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ASYMPTIOTIC BEHAVIOR FOR THE VISCOELASTIC KIRCHHOFF TYPE EQUATION WITH AN INTERNAL TIME-VARYING DELAY TERM

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Abstract. In this paper, we study the viscoelastic Kirchhoff type equation with the following nonlinear source and time-varying delay

$$u_{tt} - M(x,t,\|\nabla u(t)\|^2)\triangle u + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau$$
$$+|u|^\gamma u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-s(t)) = 0.$$

Under the smallness condition with respect to Kirchhoff coefficient and the relaxation function and other assumptions, we prove the uniform decay rate of the Kirchhoff type energy.

1. Introduction

In the present work, we are concerned with the following problem:

$$u_{tt}(x,t) - M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) + \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau \quad (1)$$
$$+|u|^\gamma u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-s(t)) = 0 \quad \text{in} \quad \Omega\times\mathbb{R}^+,$$
$$u_t(x,t-s) = z_0(x,t) \quad \text{in} \quad \Omega\times[-s(0),0), \quad (2)$$
$$u(x,t) = 0 \quad \text{on} \quad \Gamma\times\mathbb{R}^+, \quad (3)$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \quad (4)$$

where Ω be a bounded open set of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\Gamma, \gamma > 0$, and other conditions such as M, h, a be in next section. Moreover, μ_1 and μ_2 are real numbers in that μ_1 is only a positive constant, $s > 0$ represents the time-varying delay. In fact, u_0, u_1 z_0 are initially given functions belonging to suitable space and $u(x,t)$ is the transversal displacement of the strip at spatial coordinate x and time t in the real world application.

Time delays so often arise in many physical chemical, biological, thermal and economical phenomena. In recent years, the control of PDEs with time

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delay effects has become an active area of research, see for instance [1, 2] and the references therein. The presence of delay may be a source of stability. An arbitrarily small delay may destrabilize a system which is preventing like stick-slip in the mass production process for mechanical engineering.

This problem has its origin in the mathematical description of system in real world from the mathematical modeling for axially moving viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. Furthermore, sourcing effects of stability are influenced by some time-varying delay. For these reasons, there are not exist weak or strong damping term in our problem (1)-(4). Our purpose is focused on not only memory effects but also time-varying delay for the problem otherwise the previous result [3]. Recently, problems with Timoshenko or basic hyperbolic type for viscoelastic materials have been considered by many authors (See [4, 5]). Besides, many engineering devices involve the transverse vibration of axially moving strings. Axially moving string is a typical model that is widely used, especially when the subject is long and narrow enough and has a negligible flexural rigidity, to represent threads, wires, magnetic tapes, belts, band saws, and cables. Various mathematical models and simulations have been established for a better understanding with linear or nonlinear dynamic behavior of these moving continua [6, 7, 8, 9, 10, 11, 12]. The mathematical model for axially moving strings was first introduced by Kirchhoff [13] (and see Carrier [6]), and the original equation is given in the form of

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left(p_0 \ + \ \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t; E, the young's modulus; ρ, the mass density; h, the cross section area; L, the length; and p_0, the initial axial tension. Recently, problems with the extended Kirchhoff type equation which is concerning axially moving heterogeneous or non heterogeneous materials (nonlinear vibrations of beams, strings, plates, and membranes) have been considered by many authors (See [14, 15, 16]).

In this paper, we will mainly concern on an aspect of decay rate of the Kirchhoff type energy of the viscoelastic system with an internal time-varying delay term. We get its proof by using the smallness condition functions with respect to Kirchhoff coefficient, the relaxation function and internal time-varying delay. In fact, the difference of the energy consist in Kirchhoff type potential energy and internal time-varying delay.

This paper organized as follows. In Section 2, we will present some notations, material needed (assumptions, lemmas and so on) for our work and state a global existence and energy decay rate theorem (main result). Section 3 contains the proof of our main result.

2. Preliminaries and main results

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^N$

$$\langle \varphi, \psi \rangle \equiv \int_{\Omega} (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}, \|u\|_{L^1(0,+\infty)}, \ \|v\|_{L^\infty(0,+\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively.

In the sequel we state the general hypotheses.

(A$_1$) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying $h(0) > 0$, and there exists positive constant $t_0, \zeta_1, \zeta_2, \zeta_3$ such that

$$-\zeta_1 \le h'(t) \le -\zeta_2 h(t), \quad \forall t > t_0,$$
$$0 \le h''(t) \le \zeta_3 h(t), \quad \forall t > t_0.$$

(A$_2$) $a : \Omega \to \mathbb{R}^+$ is a nonnegative bounded function and $a(x) \ge a_0 > 0$ on Ω with

$$\frac{m_0}{a_0} \ge 1 - \|a\|_\infty \int_0^\infty h(s) ds = l > 0,$$

where m_0 is in (B$_2$). And also, the following smallness condition satisfy

$$\epsilon_7 < a_0^2 \int_0^t h(s) ds.$$

(A$_3$) γ satisfies

$$0 \le \gamma \le \frac{2}{n-2}, \quad n \ge 3,$$
$$\gamma \ge 0, \quad n = 1, 2.$$

(A$_4$) The initial data satisfy

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \ u_1 \in H_0^1(\Omega).$$

(B$_1$) $M(x, t, \lambda)$ is a real-valued function of class C^2 on $x \in \overline{\Omega}, \ t \ge 0, \ \lambda \le 0$.

(B$_2$) $0 < m_0 \le M(x, t, \lambda) \le C_0 f(\lambda)$ with $M(x, t, \lambda) = M_1(x, t) + M_2(x, t, \lambda)$. And also, the following smallness condition satisfy

$$f(\lambda) < \sqrt{\frac{\frac{a_0 h(t)}{2} - C_p \widetilde{C_1} + \epsilon_2 \left(m_0 - \frac{1}{2}\right)}{\epsilon_3 \epsilon_8}}.$$

(B$_3$) $\frac{\partial M_1}{\partial t} \le 0, \ \left|\frac{\partial M_2}{\partial t}\right| \le C_1 g_1(\lambda), \ \left|\frac{\partial M}{\partial \lambda}\right| \le C_2 g_2(\lambda), \ 0 < m_1 \le M_x(x, t, \lambda)$.

(B$_4$) $f, g_1, g_2 \in C^1([0, +\infty); \mathbb{R}_+)$ are strictly increasing. Furthermore, $C_i \ (i = 0, 1, 2)$ is a positive constant.

(C$_1$) There exists a non-increasing differential function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t) > 0, h'(t) \le -\zeta(t) h(t) = 0, \quad \forall t > 0.$$

For the time-varying delay, we assume as in [1] that there exist positive constants $s_0, \overline{s}$ such that

$$0 < s_0 \le s(t) \le \overline{s}, \quad \forall t > 0. \tag{5}$$

Moreover, we assume that the speed of the delay satisfies

$$s'(t) \le d < 1, \quad \forall t > 0, \tag{6}$$

which is

$$s \in W^{2,\infty}([0,T]), \quad \forall t > 0$$

and that μ_1, μ_2 satisfy

$$|\mu_2| < \sqrt{1-d}\mu_1. \tag{7}$$

As in [1], let us introduce the function

$$z(x,\varrho,t) = u_t(x, t - s(t)\varrho), \quad x \in \Omega,\ \varrho \in (0,1),\ t > 0.$$

Then, the problem (1)-(4) is equivalent to

$$\begin{aligned}
& u_{tt}(x,t) - M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) + \int_0^t h(t-\tau) div[a(x)\nabla u(\tau)]d\tau && (8)\\
& \qquad +|u|^\gamma u + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty), &&\\
& s(t)z_t(x,\varrho,t) + (1 - s'(t)\varrho)z_\varrho(x,\varrho,t) \qquad \text{in} \quad \Omega \times (0,1) \times (0,+\infty), && (9)\\
& u_t(x,t) = z(x,0,t) \qquad \text{on} \quad \Omega \times (0,+\infty), && (10)\\
& z(x,\varrho,t) = z_0(x, -\varrho s(0)) \qquad \text{in} \quad \Omega \times (0,1), && (11)\\
& u(x,t) = 0 \qquad \text{on} \quad \Gamma \times [0,+\infty), && (12)\\
& u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{in} \quad \Omega, && (13)
\end{aligned}$$

In the following, we give a lemma which will be useful in this paper.

Lemma 2.1. *Denote* $(h \diamond u)(t) = \int_0^t h(t-\tau)\|\sqrt{a(x)}(u(t)-u(\tau))\|^2 d\tau$. *Then we have*

$$\begin{aligned}
\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = & -\frac{1}{2}\frac{d}{dt}\left[(h \diamond u)(t)\right] + \frac{1}{2}(h' \diamond u)(t)\\
& + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds\right]\\
& - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.
\end{aligned} \tag{14}$$

Proof. A direct computation shows that

$$\begin{aligned}\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau &= \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau) - a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\ &\quad + \int_0^t h(t-\tau)\langle a(x)\nabla u(t), \nabla u'(t)\rangle d\tau \\ &= -\frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2\right] d\tau \\ &\quad + \frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}\nabla u(t)\|^2\right] d\tau \\ &= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau\right] \\ &\quad + \frac{1}{2}\int_0^t h'(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau \\ &\quad + \frac{1}{2}\frac{d}{dt}\int_0^t h(t-\tau)\|\sqrt{a(x)}\nabla u(t)\|^2 d\tau \\ &\quad - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.\end{aligned}$$

□

Then, we can state our result as follows.

Theorem 2.2. *Let the assumptions* $(A_1), (A_4)$, (B_1)-(B_4) *and* (C_1) *hold. Then there exists a unique solution* u *of the problem* (8)-(13) *satisfying*

$$u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)),\ u' \in L^\infty(0,T; H_0^1(\Omega)),\ u'' \in L^\infty(0,T; L^2(\Omega)),$$

and

$$u(x,t) \to u_0(x)\ \text{in}\ H_0^1(\Omega) \cap H^2(\Omega); \qquad u'(x,t) \to u_1(x)\ \text{in}\ H_0^1(\Omega);$$
$$z(x,\varrho,t) \to z_0(x)\ \text{in}\ L^2(\Omega \times (0,1)),$$

as $t \to 0$.

Proof. By using Galerkin's approximation and a routine procedure similar to that of cite [4, 15], we can the global existence result for the solution subject to (1)-(4) under the assumptions (A_1)-(A_4), (B_1)-(B_4) and (C_1). □

Theorem 2.3. *Let* u *be the global solution of the problem* (1)-(4) *with the above all conditions. We define the Kirchhoff type energy functional* $E(t)$ *as*

$$\begin{aligned}E(t) &= \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{2}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right] \\ &\quad + \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s) dx ds,\end{aligned}$$

where ζ, η are suitable positive constants.
Then the energy functional decays exponentially to zero as the time goes to infinity, that is,

$$E(t) \le \kappa e^{-\vartheta t}, \ \forall t \ge 0$$

where κ, ϑ are positive constants.

3. Proof of Theorem 2.3 (Energy decay)

Proof. Multiplying u' on both sides of Eq.(1), integrating the resulting equations over Ω, and using the Green formula and (3), we have

$$\begin{aligned}
&\langle u''(t), u'(t)\rangle + \langle M(x,t,\|\nabla u(t)\|^2)\nabla u(t), \nabla u'(t)\rangle \\
&\qquad + \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
&\qquad - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau + \langle |u|^\gamma u, u'\rangle \\
&\qquad + \langle \mu_1 u_t(x,t) + \mu_2 u_t(x,t-s(t)), u'\rangle = 0,
\end{aligned} \tag{15}$$

that is

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t} M_1(x,t)|\nabla u(x,t)|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega \frac{\partial}{\partial t} M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\
&\quad + \left[\int_\Omega \frac{\partial}{\partial \lambda} M_2(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx\right]\langle \nabla u'(t), \nabla u(t)\rangle \\
&\quad - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
&\quad - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
&\quad + \frac{\zeta}{2}\int_\Omega u_t^2(t)dx - \frac{\zeta}{2}\int_\Omega e^{-\eta s(t)} u_t^2(t-s(t))(t-s'(t))dx \\
&\quad - \frac{\eta\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)} u_t^2(s) dx ds,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
E(t) &= \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right] \\
&\quad + \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s) dx ds.
\end{aligned} \tag{17}$$

From (B_3) and Hölder inequality, and (5), (6) and some mainipulations as in [1], we obtain

$$
\begin{aligned}
E'(t) \le\, & \|u(t)\|^2 \left\{ \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\| \right\} \\
& - \langle M_x(x,t,\|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau \\
& - \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2}\right)\int_\Omega u_t^2(t)dx \\
& - \left(e^{-\eta\bar{s}}\frac{\zeta(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_\Omega u_t^2(t-s(t))dx \\
& - \frac{\eta\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)}u_t^2(s)dxds.
\end{aligned} \tag{18}
$$

By (B_3), (14) and Young's inequality, we have

$$
\begin{aligned}
E'(t) \le & \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\
& - \frac{1}{2}\frac{d}{dt}\left[(h\diamond u)(t)\right] + \frac{1}{2}(h'\diamond\nabla u)(t) \\
& + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2\int_0^t h(s)ds\right] \\
& - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2 \\
& - \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\zeta}{2}\right)\int_\Omega u_t^2(t)dx \\
& - \left(e^{-\eta\bar{s}}\frac{\zeta(1-d)}{2} - \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_\Omega u_t^2(t-s(t))dx \\
& - \frac{\eta\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)}u_t^2(s)dxds,
\end{aligned} \tag{19}
$$

where

$$
\widetilde{C_1} = \frac{C_1}{2}g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\| \tag{20}
$$

is a positive constant. And ϵ_1 is also a positive constant.
Define the new energy functional $E_1(t)$ as follows

$$
E_1(t) = E(t) + \frac{1}{2}(h\diamond\nabla u)(t) - \frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2\int_0^t h(s)ds. \tag{21}
$$

For positive constants ϵ_2 and ϵ_3, let us define the perturbed modified energy by

$$
F(t) = E_1(t) + \epsilon_2\varphi(t) + \epsilon_3\psi(t), \tag{22}
$$

where

$$\varphi(t) = \langle u'(t), u(t)\rangle. \tag{23}$$

and

$$\psi(t) = -\int_0^t h(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau. \tag{24}$$

By using the Cauchy's inequality, Hölder inequality and Poincarè inequality, there exist positive constants α_1, α_2 such that for each $t > 0$

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t). \tag{25}$$

Proposition 3.1. (Energy equivalence)

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t) \quad \textit{for all } t \geq 0,$$

where α_1 and α_2 are positive constants.

Proof. Now, we will fix ζ in the energy $E(t)$ such that

$$2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}} - \zeta > 0, \tag{26}$$

$$\zeta - \frac{|\mu_2|}{\sqrt{1-d}} > 0 \tag{27}$$

and

$$\eta < \frac{1}{s}\left|\log \frac{|\mu_2|}{\zeta\sqrt{1-d}}\right|. \tag{28}$$

Then, similar as Proposition 3.1. in [3], we can choose two constants α_1 and α_2. In fact, the existence of such a constant η is guaranteed by the assumption (7). □

Then from (A_1) and (19), and (21) and (26)-(28), we have

$$\begin{aligned} E_1'(t) \leq & \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\ & - \frac{\zeta_2}{2}(h \diamond \nabla u)(t) - \frac{1}{2}a_0 h(t)\|\nabla u(t)\|^2 \\ & - C_2\int_\Omega [u_t^2(t) + u_t^2(t-s(t))]dx \\ & - \frac{\eta\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{-\eta(s-t)}u_t^2(s)dxds \\ \leq & \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\ & - \frac{\zeta_2}{2}(h \diamond \nabla u)(t) - \frac{1}{2}a_0 h(t)\|\nabla u(t)\|^2 - C_2\int_\Omega u_t^2(t-s(t))dx, \end{aligned} \tag{29}$$

where, C_2 is some positive constant. And also, by (A_2), the energy $E_1(t)$ is a positive functional. Applying Poincarè inequality to (29), we deduce

$$\tag{30} \begin{aligned} E_1'(t) \leq & \left(C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0 h(t) \right) \|\nabla u(t)\|^2 \\ & + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 - \frac{\zeta_2}{2}(h \diamond \nabla u)(t) - C_2 \int_\Omega u_t^2(t-s(t))dx, \end{aligned}$$

where C_p is the Poincarè coefficient. Meanwhile, we note from (A_1) and (A_2) that

$$\tag{31} \begin{aligned} E_1(t) \geq & \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\ & + \frac{1}{2}\left(1 - \|a\|_\infty \int_0^t h(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(h \diamond u)(t) \\ & + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} + \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s)dxds \\ \geq & l\Big[\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\ & + \frac{\zeta}{2}\int_{t-s(t)}^t \int_\Omega e^{\eta(s-t)} u_t^2(s)dxds\Big]. \end{aligned}$$

So, we deduce the relation $0 \leq E(t) \leq l^{-1}E_1(t)$. Therefore, the uniform decay of $E(t)$ is a result of the decay of $E_1(t)$.

In fact, using (1), we have

$$\tag{32} \begin{aligned} \varphi'(t) = & \langle u''(t), u(t)\rangle + \|u'(t)\|^2. \\ = & \|u'(t)\|^2 + \Big\langle u(t), M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) \\ & - \int_0^t h(t-\tau) div[a(x)\nabla u(\tau)]d\tau - |u(t)|^\gamma u(t) \\ & - \mu_1 u_t(x,t) - \mu_2 u_t(x,t-s(t))\Big\rangle \\ = & \|u'(t)\|^2 - \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(t)|^2 dx \\ & + \int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau), \nabla u(t)\right\rangle] d\tau - |u(t)|^\gamma u(t) \\ & - \mu_1 \int_\Omega u(t)u_t(t)dx - \mu_2 \int_\Omega u(t)u_t(t-s(t))dx. \end{aligned}$$

By Cauchy inequality and Young's inequality, we have

$$
\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau),\nabla u(t)\right\rangle] d\tau\right| \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{2}\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(\tau)-\nabla u(t)|+a(x)|\nabla u(t)|)d\tau\right\|^2 \\
\le&\frac{1}{2}\|\nabla u(t)\|^2+\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(\tau)-\nabla u(t)|d\tau\right\|^2 \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2,
\end{aligned}
\tag{33}
$$

where ϵ_6 with respect to Young's inequality is a positive constant. Using the assumption (A_2) and (33), we get

$$
\begin{aligned}
&\left|\int_0^t h(t-\tau)\left\langle a(x)\nabla u(\tau),\nabla u(t)\right\rangle] d\tau\right| \\
\le&\left(\frac{1}{2}+\frac{1}{8\epsilon_6}\right)\|a\|_\infty\int_0^t h(s)ds\int_0^t h(t-\tau)\left\|\sqrt{a(x)}(\nabla u(\tau)-\nabla u(t))\right\|^2 d\tau \\
&+\left(\frac{1}{2}+\frac{\epsilon_6}{2}\right)\|\nabla u(t)\|^2\left(\|a\|_\infty\int_0^t h(s)a(x)ds\right)^2+\frac{1}{2}\|\nabla u(t)\|^2 \\
\le&\frac{1}{2}(1+(1+\epsilon_6)(1-l)^2)\|\nabla u(t)\|^2+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t).
\end{aligned}
\tag{34}
$$

Also, using Young's and Poincaré's inequalities gives

$$
-\mu_1\int_\Omega u(t)u_t(t)dx\le\varepsilon\int_\Omega|\nabla u|^2dx+C(\varepsilon)\int_\Omega u_t^2(t)dx \tag{35}
$$

$$
-\mu_2\int_\Omega u(t)u_t(t-s(t))dx\le\varepsilon\int_\Omega|\nabla u|^2dx+C(\varepsilon)\int_\Omega u_t^2(t-s(t))dx \tag{36}
$$

By combining (32) and (34)-(36), we conclude

$$
\begin{aligned}
\varphi'(t)\le&(1+C(\varepsilon))\|u'(t)\|^2+\frac{1}{2}(1-2m_0+(1+\epsilon_6)(1-l)^2+2\varepsilon)\|\nabla u(t)\|^2 \\
&+\frac{(4\epsilon_6+1)(1-l)}{8\epsilon_6}(h\diamond\nabla u)(t)-\|u(t)\|_{\gamma+2}^{\gamma+2} \\
&+C(\varepsilon)\int_\Omega u_t^2(t-s(t))dx.
\end{aligned}
\tag{37}
$$

Next, we estimate $\psi'(t)$ as follows. In fact, using (1), we have

(38)
$$
\begin{aligned}
\psi'(t) =& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau. \\
& -\int_0^t h(t-\tau)\langle a(x)u''(t), u(t)-u(\tau)\rangle d\tau - \|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds \\
=& -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau. \\
& -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau \\
& -\left\langle \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau \right\rangle \\
& +\int_0^t h(t-\tau)\langle a(x)|u|^\gamma u, u(t)-u(\tau)\rangle d\tau \\
& -\|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds \\
& +\int_\Omega \left(\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))ds\right)[\mu_1 u_t(t) + \mu_2 u_t(t-s(t))]dx.
\end{aligned}
$$

Using Cauchy inequality, Poincarè inequality and (A_1), we have

$$
\begin{aligned}
&\left| -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t)-u(\tau)\rangle d\tau \right| \\
\le& \epsilon_7 \|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}\left\| \int_0^t h(t-\tau)a(x)|u(t)-u(\tau)|d\tau \right\|^2 \\
\le& \epsilon_7 \|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}(1-l)C_p^2 (h\diamond \nabla u)(t),
\end{aligned}
\tag{39}
$$

where ϵ_7 is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Similarly, using Cauchy inequality and (B_2), we get

$$
\begin{aligned}
&\left| -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t)-\nabla u(\tau)\rangle d\tau \right| \\
\le& \epsilon_8 f^2(\|\nabla u(t)\|^2)\|u'(t)\|^2 + \frac{C_0(1-l)}{4\epsilon_8}(h\diamond \nabla u)(t)
\end{aligned}
\tag{40}
$$

and

(41)
$$\begin{aligned}
&\left|-\left\langle\int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau\right\rangle\right| \\
\leq&\epsilon_9\left\|\int_0^t h(t-\tau)(a(x)|\nabla u(t)-\nabla u(\tau)|+a(x)|\nabla u(t)|)d\tau\right\|^2 \\
&+\frac{1}{4\epsilon_9}\left(\|a\|_\infty\int_0^t h(s)ds\right)\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(t)-\nabla u(\tau))\|^2 d\tau \\
\leq&2\epsilon_9\left(\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)-\nabla u(\tau)|d\tau\right\|^2+\left\|\int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau\right\|^2\right) \\
&+\frac{1-l}{4\epsilon_9}(h\diamond\nabla u)(t) \\
\leq&\left(2\epsilon_9+\frac{1}{4\epsilon_9}\right)(1-l)(h\diamond\nabla u)(t)+2\epsilon_9(1-l)^2\|\nabla u(t)\|^2,
\end{aligned}$$

where ϵ_8, ϵ_9 are positive constants with respect to Cauchy inequality. And also, using Cauchy inequality and Poincarè inequality, we have

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau\right| \\
\leq&\epsilon_{10}\|u(t)\|_{2(\gamma+1)}^{2(\gamma+1)}+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned} \tag{42}$$

where ϵ_{10} is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Noting $H^1(\Omega)\hookrightarrow L^{2(\gamma+1)}(\Omega)$ and using Poincarè inequality, (21), (29) and (42), we get

$$\begin{aligned}
&\left|\int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau\right| \\
\leq&\epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\|\nabla u(t)\|^2+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned} \tag{43}$$

where C_p is the Poincarè coefficient. And also, we get

$$\begin{aligned}
&\left|\int_\Omega\left(\int_0^t h(t-\tau)a(x)(u(t)-u(\tau))ds\right)[\mu_1 u_t(t)+\mu_2 u_t(t-s(t))]dx\right| \\
\leq&\epsilon_{10}\int_\Omega[u_t^2(t)+u_t^2(t-s(t))]dx+\frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond\nabla u)(t),
\end{aligned} \tag{44}$$

Combining (34)-(41) and (43)-(44) and also using (A_2), we deduce

(45)
$$\begin{aligned}\psi'(t) \le &\left(\epsilon_7 - a_0^2\int_0^t h(s)ds + \epsilon_{10}\right)\|u'(t)\|^2 \\ &+ \left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right)\|\nabla u(t)\|^2 \\ &+ \left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)(1-l)(h\diamond\nabla u)(t) \\ &+ \epsilon_{10}\int_\Omega u_t^2(t-s(t))dx.\end{aligned}$$

Combining (30), (22), (37) and (45), we deduce

$$\begin{aligned}&F'(t) = E_1'(t) + \epsilon_2\varphi'(t) + \epsilon_3\psi'(t) \\ &\le w_1\|u'(t)\|^2 + w_2\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx + w_3(h\diamond\nabla u(t)) \\ &\quad - \|u(t)\|_{\gamma+2}^{\gamma+2} + w_4\int_\Omega u_t^2(t-s(t))dx,\end{aligned} \tag{46}$$

where

$$w_1 = \frac{m_1}{4\epsilon_1} + (1+C(\varepsilon))\epsilon_2 + \epsilon_3\left(\epsilon_7 - a_0^2\int_0^t h(s)ds + \epsilon_{10}\right),$$

$$\begin{aligned}w_2 =& f(\|\nabla u(t)\|^2)C_0\left[C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0h(t)\right] \\ &+ \frac{\epsilon_2 f(\|\nabla u(t)\|^2)C_0}{2}(1 - 2m_0 + (1+\epsilon_6)(1-l)^2 + 2\varepsilon) \\ &+ \epsilon_3 f(\|\nabla u(t)\|^2)C_0\left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right),\end{aligned}$$

$$w_3 = -\frac{\zeta_2}{2} + \left[\frac{\epsilon_2(4\epsilon_6+1)}{8\epsilon_6} + \epsilon_3\left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)\right](1-l),$$

$$w_4 = \epsilon_2 C(\varepsilon) + \epsilon_3\epsilon_{10} - C_2$$

By using the smallness condition in (A_2) and (B_2), for the fixed $\epsilon_i, i = 1, 4, \cdots, 10$, we choose $\epsilon_j > 0, j = 2, 3$ and ε small enough such that $w_k < 0, k = 1, 2, 3, 4$. According to (21) and (46), there exist a positive constant s such that

$$F(t) \le -sE_1(t) \tag{47}$$

for all t which is larger than the fixed time T_0. We conclude from (25) and (47) that

$$F(t) \le -s\alpha_1 F(t)$$

for all t which is larger than the fixed time T_0. That is, for all t which is larger than the fixed time T_0,

$$F(t) \le F(T_0)e^{s\alpha_1 T_0}e^{-s\alpha_1 t}. \tag{48}$$

Therefore, we deduce from (25), (31) and (48) that there are positive constants κ and ϑ such that

$$E(t) \leq \kappa \exp\{-\vartheta t\} \quad for\ all\ t \geq 0\ and\ as\ t \to +\infty.$$

□

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