

A NOTE ON SPACES DETERMINED BY CLOSURE-LIKE OPERATORS

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ABSTRACT. In this paper, we study some classes of spaces determined by closure-like operators $[\cdot]_s$, $[\cdot]_c$ and $[\cdot]_k$ etc. which are wider than the class of Fréchet-Urysohn spaces or the class of sequential spaces and related spaces. We first introduce a WADS space which is a generalization of a sequential space. We show that X is a WADS and k -space iff X is sequential and every WADS space is C -closed and obtained that every WADS and countably compact space is sequential as a corollary. We also show that every WAP and countably compact space is countably sequential and obtain that every WACP and countably compact space is sequential as a corollary. And we show that every WAP and weakly k -space is countably sequential and obtain that X is a WACP and weakly k -space iff X is sequential as a corollary.

1. Introduction

All spaces considered here are assumed to be infinite and T_2 . Our terminology is standard and follows [19].

Let us recall some definitions.

A space X is *Fréchet-Urysohn*[2](also called *Fréchet*[8] or *closure sequential*[19]) iff for each subset A of X and each $x \in \overline{A} \setminus A$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of A which converges to x . X is *sequential*[8] iff for each non-closed subset A of X , there exist $x \in \overline{A} \setminus A$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of A such that $\{x_n\}_{n \in \mathbb{N}}$ converges to x in X . X has *countable tightness*[2](also called *closure countable*[19] or *c-space*[17]) iff for each subset A of X and each $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$. X is a *k-space*[1] iff for each subset A of X , $A = \{x \in X \mid x \in \overline{A \cap K}^K \text{ for some compact subset } K \text{ of } X\}$ implies $A = \overline{A}$, where $\overline{A \cap K}^K$ is the closure of $A \cap K$ in the subspace K of X .

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It is well-known that every first countable space is Fréchet-Urysohn, every Fréchet-Urysohn space is sequential and every sequential space is a k -space having countable tightness.

Many topologists have studied on spaces mentioned above and relationships among the spaces. In particular, A. V. Arhangel'skii and D. N. Stavrova characterized these spaces[1, 4] and A. Bella, V. V. Tkachuk and I. V. Yaschenko discussed the relationships among these spaces and AP or WAP spaces[5, 6, 18].

The purpose of this paper is to study some classes of spaces determined by closure-like operators $[\cdot]_s$, $[\cdot]_c$ and $[\cdot]_k$ etc. which are wider than the class of Fréchet-Urysohn spaces or the class of sequential spaces and related spaces.

We first introduce a WADS space which is a generalization of a sequential space and prove that X is a WADS and k -space iff X is sequential and every WADS space is C -closed and obtained that every WADS and countably compact space is sequential as a corollary. We also show that every WAP and countably compact space is countably sequential and obtain that every WACP and countably compact space is sequential as a corollary.

Next, we introduce kq -spaces, kq_1 -spaces, sc -spaces and sc_1 -spaces which are generalizations of Fréchet-Urysohn spaces and sequential spaces and determined by $[\cdot]_s$, $[\cdot]_c$ and $[\cdot]_k$.

We show that X is a kq - and k -space(a kq_1 - and k_1 -space) iff X is sequential(resp. Fréchet-Urysohn). And we show that every sc -space is countably sequential and X is an sc -space(an sc_1 -space) having countable tightness iff X is sequential(resp. Fréchet-Urysohn).

Finally, we introduce spaces having weakly countable tightness and weakly k -spaces which are generalizations of spaces having countable tightness and k -spaces, respectively and determined by $[\cdot]_c$ and $[\cdot]_k$.

We show that every weakly k -space having countable tightness is a k -space and every WAP and weakly k -space is countably sequential and obtain that X is a WACP and weakly k -space iff X is sequential as a corollary.

Moreover, we show that if X has countable tightness(or X is an AP space) and for each countable subset C of X , \overline{C} is compact, then for each subset A of X , $[A]_k = [A]_c$ and if for each subset A of a space X , $[A]_k = [A]_c$, then X is a weakly k -space having weakly countable tightness.

2. Results

Recall well-known operators $[\cdot]_s$ and $[\cdot]_c$ from the power set $\wp(X)$ of X into $\wp(X)$ itself defined as follows: for each subset A of a space X ,

$[A]_s = \{x \in X \mid \{x_n\}_{n \in \mathbb{N}}$ converges to x for some sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of $A\}$,

and

$[A]_c = \{x \in X \mid x \in \overline{C}$ for some countable subset C of $A\}$.

It is obvious that for each subset A of X , $A \subset [A]_s \subset [A]_c \subset \overline{A}$ and for each countable subset C of X , $[C]_c = \overline{C}$.

We now introduce operators $[\cdot]_{k^*}$, $[\cdot]_{k'}$ and $[\cdot]_k$ from the power set $\wp(X)$ of X into $\wp(X)$ itself defined as follows: for each subset A of a space X ,

$$[A]_{k^*} = \{x \in X \mid x \in \overline{A \cap \overline{K}^K} \text{ for some compact subset } K \text{ of } X\},$$

$$[A]_{k'} = \{x \in X \mid x \in \overline{A \cap \overline{K}} \text{ for some compact subset } K \text{ of } X\},$$

and

$$[A]_k = \{x \in X \mid \text{there exists a subset } K \text{ of } A \text{ such that } x \in \overline{K} \text{ and } \overline{K} \text{ is compact in } X\}.$$

We begin by showing the relation among $[\cdot]_{k^*}$, $[\cdot]_{k'}$ and $[\cdot]_k$.

Theorem 2.1. For each subset A of a space X , $[A]_{k^*} = [A]_{k'} = [A]_k$; i.e., $[\cdot]_{k^*} = [\cdot]_{k'} = [\cdot]_k$.

Proof. Let $x \in [A]_{k^*}$. Then there exists a compact subset K of X such that $x \in \overline{A \cap \overline{K}^K}$. Note that $\overline{A \cap \overline{K}^K} = \overline{A \cap \overline{K}} \cap K$. Thus $x \in \overline{A \cap \overline{K}}$. Hence $x \in [A]_{k'}$, and therefore $[A]_{k^*} \subset [A]_{k'}$.

Let $x \in [A]_{k'}$. Then there exists a compact subset K of X such that $x \in \overline{A \cap \overline{K}}$. Since X is T_2 , K is closed in X , and hence, $x \in \overline{A \cap \overline{K}} \subset \overline{K} = K$. By the compactness of K , $\overline{A \cap \overline{K}}$ is compact in X . Thus there exists a subset $A \cap K$ of A such that $x \in \overline{A \cap \overline{K}}$ and $\overline{A \cap \overline{K}}$ is compact and so $x \in [A]_k$. Therefore $[A]_{k'} \subset [A]_k$.

Let $x \in [A]_k$. Then there exists a subset K of A such that $x \in \overline{K}$ and \overline{K} is compact in X . It is enough to show that $x \in \overline{A \cap \overline{K}^K}$. Let U be an open subset of X with $x \in U$. Then $U \cap K \neq \emptyset$. Since $K \subset A$, $U \cap K \subset U \cap (A \cap \overline{K})$. Thus $U \cap (A \cap \overline{K}) \neq \emptyset$. Hence $x \in \overline{A \cap \overline{K}}$. Since $x \in \overline{K}$, $x \in \overline{A \cap \overline{K}} \cap \overline{K} = \overline{A \cap \overline{K}^K}$, and so $x \in [A]_{k^*}$. Therefore $[A]_k \subset [A]_{k^*}$. □

It is easy to check that for each subset A of a space X , $A \subset [A]_s \subset [A]_k \subset \overline{A}$ and $[\cdot]_c$ satisfies the Kuratowski topological closure axioms, but $[\cdot]_s$ and $[\cdot]_k$ do not satisfy, in general.

Recall that a space X is a k_1 -space [1, 4] iff for each subset A of X , the set $\{x \in X \mid x \in \overline{A \cap \overline{K}} \text{ for some compact subset } K \text{ of } X\}$ is closed in X ; i.e., for each subset A of X , $[A]_{k'} = \overline{A}$.

Every k_1 -space is obviously a k -space, but the converse is not true in general [1].

Using $[\cdot]_s$, $[\cdot]_c$ and $[\cdot]_k$ above, we have directly the following:

- (1) X is Fréchet-Urysohn iff for each subset A of X , $[A]_s = \overline{A}$.
- (2) X is sequential iff for each subset A of X , $A = [A]_s$ implies $A = \overline{A}$.
- (3) X is a k_1 -space iff for each subset A of X , $[A]_k = \overline{A}$.
- (4) X is a k -space iff for each subset A of X , $A = [A]_k$ implies $A = \overline{A}$.
- (5) X has countable tightness iff for each subset A of X , $A = [A]_c$ implies $A = \overline{A}$ iff $[A]_c = \overline{A}$ [11].

From the facts above, it is obvious that every Fréchet-Urysohn space is a k_1 -space and every k_1 -space is a k -space.

Recall that a space X is AP (standing for Approximation by Points)[6, 9, 15] (also called *Whyburn*[16]) iff for each subset A of X , $[A]_{AP} = \overline{A}$, where $[A]_{AP} = A \cup \{x \in \overline{A} \setminus A \mid \overline{B} = B \cup \{x\} \text{ for some subset } B \text{ of } A\}$. X is WAP (standing for Weak Approximation by Points)[6, 9, 15](also called *weakly Whyburn*[16]) iff for each subset A of X , $A = [A]_{AP}$ implies $A = \overline{A}$. X is $WACP$ (standing for Weak Approximation by Countable Points)[9] iff for each subset A of X , $A = [A]_{ACP}$ implies $A = \overline{A}$, where $[A]_{ACP} = A \cup \{x \in \overline{A} \setminus A \mid \overline{C} = C \cup \{x\} \text{ for some countable subset } C \text{ of } A\}$.

It is not difficult to see that every closed subspace of a $WACP$ space is $WACP$, every $WACP$ space is a WAP space having countable tightness. Every sequential space is $WACP$ and every AP space is WAP , but the converses are not true in general[9, 18].

In [18, Coro 2.3], the authors proved that X is an AP and k -space iff X is Fréchet-Urysohn.

From the following example, we see that a WAP and k -space need not be sequential, in general.

Example 2.2. The space of ordinals $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, is compact Hausdorff but not sequential[9]. Since X is compact, clearly, X is a k -space. Note that every scattered space(i.e., every subspace of the space has an isolated point) is WAP [18, Thm 2.7]. It is well-known that X is scattered. Thus, X is a WAP and k -space, but not sequential.

We recall that a space X is *discretely generated*[7] iff for each subset A of X and each $x \in \overline{A}$, there exists a discrete subset D of A such that $x \in \overline{D}$. X is *weakly discretely generated*[7] iff for each non-closed subset A of X , there exist $x \in \overline{A} \setminus A$ and a discrete subset D of A such that $x \in \overline{D}$. Using the following operator $[\cdot]_{DG}$ defined by for each subset A of a space X , $[A]_{DG} = \{x \in X \mid x \in \overline{D} \text{ for some discrete subset } D \text{ of } A\}$, we have that a space X is discretely generated(weakly discretely generated) iff for each subset A of X , $[A]_{DG} = \overline{A}$ (resp. $A = [A]_{DG}$ implies $A = \overline{A}$).

Let us introduce a generalization of a sequential space.

Definition 2.3. A space X is $WADS$ (standing for Weak Approximation by Discrete Subsets) iff for each subset A of X , $A = [A]_{ADS}$ implies $A = \overline{A}$, where $[A]_{ADS} = A \cup \{x \in \overline{A} \setminus A \mid \overline{D} = D \cup \{x\} \text{ for some discrete subset } D \text{ of } A\}$.

It is clear that every sequential space is $WADS$, every weakly discretely generated AP space is $WADS$ and every $WADS$ space is WAP and weakly discretely generated.

From example below, we have that a $WADS$ space is neither $WACP$ nor sequential, in general.

Example 2.4. Let $X = \{z\} \cup \mathbb{R}$, where \mathbb{R} is the set of real numbers with $z \notin \mathbb{R}$. We define a topology τ on X by for each $x \in \mathbb{R}$, $\{x\} \in \tau$ and $z \in U \in \tau$ iff $\mathbb{R} \setminus U$ is countable. Then X is Hausdorff, but X is neither $WACP$ nor sequential and

z is a unique non-isolated point in X [9, Exam 1.1(2)]. We now show that X is a WADS space. Let A be a subset of X and $z \in \bar{A} \setminus A$. Then, clearly, $\bar{A} = A \cup \{z\}$ and so $A \subset \mathbb{R}$. It follows that A is uncountable and discrete, and hence X is WADS.

We now study some properties of sequential spaces, k -spaces, WAP spaces, WACP spaces and WADS spaces and relationships among these spaces.

Theorem 2.5. X is a WADS and k -space iff X is sequential.

Proof. Let A be a subset of X with $\bar{A} \setminus A \neq \emptyset$. Then since X is a k -space, there exists $x \in [A]_k \setminus A$ and hence there exists a subset K of A such that $x \in \bar{K}$ and \bar{K} is compact. Put $K' = \bar{K} \cap A$. Then we have that $\bar{K}' = \bar{K}$ and $\bar{K}' \setminus K' = \bar{K}' \setminus A \subset \bar{A} \setminus A$. Since $\bar{K}' \setminus K' \neq \emptyset$ and X is WADS, there exist $y \in \bar{K}' \setminus K'$ and a discrete subset $D \subset K'$ such that $\bar{D} = D \cup \{y\}$. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence of distinct points of D . It is enough to show that $\{y_n\}_{n \in \mathbb{N}}$ converges to y . Suppose that $\{y_n\}_{n \in \mathbb{N}}$ does not converges to y . Then there exists an open set U in X such that $y \in U$ and $\{y_n\}_{n \in \mathbb{N}}$ is not eventually in U . Hence we can construct a subsequence $\{y_{\phi(n)}\}_{n \in \mathbb{N}}$ of $\{y_n\}_{n \in \mathbb{N}}$ such that $y_{\phi(n)} \notin U$ for all $n \in \mathbb{N}$. Since \bar{D} is compact and so countably compact, $\{y_{\phi(n)}\}_{n \in \mathbb{N}}$ has an accumulation point y_0 in \bar{D} . Since D is discrete and $\bar{D} = D \cup \{y\}$, we have easily that $y = y_0$. Thus, since $y_0 (= y)$ is an accumulation point of $\{y_{\phi(n)}\}_{n \in \mathbb{N}}$, $U \cap \{y_{\phi(n)} | n \in \mathbb{N}\} \neq \emptyset$. This is a contradiction.

The converse is trivial. □

Recall that a space X is C -closed [10, 13] iff every countably compact subset of X is closed.

It is well-known that every sequential space is C -closed and every subspace of a C -closed space is C -closed [13]. In [10, Thm 2.11], the author proved that every weakly discretely generated AP space is C -closed.

We now give a generalization of the above result of [10].

Theorem 2.6. Every WADS space is C -closed.

Proof. Suppose that there exists a non-closed countably compact subset A of a WADS space X . Then, since X is WADS, there exist $x \in \bar{A} \setminus A$ and a discrete subset D of A such that $\bar{D} = D \cup \{x\}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of distinct points of D . Then since A is countably compact, $\{x_n\}_{n \in \mathbb{N}}$ has an accumulation point x_0 in A . Since D is discrete and $\bar{D} = D \cup \{x\}$, we have clearly that $x = x_0$ and so $x \in A$. This is a contradiction. □

Clearly every weakly discretely generated AP space is WADS. From Thm 2.6, we obtain directly the following corollary and hence we omit the proof.

Corollary 2.7. [10] Every weakly discretely generated AP space is C -closed.

In [10], the author showed that every countably compact WAP and C -closed space is sequential. From Thm 2.6, we have immediately the following corollary and hence we omit the proof.

Corollary 2.8. Every WADS and countably compact space is sequential.

From the example below, we have that a weakly discretely generated WAP space need not be WADS, in general.

Example 2.9. The space X in Exam 2.2 is WAP and compact Hausdorff, but it is not sequential. Since X is compact and not sequential, by Coro 2.8, X is not WADS. In [7, Prop 4.1], the authors showed that every compact space is weakly discretely generated. Thus, X is weakly discretely generated and WAP, but not WADS.

We recall that a space X is *countably sequential*[12] iff for each non-closed countable subset C of X , there exist $x \in \overline{C} \setminus C$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of C such that $\{x_n\}_{n \in \mathbb{N}}$ converges to x ; i.e., for each countable subset C of X , $C = [C]_s$ implies $C = \overline{C}$.

It is clear that every sequential space is countably sequential. Every WACP and countably sequential space is sequential[12, Thm 2.7].

We give some sufficient conditions for a space X to be countably sequential.

It is well-known that every AP and compact space is Fréchet-Urysohn([6, Thm 1.1] and [18, Thm 2.2]). In Exam 2.2, the space X is WAP and compact, but not sequential. Thus, we know that a WAP and compact space need not be sequential, in general.

Theorem 2.10. Every WAP and countably compact space is countably sequential.

Proof. Let A be a countable subset of a WAP and countably compact space X with $\overline{A} \setminus A \neq \emptyset$. Since X is WAP, there exist $x \in \overline{A} \setminus A$ and a subset C of A such that $\overline{C} = C \cup \{x\}$. Since X is countably compact and \overline{C} is countable, \overline{C} is compact. It is well-known that every countable, locally compact and T_3 space is second countable[19, p.88]. It follows that \overline{C} is first countable. Thus, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of C such that $\{x_n\}_{n \in \mathbb{N}}$ converges to x . Therefore X is countably sequential. \square

From Thm 2.10, we have the following corollary.

Corollary 2.11. Every WACP and countably compact is sequential.

Proof. Let X be a WACP and countably compact space. Then, by Thm 2.10, X is countably sequential. By [12, Thm 2.7], X is sequential. \square

Remark 2.12. In [9], the author raised a question "Is every WACP and countably compact(or compact) space sequential?" From Coro 2.11, we obtain that the answer for this question is affirmative.

We now introduce some generalizations of a sequential space and a Fréchet-Urysohn space and study on properties of these spaces and relations among the spaces and related spaces.

Definition 2.13. (1) A space X is a kq -space[14] iff for each subset A of X , $A = [A]_s$ implies $A = [A]_k$.

(2) A space X is a kq_1 -space iff for each subset A of X , $[A]_s = [A]_k$.

(3) A space X is an sc -space iff for each subset A of X , $A = [A]_s$ implies $A = [A]_c$.

(4) A space X is an sc_1 -space iff for each subset A of X , $[A]_s = [A]_c$.

It is easy to check that every Fréchet-Urysohn space is a kq_1 -space, every sequential space is a kq -space and every kq_1 -space is a kq -space. It is clear that every sc_1 -space is an sc -space, every sequential space is an sc -space and every Fréchet-Urysohn space is an sc_1 -space.

Theorem 2.14. (1) X is a kq - and k -space iff X is sequential.

(2) X is a kq_1 - and k_1 -space iff X is Fréchet-Urysohn.

Proof. (1) See [14, Lemma 3.2].

(2) Let A be a subset of X . Then since X is a k_1 -space, $[A]_k = \overline{A}$. And since X is a kq_1 -space, $[A]_s = [A]_k$, and hence $[A]_s = \overline{A}$. Thus X is Fréchet-Urysohn.

The converse is trivial. □

It is obvious that every k_1 - and kq -space is sequential and every k - and kq_1 -space is sequential.

From Thm 2.14, we obtain the following corollary.

Corollary 2.15. (1) Let X be a k -space. Then, X is WADS iff X is a kq -space.

(2) Let X be a k_1 -space. Then, X is AP iff X is a kq_1 -space.

Proof. (1) From Thms 2.5 and 2.14(1), it follows.

(2) From [18, Coro 2.3], we have clearly that X is an AP and k_1 -space iff X is Fréchet-Urysohn. Thus, by Thm 2.14(2), it holds. □

We now study relations among sc -spaces, sc_1 -spaces, sequential spaces, Fréchet-Urysohn spaces and countably sequential spaces.

Theorem 2.16. Every sc -space is countably sequential.

Proof. Let C be a countable subset of an sc -space X with $\overline{C} \setminus C \neq \emptyset$. Then since C is countable, $[C]_c = \overline{C}$ and so $[C]_c \setminus C \neq \emptyset$. Since X is an sc -space, $[C]_s \setminus C \neq \emptyset$. Thus, it holds. □

Theorem 2.17. (1) X is an sc -space having countable tightness iff X is sequential.

(2) X is an sc_1 -space having countable tightness iff X is Fréchet-Urysohn.

Proof. (1) Let A be a subset of X with $\overline{A} \setminus A \neq \emptyset$. Then since X has countable tightness, we have that $[A]_c = \overline{A}$ and hence $[A]_c \setminus A \neq \emptyset$. Since X is an sc -space, $[A]_s \setminus A \neq \emptyset$. Thus X is sequential.

The converse is trivial.

(2) Let A be a subset of X . Then since X has countable tightness, $[A]_c = \overline{A}$. And since X is an sc_1 -space, $[A]_s = [A]_c$ and hence $[A]_s = \overline{A}$. Thus X is Fréchet-Urysohn.

The converse is trivial. \square

Next, we introduce generalizations of a space having countable tightness and a k -space and give some properties of the spaces and related spaces.

Definition 2.18. (1) A space X has *weakly countable tightness*[11] iff for each subset A of X , $A = [A]_c$ implies $A = [A]_k$.

(2) A space X is a *weakly k -space* iff for each subset A of X , $A = [A]_k$ implies $A = [A]_c$.

It is obvious that every space having countable tightness has weakly countable tightness, every k -space is a weakly k -space, every kq -space has weakly countable tightness and every sc -space is a weakly k -space.

In [11, Exam 2.15], the author showed that a space having weakly countable tightness need not have countable tightness in general.

For the space X , in Exam 2.4, we have that every countable subset C of X is closed and hence for each subset A of X , $A = [A]_c$. Thus, X is a weakly k -space. Note that only finite subsets of X are compact. Hence, $z \notin [\mathbb{R}]_k$; i.e., $\mathbb{R} = [\mathbb{R}]_k$. But, $\overline{\mathbb{R}} = X$. Thus, X is not a k -space. Therefore, we know that a weakly k -space need not be a k -space, in general.

From the following example, we have that a space having countable tightness need not be a weakly k -space in general.

Example 2.19. Let $C_p([0, 1])$ be the space of all real-valued continuous functions on $[0, 1]$ endowed with the topology of pointwise convergence. In [3], the author showed that $C_p([0, 1])$ has countable tightness[3, II.1.4.Coro], but it is not sequential[3, II.3.5. Lem]. He also showed that $C_p([0, 1])$ is sequential iff it is a k -space[3, II.3.7.Thm]. Hence we know that $C_p([0, 1])$ is not a k -space. Note that for each subset A of $C_p([0, 1])$, $\overline{A} = [A]_c$, but there exists a subset $B \subset C_p([0, 1])$ such that $B = [B]_k$ and $B \neq \overline{B}$. Thus $B = [B]_k$ does not imply $B = [B]_c$. Therefore $C_p([0, 1])$ is not a weakly k -space.

By definitions above, we have immediately the following theorem and hence we omit the proofs.

Theorem 2.20. (1) Every k -space having weakly countable tightness has countable tightness[11, Thm 2.18].

(2) Every weakly k -space having countable tightness is a k -space.

(3) Every sc -space having weakly countable tightness is a kq -space.

(4) Every kq - and weakly k -space is an sc -space.

Theorem 2.21. Every WAP and weakly k -space is countably sequential.

Proof. Let C be a countable subset of a WAP and weakly k -space X with $\overline{C} \setminus C \neq \emptyset$. Since X is WAP, there exist $x \in \overline{C} \setminus C$ and a subset B of C such

that $\overline{B} = B \cup \{x\}$. Since B is countable, $[B]_c = \overline{B}$. So $x \in [B]_c \setminus B$. Since X is a weakly k -space, $[B]_k \setminus B \neq \emptyset$. Hence there exists a subset $K \subset B$ such that $x \in \overline{K}$ and \overline{K} is compact. Since \overline{B} is countable and $\overline{K} \subset \overline{B}$, \overline{K} is also countable. By the similar argument of the proof of Thm 2.10, \overline{K} is first countable. Thus, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of K such that $\{x_n\}_{n \in \mathbb{N}}$ converges to x . Therefore X is countably sequential. \square

From Thm 2.21, we have the following corollaries.

Corollary 2.22. X is a WACP and weakly k -space iff X is sequential.

Proof. Since every WACP space is WAP, from Thm 2.21, we have that every WACP and weakly k -space is countably sequential. Note that every WACP and countably sequential space is sequential [12, Thm 2.7]. It follows that every WACP and weakly k -space is sequential.

The converse is trivial. \square

From Coro 2.22, we have immediately the following corollary.

Corollary 2.23. (1) X is a WACP and k -space iff X is sequential.

(2) Let X be a k -space. Then, X is WACP iff X is WADS iff X is a kq -space.

Proof. (1) By Coro 2.22, it is obvious.

(2) From Thm 2.5, Coro 2.15(1) and Coro 2.23(1), it follows. \square

Finally, we study on some properties of a space X satisfying the following property: for each subset A of X , $[A]_k = [A]_c$.

It is obvious that every Fréchet-Urysohn space, every kq_1 - and sc_1 -space, and every k_1 -space having countable tightness have the property above.

Theorem 2.24. (1) If X has countable tightness and for each countable subset C of X , \overline{C} is compact, then for each subset A of X , $[A]_k = [A]_c$.

(2) If X is an AP space and for each countable subset C of X , \overline{C} is compact, then for each subset A of X , $[A]_k = [A]_c$.

Proof. (1) Let A be a subset of X . Then since X has countable tightness, $[A]_c = \overline{A}$ and so, $[A]_k \subset [A]_c$.

Let $x \in [A]_c \setminus A$. Then there exists a countable subset C of A such that $x \in \overline{C}$. By hypothesis, \overline{C} is compact, and hence, $x \in [A]_k$. Thus, $[A]_c \subset [A]_k$.

(2) Let $x \in [A]_k \setminus A$. Then there exists a subset K of A such that $x \in \overline{K}$ and \overline{K} is compact. Since X is AP, \overline{K} is AP. It is well-known that every compact and AP space is Fréchet-Urysohn, and hence there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of K such that $\{x_n\}_{n \in \mathbb{N}}$ converges to x . It follows that $x \in \overline{\{x_n \mid n \in \mathbb{N}\}}$, and so $x \in [A]_c$. Thus, $[A]_k \subset [A]_c$.

By (1) above, we know that $[A]_c \subset [A]_k$. \square

Remarks 2.25. (1) It is well-known that having countable tightness and being AP are independent [9, Exam 1.1(2)].

(2) It is obvious that if for each subset A of a space X , $[A]_k \subset [A]_c$, then X has weakly countable tightness, and if for each subset A of X , $[A]_c \subset [A]_k$, then X is a weakly k -space. We have that if for each subset A of a space X , $[A]_k = [A]_c$, then X is a weakly k -space having weakly countable tightness.

(3) From Thm 2.24(1), we have that if X is a compact space having countable tightness, then for each subset A of X , $[A]_k = [A]_c$. But, the converse is not true. For, for each subset A of the space X , in Exam 2.4, $[A]_k = [A]_c$, but X is neither compact nor having countable tightness.

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References

- [1] A. V. Arhangel'skii, A characterization of very k -spaces, Czechoslovak Math. J. 18(93)(1968), 392-395.
- [2] A. V. Arhangel'skii and L.S.Pontryagin(Eds.), General Topology I, Encyclopaedia of Mathematical Sciences, vol. 17, Springer-Verlage, Berlin, 1990.
- [3] A. V. Arhangel'skii, Topological function spaces, Mathematics and its application, Vol.78, Kluwer Academic Publishers, 1992.
- [4] A. V. Arhangel'skii and D. N. Stavrova, On a common generalization of k -spaces and spaces with countable tightness, Top. and its Appl. 51(1993), 261-268.
- [5] A. Bella, On spaces with the property of weak approximation by points, Comment. Math. Univ. Carolinae 35(2)(1994), 357-360.
- [6] A. Bella and I. V. Yaschenko, On AP and WAP spaces, Comment. Math. Univ. Carolinae 40(3)(1999), 531-536.
- [7] A. Dow, M. G. Tkachenko, V. V. Tkachuk and R. G. Wilson, Topologies generated by discrete subspaces, Glansnik Math. 37(57)(2002), 187-210.
- [8] S. P. Franklin, Spaces in which sequences suffice, Fund. Math. 57(1965), 107-115.
- [9] W. C. Hong, Generalized Fréchet-Urysohn spaces, J. Korean Math. Soc. 44(2)(2007), 261-273.
- [10] W. C. Hong, On spaces in which compact-like sets are closed, and related spaces, Commun. Korean Math. Soc. 22(2)(2007), 297-303.
- [11] W. C. Hong, On spaces which have countable tightness and related spaces, Honam Math. J. 34(2)(2012), 199-208.
- [12] W. C. Hong and S. Kwon, A generalization of a sequential space and related spaces, Honam Math. J. 36(2)(2014), 425-434.
- [13] M. Ismail and P. Nyikos, On spaces in which countably compact sets are closed, and hereditary properties, Top. and its Appl. 11(1980), 281-292.
- [14] S. Lin and C. Zheng, The k -quotient images of metric spaces, Commun. Korean Math. Soc. 27(2012), 377-384.
- [15] M. A. Moon, M. H. Cho and J. Kim, On AP spaces in concern with compact-like sets and submaximality, Comment. Math. Univ. Carolinae 52(2)(2011), 293-302.
- [16] J. Pelant, M. G. Tkachenko, V. V. Tkachuk and R. G. Wilson, Pseudocompact Whyburn spaces need not be Fréchet, Proc. Amer. Math. Soc. 131(2003), no.10,3257-3265.
- [17] Y. Tanaka, Necessary and sufficient conditions for products of k -spaces, Top. Proceedings 14(1989), 281-313.
- [18] V. V. Tkachuk and I. V. Yaschenko, Almost closed sets and topologies they determine, Comment. Math. Univ. Carolinae 42(2)(2001), 395-405.
- [19] A. Wilansky, Topology for analysis, Ginn and Company 1970.

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