*\kappa_2||\hat{s}||^2\Big) \\ +\hat{\lambda}\int\Big(4\phi\overline{\hat{u}} - \eta^*\kappa_1\xi\overline{\hat{v}} + \mu\eta^*\kappa_2\zeta\overline{\hat{s}}\Big) = 0\end{aligned}$$

and from (28), we have

$$\begin{cases} 4||A^{1/2}\hat{u}||^2 - \eta^*\kappa_1||A^{1/2}\hat{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\hat{s}||^2 = 0 \\ 4||\hat{u}||^2 - \eta^*\kappa_1||\hat{v}||^2 + \mu\eta^*\kappa_2||\hat{s}||^2 = 0. \end{cases} \tag{29}$$

Multiplying (19) by $\overline{\phi}$ and (20) by $\overline{\xi}$, we then get

$$||A^{1/2}\phi||^2 + i\beta||\phi||^2 = -\mu\overline{\phi}(\eta^*) \quad \text{and} \quad ||A^{1/2}\xi||^2 + i\beta||\xi||^2 = \frac{\mu}{\eta^*}\,\overline{\xi}(\eta^*)$$

and applying (24) to the above equation, we have

$$||A^{1/2}\phi||^2 = (\eta^*)^2||A^{1/2}\xi||^2 \quad \text{and} \quad ||\phi||^2 = (\eta^*)^2||\xi||^2. \tag{30}$$

Now, multiplying (14) by $2i\beta\overline{\hat{u}}$ and (19) by $\hat{\lambda}\overline{\hat{u}}$ and subtracting the resultants from each, we now obtain

$$2i\beta\big(||A^{1/2}\hat{u}||^2-(\eta^*)^2||A^{1/2}\hat{v}||^2\big)-2\beta^2\big(||\hat{u}||^2-(\eta^*)^2||\hat{v}||^2\big)+\hat{\lambda}\big(||\phi||^2-(\eta^*)^2||\xi||^2\big).$$

Applying (30) to the above equation, we have

$$||A^{1/2}\hat{u}||^2 - (\eta^*)^2||A^{1/2}\hat{v}||^2 = 0 \quad \text{and} \quad ||\hat{u}||^2 - (\eta^*)^2||\hat{v}||^2 = 0$$

and thus (29) implies:

$$\big(4 - \frac{\kappa_1}{\eta^*}\big)||\hat{u}||^2 + \mu\kappa_2\,\eta^*\,||\hat{s}||^2 = 0.$$

Since $4 - \frac{\kappa_1}{\eta^*} > 0$, we have $\hat{u} = 0$ and $\hat{s} = 0$ and so, $\hat{v} = 0$. By (13) and (16), we have $\hat{\lambda} = 0$ and $\hat{w} = 0$. □

THEOREM 4.3. *Under the same condition as in Theorem 4.2,* $(0, 0, 0, \eta^*, \tau^*)$ *satisfies the transversality condition. Hence, it is a Hopf point for (8).*

Proof. By implicit differentiation of $E(\psi_0(\tau), v_0(\tau), w_0(\tau), s_0(\tau), \lambda(\tau), \tau) = 0$, we find

$$D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\psi_0'(\tau^*), v_0'(\tau^*), w_0'(\tau^*), s_0'(\tau^*), \lambda'(\tau^*))$$

$$= \begin{pmatrix} \mu\, G(\cdot,\eta^*)\big(4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1\,(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*)\big) \\ -\frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)\big(4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1\,(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*)\big) \\ J(\cdot,\eta^*)\big(4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1\,(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*))\big) \\ -\frac{1}{\eta^*}\,\hat{J}(\cdot,\eta^*)\big(4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1\,(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*)\big) \\ 4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1\,(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*) \end{pmatrix}.$$

This means that the functions $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{v} := v_0'(\tau^*)$, $\tilde{w} := w_0'(\tau^*)$, $\tilde{s} := s_0'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$(31)\quad \begin{cases} (A+i\beta)\tilde{u} + \tilde{\lambda}\psi_0 - \tau^*\mu G(\cdot,\eta^*)\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) \\ \qquad = \mu G(\cdot,\eta^*)Z(\eta^*), \\ (A+i\beta)\tilde{v} + \tilde{\lambda}\xi_0 + \tau^*\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) \\ \qquad = -\frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)Z(\eta^*) \\ (A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho_0 - \tau^* J(\cdot,\eta^*)\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) \\ \qquad = J(\cdot,\eta^*)Z(\eta^*), \\ (A_0+i\beta)\tilde{s} + \tilde{\lambda}\zeta_0 + \tau^*\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) \\ \qquad = -\frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)Z(\eta^*) \\ \tilde{\lambda} - \tau^*\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) = Z(\eta^*), \end{cases}$$

where $Z(\eta^*) = 4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \hat{\gamma}'(\eta^*)) - \kappa_2(s_0(\eta^*) + \hat{\alpha}'(\eta^*))$.

By letting $\phi := \psi_0 - \mu G(\cdot,\eta^*)$, $\xi = v_0 + \frac{\mu}{\eta^*}\hat{G}(\cdot,\eta^*)$, $\rho = w_0 - J(\cdot,\eta^*)$ and $\zeta = s_0 - \hat{J}(\cdot,\eta^*)$ as before, we obtain

$$(32)\qquad (A+i\beta)\tilde{u} + \tilde{\lambda}\phi = 0,$$

$$(33)\qquad (A+i\beta)\tilde{v} + \tilde{\lambda}\xi = 0,$$

$$(34)\qquad (A_0+i\beta)\tilde{w} + \tilde{\lambda}\rho = 0,$$

$$(35)\qquad (A_0+i\beta)\tilde{s} + \tilde{\lambda}\zeta = 0,$$

$$(36)\qquad \tilde{\lambda} - \tau^*\big(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)\big) = \tfrac{i\beta}{\tau^*}.$$

Multiplying (32) by $4\overline{\phi}$, (33) by $-\eta^*\kappa_1\overline{\xi}$ and (34) by $\mu\eta^*\kappa_2\overline{\zeta}$ and adding the resultants to each, we now obtain

$$-4\mu\tilde{u}(\eta^*) - \kappa_1\mu\,\tilde{v}(\eta^*) + \mu\eta^*\kappa_2\tilde{s}(\eta^*) + \tilde{\lambda}(4||\phi||^2 - \eta^*\kappa_1||\xi||^2 + \mu\eta^*\kappa_2||\zeta||^2)$$
$$+2i\beta\int\big(4\tilde{u}\overline{\phi} - \eta^*\kappa_1\tilde{v}\overline{\xi} + \mu\eta^*\kappa_2\tilde{s}\,\overline{\zeta}\big) = 0.$$

From (27) and (36), the above equation implies that

$$(37)\qquad i\beta\frac{\mu}{(\tau^*)^2} + 2i\beta\int\Big(4\tilde{u}\overline{\phi} - \eta^*\kappa_1\tilde{v}\overline{\xi} + \mu\eta^*\kappa_2\tilde{s}\,\overline{\zeta}\Big) = 0.$$

Multiplying (32) by $4\overline{\tilde{u}}$, (33) by $-\eta^*\kappa_1\overline{\tilde{v}}$ and (35) by $\mu\eta^*\kappa_2\overline{\tilde{s}}$ and summing their resultants to each, we now obtain

$$\begin{aligned}4||A^{1/2}\tilde{u}||^2 - \eta^*\kappa_1||A^{1/2}\tilde{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\tilde{s}||^2 \\ +i\beta\Big(4||\tilde{u}||^2 - \eta^*\kappa_1||\tilde{v}||^2 + \mu\eta^*\kappa_2||\tilde{s}||^2\Big) \\ +\tilde{\lambda}\int\big(4\tilde{u}\overline{\phi} - \eta^*\kappa_1\tilde{v}\overline{\xi} + \mu\eta^*\kappa_2\tilde{s}\,\overline{\zeta}\big) = 0.\end{aligned}$$

From (37), we have

$$\begin{aligned}4||A^{1/2}\tilde{u}||^2 - \eta^*\kappa_1||A^{1/2}\tilde{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\tilde{s}||^2 \\ +i\beta\Big(4||\tilde{u}||^2 - \eta^*\kappa_1||\tilde{v}||^2 + \mu\eta^*\kappa_2||\tilde{s}||^2\Big) = \tfrac{\mu}{2(\tau^*)^2}\,\tilde{\lambda}.\end{aligned}$$

The real part of the above is given by

$$4||A^{1/2}\tilde{u}||^2 - \eta^*\kappa_1||A^{1/2}\tilde{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\tilde{s}||^2 = \frac{\mu}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda}. \tag{38}$$

Now, multiplying (32) by $2i\beta\overline{\tilde{u}}$ and (33) by $\tilde{\lambda}\overline{\tilde{u}}$ and subtracting resultants from each other, we obtain

$$||A^{1/2}\tilde{u}||^2 - (\eta^*)^2||A^{1/2}\tilde{v}||^2 = 0 \quad\text{and}\quad ||\tilde{u}||^2 - (\eta^*)^2||\tilde{v}||^2 = 0$$

by (30). Thus (38) implies that

$$\frac{\mu}{2(\tau^*)^2}\mathrm{Re}\tilde{\lambda} = \big(4 - \frac{\kappa_1}{\eta^*}\big)||A^{1/2}\tilde{u}||^2 + \mu\kappa_2\,\eta^*||A_0^{1/2}\tilde{s}||^2$$

which is positive since $4 - \frac{\kappa_1}{\eta^*} > 0$. We have $\mathrm{Re}\lambda'(\tau^*) > 0$ for $\beta > 0$, and thus, by the Hopf-bifurcation theorem in [4], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. □

We shall now show that there exists a unique $\tau^* > 0$ such that $(0, 0, 0, \eta^*, \tau^*)$ is a Hopf point; thus τ^* is the origin of a branch of nontrivial periodic orbits.

LEMMA 4.4. *Suppose that* $4 - \frac{\kappa_1}{\eta^*} > 0$. *Let* G_β *and* $\hat{G}_\beta$ *be Green functions of the differential operator* $A + i\beta$ *satisfying (19) and (20), respectively. Then, the expression* $4\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*)) - \frac{\kappa_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$ *and* $\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ *are strictly decreasing in* $\beta \in \mathbb{R}^+$ *with*

$$\mathrm{Re}\,G_0(\eta^*,\eta^*) = G(\eta^*,\eta^*), \quad \lim_{\beta\to\infty}\mathrm{Re}\,G_\beta(\eta^*,\eta^*) = 0\,.$$

Moreover, $-4\,\mathrm{Im}G_\beta(\eta^*,\eta^*)+\frac{\kappa_1}{\eta^*}\,\mathrm{Im}\,\hat{G}_\beta(\eta^*,\eta^*)-\frac{\kappa_2}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*))>0$ *for* $\beta>0$.

Proof. First, we have $(A+i\beta)^{-1}=(A-i\beta)(A^2+\beta^2)^{-1}$, so if $L(\beta):=\mathrm{Re}\,(A+i\beta)^{-1}$, then $L(\beta)=A(A^2+\beta^2)^{-1}$. Moreover, $L(\beta)\longrightarrow A^{-1}$ as $\beta\to 0$ and $L(\beta)\longrightarrow 0$ as $\beta\to\infty$, which results from the corresponding limiting behavior for $\mathrm{Re}\,\big(G_\beta(\eta^*,\eta^*)\big)$.

Now to show that $\beta\mapsto(4\,\mathrm{Re}\,(G_\beta(\eta^*,\eta^*))-\frac{\kappa_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*))$ is strictly decreasing, define $h(\beta)(x):=4G_\beta(x,\eta^*)-\frac{\kappa_1}{\eta^*}\,\hat{G}_\beta(x,\eta^*)-4\,G(x,\eta^*)+\frac{\kappa_1}{\eta^*}\,\hat{G}(x,\eta^*)$. Then (in the weak sense initially)

$$(A+i\beta)h(\beta)=-4i\beta\,G(\cdot,\eta^*)+i\beta\tfrac{\kappa_1}{\eta^*}\hat{G}(\eta^*,\eta^*). \tag{39}$$

As a result $h(\beta)\in D(A)_{\mathbf{C}}$ and $h:\mathbb{R}^+\to D(A)_{\mathbf{C}}$ is differentiable with $ih(\beta)+(A+i\beta)h'(\beta)=-i\,G(\cdot,\eta^*)$, therefore

$$(A+i\beta)h'(\beta)=-i\big(4\,G_\beta(\cdot,\eta^*)-\tfrac{\kappa_1}{\eta^*}\hat{G}_\beta(\cdot,\eta^*)\big).$$

Thus, we get

$$\begin{aligned}-i\,(4-\tfrac{\kappa_1}{\eta^*})\overline{h'(\beta)(\eta^*)} &= \textstyle\int(A+i\beta)^2h'(\beta)\overline{h'(\beta)(r)}\,dr\\ &= \textstyle\int(A+i\beta)h'(\beta)\cdot(A+i\beta)\overline{h'(\beta)}\,dr\\ &= \textstyle\int|Ah'(\beta)|^2-\beta^2|h'(\beta)|^2\,dr+2i\beta\int Ah'(\beta)\overline{h'(\beta)}\,dr\,.\end{aligned} \tag{40}$$

From (40) it follows that

$$-(4-\tfrac{\kappa_1}{\eta^*})\mathrm{Re}\,(h'(\beta)(\eta^*))=2\beta\textstyle\int|A^{1/2}h'(\beta)|^2>0$$

and thus $\mathrm{Re}\,(h'(\beta)(\eta^*))<0$ if $4>\frac{\kappa_1}{\eta^*}$. In order to show $4\,\mathrm{Im}(G_\beta(\eta^*,\eta^*))-\frac{\kappa_1}{\eta^*}\,\mathrm{Im}(\,\hat{G}_\beta(\eta^*,\eta^*))<0$ for $\beta>0$, we multiply (39) by $\overline{h(\beta)(r)}$ and integrate the resulting equation, then

$$\begin{aligned}-i\,\beta(4-\tfrac{\kappa_1}{\eta^*})\overline{h(\beta)(\eta^*)} &= \int A(A+i\beta)h(\beta)(r)\overline{h(\beta)(r)}\,dr\\ &= \int|Ah(\beta)|^2+i\beta\int A|h(\beta)|^2,\end{aligned}$$

which implies that $-\beta(4-\frac{\kappa_1}{\eta^*})\mathrm{Im}h(\beta)(\eta^*)=\int|Ah(\beta)|^2>0$. Since $(4-\frac{\kappa_1}{\eta^*})>0$, we have $\mathrm{Im}h(\beta)(\eta^*)<0$ for $\beta>0$.

Similarly, we define $k(\beta)(r) := \hat{J}_\beta(r,\eta^*) - \hat{J}(r,\eta^*)$. Then

$$\begin{cases} \mathrm{Re}\,(k'(\beta)(\eta^*)) = 2\beta||A_0^{1/2}k'(\beta)||^2 > 0 \\ \mathrm{Im}k(\beta)(\eta^*) = ||A_0 k(\beta)||^2 > 0. \end{cases}$$

We have $\mathrm{Re}k'(\beta)(\eta^*) < 0$ and $\mathrm{Im}k(\beta)(\eta^*) < 0$ for $\beta > 0$. Thus $4\,\mathrm{Re}\,(G_\beta(\eta^*, \eta^*)) - \frac{\kappa_1}{\eta^*}\mathrm{Re}\,(\hat{G}_\beta(\eta^*,\eta^*)) + \frac{\kappa_2}{\eta^*}\mathrm{Re}(\hat{J}_\beta(\eta^*,\eta^*))$ is a strictly decreasing function of $\beta > 0$ and $-4\,\mathrm{Im}G_\beta(\eta^*,\eta^*) + \frac{\kappa_1}{\eta^*}\,\mathrm{Im}\,\hat{G}_\beta(\eta^*,\eta^*) - \frac{\kappa_2}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*)) > 0$ for $\beta > 0$ if $4 - \frac{\kappa_1}{\eta^*} > 0$. □

THEOREM 4.5. *Under the same condition as in Theorem 4.2, for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (11) with $\beta > 0$.*

Proof. We only need to show that the function $(u,v,w,s,\beta,\tau) \mapsto E(u,v,w,s,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system of equations (11) with $\lambda = i\beta$, $u = a - \mu G(\cdot,\eta^*)$, $v = p + \frac{\mu}{\eta^*}\,\hat{G}(\cdot,\eta^*)$, $w = b - J(\cdot,\eta^*)$ and $s = q + \frac{1}{\eta^*}\hat{J}(\cdot,\eta^*)$,

$$\begin{cases} (A+i\beta)a = -\mu\,\delta_{\eta^*}, \\ (A+i\beta)p = \frac{\mu}{\eta^*}\,\delta_{\eta^*}, \\ (A_0+i\beta)b = -\delta_{\eta^*}, \\ (A_0+i\beta)q = \frac{1}{\eta^*}\delta_{\eta^*}, \\ \frac{i\beta}{\tau^*} = 4(a(\eta^*) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1(p(\eta^*) - \frac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)) \\ \qquad -\kappa_2(q(\eta^*) - \frac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)). \end{cases}$$

The real and imaginary parts of the above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = 4\,\mathrm{Im}(-\mu G_\beta(\eta^*,\eta^*)) + \frac{\mu\kappa_1}{\eta^*}\mathrm{Im}(\hat{G}_\beta(\eta^*,\eta^*)) - \frac{\kappa_2}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*)) \\ 0 = 4(\mathrm{Re}(-\mu G_\beta(\eta^*,\eta^*)) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1\big(\mathrm{Re}(\frac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) \\ \qquad -\frac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)) - \kappa_2(\mathrm{Re}(\frac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) \\ \qquad -\frac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)). \end{cases}$$

Since $4\mathrm{Im}(-\mu G_\beta(\eta^*,\eta^*)) + \kappa_1\frac{\mu}{\eta^*}\,\mathrm{Im}(\,\hat{G}_\beta(\eta^*,\eta^*)) - \frac{\kappa_2}{\eta^*}\mathrm{Im}(\hat{J}_\beta(\eta^*,\eta^*)) > 0$ by Lemma 4.4, there is a critical point τ^*, provided the existence of β.

We now define

$$\begin{aligned} T(\beta) = 4(\mathrm{Re}(-\mu G_\beta(\eta^*,\eta^*)) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) \\ +\kappa_1(\mathrm{Re}(\tfrac{\mu}{\eta^*}\hat{G}_\beta(\eta^*,\eta^*)) - \tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)) \\ -\kappa_2(\mathrm{Re}(\tfrac{1}{\eta^*}\hat{J}_\beta(\eta^*,\eta^*)) - \tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)). \end{aligned}$$

Using Lemma 4.4, we have $T'(\beta) > 0$ for $\beta > 0$ and $T(0) = 4\gamma'(\eta^*) + \kappa_1\hat{\gamma}'(\eta^*) - \kappa_2\hat{\alpha}'(\eta^*) < 0$ by assumption.

$$\begin{aligned} \lim_{\beta\to\infty} T(\beta) &= 4(\mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1(-\tfrac{\mu}{\eta^*}\hat{G}(\eta^*,\eta^*) + \hat{\gamma}'(\eta^*)) \\ &\quad -\kappa_2(-\tfrac{1}{\eta^*}\hat{J}(\eta^*,\eta^*) + \hat{\alpha}'(\eta^*)) \\ &= (4 - \tfrac{\kappa_1}{\eta^*})(\mu G(\eta^*,\eta^*) + \gamma'(\eta^*)) + \kappa_1\hat{\gamma}'(\eta^*) + \tfrac{\kappa_2}{\eta^*}(J(\eta^*,\eta^*) \\ &\quad +\alpha'(\eta^*)) - \kappa_2\hat{\alpha}'(\eta^*) \end{aligned}$$

is positive since $4 > \frac{\kappa_1}{\eta^*}$. Hence, there exists a unique $\beta > 0$. □

The following theorem summarizes the results above.

THEOREM 4.6. *Suppose that* $\frac{\mu}{2(1+\mu)} < \frac{1}{2} - a_0 < \frac{1}{1+\mu}$ *and* $\mu\kappa_1 < \kappa_2$. *Then (8) and (6) have at least one stationary solution* $(u^*, v^*, w^*, s^*, \eta^*)$ *where* $u^* = v^* = w^* = s^* = 0$ *and* $(a^*, p^*, b^*, q^*, \eta^*)$ *for all* τ, *respectively. Moreover, assume that* $4 > \frac{\kappa_1}{\eta^*}$. *Then there exists a unique* τ^* *such that the linearization* $-\widetilde{A} + \tau^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0,0,0,0,\eta^*,\tau^*)$ *is then a Hopf point for (8), and there exists a* C^0*-curve of nontrivial periodic orbits for (8) and (6), bifurcating from* $(0,0,0,0,\eta^*,\tau^*)$ *and* $(u^*, v^*, w^*, s^*, \eta^*, \tau^*)$, *respectively.*

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YoonMee Ham
Department of Mathematics
Kyonggi University
Suwon 443-760, Republic of Korea
E-mail: ymham@kyonggi.ac.kr

Sang-Gu Lee
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Republic of Korea
E-mail: sglee@skku.edu