East Asian Math. J.
Vol. 32 (2016), No. 3, pp. 319–332
http://dx.doi.org/10.7858/eamj.2016.024

EXISTENCE OF FIXED POINTS OF SET-VALUED MAPPINGS IN b-METRIC SPACES

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Abstract. In this paper, we introduce the notion of generalized $\alpha - \psi$-Geraghty multivalued mappings and investigate the existence of a fixed point of such multivalued mappings. We present a concrete example and an application on integral equations illustrating the obtained results.

1. Introduction

The notion of b-metric was proposed by Czerwik [10, 11] to generalize the concept of a distance. The analog of the famous Banach fixed point theorem was proved by Czerwik in the frame of complete b-metric spaces. Following these initial papers, the existence and the uniqueness of (common) fixed points for the classes of both singlevalued and multivalued operators in the setting of (generalized) b-metric spaces have been investigated extensively, (see e.g. [1, 3, 4, 5, 9, 13, 14, 15, 16, 24, 26, 27] and related references therein.)

Recently, Samet *et al.* [25] introduced the notion of α-admissible mappings to combine some existing fixed point results in distinct setting. This idea was extended by Karapınar and Samet in [17] by introducing the notion of generalized $\alpha - \psi$-contractive type mappings. Following this trend several interesting results have been reported, see e.g. [2, 6, 7, 8, 18, 19, 20, 21, 22] and related references therein.

In this manuscript, we introduce the notion of generalized $\alpha - \psi$-Geraghty multivalued mappings in the context of complete b-metric spaces and examine the existence of fixed points for such mappings.

Throughout the paper, the standard letters $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}_0$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers, the set of all nonnegative integer numbers and the set of all positive integer numbers, respectively.

In what follows, we collect some basic notions, notations and fundamental results in the literature.

Received July 15, 2015; Accepted February 22, 2016.
2010 *Mathematics Subject Classification.* 46T99, 47H10, 54H25.
Key words and phrases. b-metric space, generalized $\alpha-\psi$-Geraghty multivalued mappings, fixed point.

(pISSN 1226-6973, eISSN 2287-2833)

Definition 1. [11] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to \mathbb{R}_0^+$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:
(bM_1) $d(x, y) = 0$ if and only if $x = y$;
(bM_2) $d(x, y) = d(y, x)$;
(bM_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
In this case, the pair (X, d) is called a b-metric space (with constant s).

Remark 1. Since a metric space turns into a b-metric space by taking the constant $s = 1$, the class of b-metric spaces is larger than the class of metric spaces.

The following example shows that there exists a b-metric which is not a metric.

Example 1.1. *Let $X = \{a, b, c\}$ with $0 < a < 2b < c$ and $d : X \times X \to [0, \infty)$ be defined by*
$$d(a, b) = b, \quad d(a, c) = \frac{b}{2} \quad \textit{and} \quad d(b, c) = c,$$
with $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Notice that d is not a metric since $d(b, c) > d(a, b) + d(a, c)$. However, it is easy to see that d is a b-metric space with $s \geq 2$.

One of the most interesting example of b-metric is the following.

Example 1.2. *Let $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined by*
$$d(a, b) = |x^2 - y^2|, \text{ for all } x, y \in X.$$
It is clear that d is not a metric, but it is easy to see that d is a b-metric space with $s \geq 2$.

Let (X, d) be a b-metric space. Take $P_{b,cl}(X)$ the set of bounded and closed sets in X. For $x \in X$ and $A, B \in P_{b,cl}(X)$, as in [10], we define
$$D(x, A) = \inf_{a \in A} d(x, a),$$
$$D(A, B) = \sup_{a \in A} D(a, B).$$
Define a mapping $H : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, \infty)$ such that
$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, B)\},$$
for every $A, B \in CB(X)$. Then, the mapping H forms a b-metric and it is called as the Hausdorff b-metric induced by the b-metric d.

Again, from [10], we cite the following lemmas.

Lemma 1.3. [10] *Let (X, d) be a b-metric space. For any $A, B \in P_{b,cl}(X)$ and any $x, y \in X$, we have the following:*
(1) *$D(x, B) \leq d(x, b)$ for any $b \in B$,*
(2) *$D(x, B) \leq H(A, B)$,*
(3) *$D(x, A) \leq s(d(x, y) + D(y, B)$.*

Lemma 1.4. [10] *Let A and B be nonempty closed and bounded subsets of a b-metric space (X, d) and $q > 1$. Then, for all $a \in A$, there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

Definition 2. [23] Let (X, d) be a b-metric space. X is said α-regular, if for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Definition 3. [21] Let $T : X \to P_{b,cl}(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a given function. Then T is said to be α-admissible if

$$(T3) \;\; \alpha(x, y) \geq 1 \text{ for all } y \in Tx, \Rightarrow \alpha(y, z) \geq 1, \text{ for all } z \in Ty.$$

In this paper, we introduce a generalized multivalued contraction via α-admissible mappings in the setting of b-metric spaces and we establish several fixed point results: existence and uniqueness. A concrete example and an application on integral equation are also provided illustrating the obtained results.

2. Main results

In this section, we introduce first the notion of a generalized $\alpha - \psi$-Geraghty [12] contraction type multivalued mapping in the setting of b-metric spaces. After then, we state and prove our main result.

Let Ψ be set of all increasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following property: $\psi(ct) \leq c\psi(t)$ for all $c > 1$. We denote by $\mathcal{F}$ the family of all functions $\beta : [0, \infty) \to [0, \frac{1}{s^2})$ for some $s \geq 1$.

Definition 4. Let (X, d) be a b-metric space and $T : X \to P_{b,cl}(X)$ be a multivalued mapping. We say that T is a generalized $\alpha{-}\psi$-Geraghty contraction type multivalued mapping whenever there exist $\alpha : X \times X \to [0, \infty)$ and some $L \geq 0$ such that for

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\} \tag{1}$$

$$\text{and } N(x, y) = \min\{D(x, Tx), D(y, Tx)\}, \tag{2}$$

we have

$$\alpha(x, y)\psi(s^3 H(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y)), \tag{3}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$.

Remark 2. The functions belonging to $\mathcal{F}$ are strictly smaller than $\frac{1}{s^2}$. Then, the expression $\beta(\psi(M(x, y)))$ in (3) satisfies

$$\beta(\psi(M(x, y))) < \frac{1}{s^2} \text{ for any } x, y \in X \text{ with } x \neq y.$$

Theorem 2.1. *Let (X, d) be a complete b-metric space and $T : X \to P_{b,cl}(X)$ be a generalized $\alpha-\psi$-Geraghty contraction type multivalued mapping such that*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) T is continuous .
Then, T has a fixed point.

Proof. For $s = 1$, the proof is too similar to a paper of Mohammadi et al. [22]. So, from now on, we suppose that $s > 1$. By condition (ii), there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_1 = x_0$, then we have nothing to prove. Let $x_1 \neq x_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Now, assume that $x_1 \notin Tx_1$. Let us take a real q such that $1 < q < s$. Then

$$0 < \psi(D(x_1, Tx_1)) \leq \alpha(x_0, x_1)\psi(H(Tx_0, Tx_1)) < q\alpha(x_0, x_1)\psi(s^3 H(Tx_0, Tx_1)).$$

Hence, there exists $x_2 \in Tx_1$ such that

$$\begin{aligned} \psi(d(x_1, x_2)) &< q\alpha(x_0, x_1)\psi(s^3 H(Tx_0, Tx_1)) \\ &\leq q\beta(\psi(M(x_0, x_1)))\psi(M(x_0, x_1)) + qL\phi(N(x_0, x_1)) \\ &< \frac{q}{s^2}\psi(M(x_0, x_1)) + qL\phi(N(x_0, x_1)), \end{aligned} \tag{4}$$

where

$$\begin{aligned} M(x_0, , x_1) &= \max\{d(x_0, , x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2s}\} \\ &\leq \max\{d(x_0, , x_1), D(x_1, Tx_1), \frac{D(x_0, Tx_1)}{2s}\} \\ &\leq \max\{d(x_0, , x_1), D(x_1, Tx_1), \frac{D(x_0, Tx_1)}{2s}\} \end{aligned}$$

and

$$\begin{aligned} N(x_0, , x_1) &= \min\{D(x_0, Tx_0), D(x_1, Tx_0)\} \\ &\leq \min\{d(x_0, , x_1), d(x_1, x_1)\} = 0. \end{aligned}$$

Since

$$\frac{D(x_0, Tx_1)}{2s} \leq \frac{s[d(x_0, x_1) + D(x_1, Tx_1)]}{2s} \leq \max\{d(x_0, x_1), D(x_1, Tx_1)\},$$

then we get

$$M(x_0, , x_1) \leq \max\{d(x_0, , x_1), D(x_1, Tx_1)\}.$$

If $\max\{d(x_0, , x_1), D(x_1, Tx_1)\} = D(x_1, Tx_1)$, then by (4), we have

$$\psi(D(x_1, Tx_1)) \leq \psi(d(x_1, x_2)) < \frac{q}{s^2}\psi(D(x_1, Tx_1)) < \psi(D(x_1, Tx_1)),$$

which is a contradiction. Hence, we obtain $\max\{d(x_0, , x_1), D(x_1, Tx_1)\} = d(x_0, x_1)$ and then by (4),

$$\psi(d(x_1, x_2)) \leq \frac{q}{s^2}\psi(d(x_0, x_1)).$$

Having in mind that $\psi \in \Psi$ and $\frac{q}{s^2} < 1$, hence

$$\psi(\frac{s^2}{q}d(x_1,x_2)) \leq \frac{s^2}{q}\psi(d(x_1,x_2)) < \psi(d(x_0,x_1)). \tag{5}$$

Since ψ is increasing, we have

$$d(x_1,x_2) \leq \frac{q}{s^2}d(x_0,x_1).$$

Recall that $x_2 \in Tx_1$ and $x_1 \notin Tx_1$, so it is clear that $x_2 \neq x_1$. Put

$$q_1 = \frac{\frac{q}{s^2}\psi(d(x_0,x_1))}{\psi(d(x_1,x_2))}.$$

By (5), we have $q_1 > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Assume that $x_2 \notin Tx_2$. Then,

$$0 < \psi(d(x_2,Tx_2)) \leq \alpha(x_1,x_2)\psi(H(Tx_1,Tx_2)) < q_1\alpha(x_1,x_2)\psi(s^3H(Tx_1,Tx_2)).$$

Hence, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned}\psi(d(x_2,x_3)) &< q_1\alpha(x_1,x_2)\psi(s^3H(Tx_1,Tx_2))\\ &\leq q_1\beta(\psi(M(x_1,x_2)))\psi(M(x_1,x_2)) + q_1L\phi(N(x_1,x_2))\\ &< \frac{q_1}{s^2}\psi(M(x_1,x_2)) + q_1L\phi(N(x_1,x_2)).\end{aligned}$$

Similarly, $M(x_1,x_2) \leq d(x_1,x_2)$ and $N(x_1,x_2) = 0$. So in addition to (4),

$$\psi(d(x_2,x_3)) \leq \frac{q_1}{s^2}\psi(d(x_1,x_2)) \leq (\frac{q}{s^2})^2\psi(d(x_0,x_1)).$$

Again by (5), we obtain

$$d(x_2,x_3) \leq (\frac{q}{s^2})^2 d(x_0,x_1)$$

It is clear that $x_2 \neq x_1$. Put

$$q_2 = \frac{(\frac{q}{s^2})^2\psi(d(x_0,x_1))}{\psi(d(x_2,x_3))}.$$

Then $q_2 > 1$. If $x_3 \in Tx_3$, then x_3 is a fixed point of T. Assume that $x_3 \notin Tx_3$. Then,

$$0 < \psi(d(x_3,Tx_3)) \leq \alpha(x_2,x_3)\psi(H(Tx_2,Tx_3)) < q_2\alpha(x_2,x_3)\psi(s^3H(Tx_2,Tx_3)).$$

Thus, there exists $x_4 \in Tx_3$ such that

$$\begin{aligned}\psi(d(x_3,x_4)) &< q_2\alpha(x_2,x_3)\psi(s^3H(Tx_2,Tx_3))\\ &\leq q_2\beta(\psi(M(x_2,x_3)))\psi(M(x_2,x_3)) + q_2L\phi(N(x_2,x_3))\\ &< \frac{q_2}{s^2}\psi(M(x_2,x_3)) + q_2L\phi(N(x_2,x_3)).\end{aligned} \tag{6}$$

Similarly $M(x_2,x_3) \leq d(x_2,x_3)$ and $N(x_2,x_3) = 0$. So by (6),

$$\psi(d(x_3,x_4)) \leq \frac{q_2}{s^2}\psi(d(x_2,x_3)) \leq (\frac{q}{s^2})^3\psi(d(x_0,x_1)).$$

Similarly, from (5), we obtain

$$d(x_3,x_4) \le (\frac{q}{s^2})^3 d(x_0,x_1).$$

It is clear that $x_3 \neq x_2$. Put

$$q_3 = \frac{(\frac{q}{s^2})^3 \psi(d(x_0,x_1))}{\psi(d(x_2,x_3))}.$$

Then $q_3 > 1$. By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n,x_{n+1}) < (\frac{q}{s^2})^n d(x_0,x_1)$ for all n.
For $n < m$, by the triangle inequality

$$\begin{aligned} d(x_n,x_m) &\le sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \dots \\ &+ s^{m-n-2}[d(x_{m-2},x_{m-1}) + d(x_{m-1},x_m)]) \\ &\le s(\frac{q}{s^2})^n (1 + s(\frac{q}{s^2}) + s^2(\frac{q}{s^2})^2 + \dots) d(x_0,x_1) \\ &= [\frac{s(\frac{q}{s^2})^n}{1 - s(\frac{q}{s^2})}] d(x_0,x_1) \to 0 \ as \ n \to \infty. \end{aligned}$$

Therefore, for $n < m$, we obtain

$$d(x_n,x_m) \to 0 \ as \ n \to \infty. \tag{7}$$

Therefore

$$\lim_{m,n\to\infty} d(x_n,x_m) = 0.$$

We deduce that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete b-metric space, so there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. The mapping T is continuous, so

$$D(x^\star, Tx^\star) = \lim_{n\to\infty} D(x_{n+1}, Tx^\star) \le \lim_{n\to\infty} H(Tx_n, Tx^\star) = 0$$

and so $x^\star \in Tx^\star$. □

It is possible to remove the continuity of the mapping T in the above theorem by replacing it with a suitable new condition, that is, X is α-regular.

Theorem 2.2. *Let (X,d) be a complete b-metric space and $T : X \to P_{b,cl}(X)$ be a generalized $\alpha - \psi$-Geraghty contraction type multivalued mapping such that*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;
(iii) X is α-regular.
Then T has a fixed point.

Proof. Following the lines in the proof of Theorem 2.1, we conclude that $\lim_{n\to\infty} x_n = x^*$. If X is α-regular, then since $\alpha(x_n,x_{n+1}) \ge 1$, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, x^*) \ge 1, \tag{8}$$

for all k. By triangular inequality

$$\begin{aligned} D(x^*, Tx^*) &\leq sd(x^*, x_{n_k+1}) + sD(x_{n_k+1}, Tx^*) \\ &\leq sd(x^*, x_{n_k+1}) + sH(Tx_{n_k}, Tx^*). \end{aligned}$$

Let k tend to infinity

$$D(x^*, Tx^*) \leq \lim_{k\to\infty} sH(Tx_{n_k}, Tx^*). \tag{9}$$

Having $\psi \in \Psi$, (8) and (9), so

$$\begin{aligned} \psi(s^2 D(x^*, Tx^*)) \leq \lim_{k\to\infty} \psi(s^3 H(Tx_{n_k}, Tx^*)) &\leq \lim_{k\to\infty} \alpha(x_{n_{k+1}}, x^*)\psi(sH(Tx_{n_k}, Tx^*)) \\ &\leq \lim_{k\to\infty} [\beta(\psi(M(x_{n_k}, x^*)))\psi(M(x_{n_k}, x^*)) + L\phi(N(x_{n_k}, x^*))]. \end{aligned} \tag{10}$$

We have

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*), \frac{D(x_{n_k}, Tx^*) + D(x^*, Tx_{n_k})}{2s}\} \\ &\leq \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_{k+1}}), D(x^*, Tx^*), \frac{D(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} N(x_{n_k}, x^*) &= \min\{D(x_{n_k}, Tx_{n_k}), D(x^*, Tx_{n_k})\} \\ &\leq \min\{d(x_{n_k}, x_{n_{k+1}}), d(x^*, x_{n_{k+1}})\}. \end{aligned}$$

Recall that

$$\frac{D(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \leq \frac{sd(x_{n_k}, x^*) + sD(x^*, Tx^*) + d(x^*, x_{n_{k+1}})}{2s}.$$

Then, by (7), we get that

$$\limsup_{k\to\infty} \frac{D(x_{n_k}, Tx^*) + d(x^*, x_{n_{k+1}})}{2s} \leq \frac{D(x^*, Tx^*)}{2}.$$

When k tends to infinity, we deduce

$$\lim_{k\to\infty} M(x_{n_k}, x^*) = D(x^*, Tx^*),$$

and

$$\lim_{k\to\infty} N(x_{n_k}, x^*) = 0.$$

Since $\lim_{k\to\infty} \beta(\psi(M(x_{n_k}, x^*))) \leq \dfrac{1}{s^2}$, so by (10)

$$\psi(s^2 D(x^*, Tx^*)) \leq \frac{1}{s^2}\psi(D(x^*, Tx^*)).$$

Since $\psi \in \Psi$, the above holds unless $D(x^*, Tx^*) = 0$, that is, $x^* \in Tx^*$ and x^* is a fixed point of T. $\square$

For the uniqueness of a fixed point of a generalized $\alpha-\psi$ contractive mapping, we will consider the following hypothesis.

(H) For all $x, y \in \mathrm{Fix}(T)$, either $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$.
Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.3. *Adding condition (H) to hypotheses of Theorem 2.1 (respectively, Theorem 2.2), we obtain uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed points of T. Then, it is obvious that $M(x^*,y^*) = d(x^*,y^*)$ and $N(x^*,y^*) = 0$. So

$$\begin{aligned}
\psi(d(x^*,y^*)) &\leq \psi(s^3 H(Tx^*,Ty^*)) \\
&\leq \alpha(x^*,y^*)\psi(s^3 H(Tx^*,Ty^*)) \\
&\leq \beta(\psi(M(x^*,y^*)))\psi(M(x^*,y^*)) + L\phi(N(x^*,y^*)) \\
&< \frac{1}{s^2}\psi(d(x^*,y^*)) < \psi(d(x^*,y^*)),
\end{aligned}$$

which is contradiction. □

The following result can be derived from Theorem 2.3, by taking $L = 0$.

Corollary 2.4. *Let (X,d) be a complete b-metric space and $T : X \to P_{b,cl}(X)$ be a multivalued mapping. Suppose that there exist $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)\psi(s^3 H(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)), \tag{11}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$, $\psi, \phi \in \Psi$ and

$$M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2s}\}. \tag{12}$$

Suppose also that
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;
(iii) T is continuous or X is α-regular.
Then T has a fixed point. Moreover, if (H) is satisfied, then the obtained fixed point is unique.

Corollary 2.5. *Let (X,d) be a complete b-metric space and $T : X \to P_{b,cl}(X)$ be a multivalued mapping. Suppose that there exist $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)\psi(s^3 H(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)), \tag{13}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$. Suppose also that (i) T is α-admissible;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;
(iii) T is continuous or X is α-regular.
Then, T has a fixed point. Moreover, if (H) is satisfied, then the obtained fixed point is unique.

Regarding the analogy of the proof with Theorem 2.1, Theorem 2.2 and Theorem 2.3, we omit the proof

3. Consequences

Definition 5. Let (X, d) be a b-metric space and $T : X \to X$ be a mapping. We say that T is a generalized $\alpha - \psi$-Geraghty contraction type mapping whenever there exist $\alpha : X \times X \to [0, \infty)$ and some $L \geq 0$ such that for

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\} \tag{14}$$

$$\text{and } N(x, y) = \min\{d(x, Tx), d(y, Tx)\}, \tag{15}$$

we have

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y)), \tag{16}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$.

Corollary 3.1. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a generalized $\alpha - \psi$-Geraghty contraction type mapping such that*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either T is continuous or X is α-regular.
Then, T has a fixed point. Moreover, if (H) is satisfied, then the obtained fixed point is unique.

Corollary 3.2. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping. If there exist $\alpha : X \times X \to [0, \infty)$ and some $L \geq 0$ such that*

$$\alpha(x, y)\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) + L\phi(N(x, y)), \tag{17}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$ and

$$\text{and } N(x, y) = \min\{d(x, Tx), d(y, Tx)\}. \tag{18}$$

Suppose also that (i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either T is continuous or X is α-regular.
Then, T has a fixed point. Moreover, if (H) is satisfied, then the obtained fixed point is unique.

By letting $\alpha(x, y) = 1$ for all $x, y \in X$, we get the following consequences:

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) + L\phi(N(x, y)), \tag{19}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$.

Corollary 3.3. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a generalized $\alpha - \psi$-Geraghty contraction type mapping. If T is continuous, then T has a fixed point.*

Corollary 3.4. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a continuous mapping. If there exists $L \geq 0$ such that*

$$\psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)) + L\phi(N(x, y)), \tag{20}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$ and

$$\text{and } N(x,y) = \min\{d(x,Tx), d(y,Tx)\}. \tag{21}$$

Then, T has a fixed point.

If in (20) we let $L = 0$ then we obtain the following sequence.

Corollary 3.5. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a continuous mapping such that*

$$\psi(s^3 d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)), \tag{22}$$

for all $x, y \in X$, where $\beta \in \mathcal{F}$ and $\psi, \phi \in \Psi$. Then, T has a fixed point.

We present the following example.

Example 3.6. *Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |x(t)|dt < 1$. Define $d : X \times X :\to [0, \infty)$ by*

$$d(x,y) = \int_0^1 |x(t) - y(t)|^2 dt.$$

Then, d is a b-metric on X, with $s = 2$. The operator $T : X \to X$ is defined by

$$Tx(t) = \frac{1}{8} ln(1 + |x(t)|).$$

Consider the mapping $\alpha : X \times X \to [0, \infty)$

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \geq y, \\ 0 \text{ otherwise.} \end{cases}$$

We take $\beta : [0, \infty) \to [0, \frac{1}{4})$ and $\psi : [0, \infty) \to [0, \infty)$ as

$$\psi(t) = t \quad \text{and} \quad \beta(t) = \frac{t^2 + 1}{4t^2 + 8}.$$

Evidently, $\psi \in \Psi$ and $\beta \in \mathcal{F}$. Moreover, T is α-admissible mapping and $\alpha(1, T1) \geq 1$. Now, we prove that T is a generalized $\alpha - \psi$-Geraghty contraction type mapping.

$$\begin{aligned}
&\alpha(x(t), y(t))\psi(s^3 d(Tx(t), Ty(t))) \leq 2^3 (\int_0^1 |Tx(t) - Ty(t)|^2 dt) \\
&= 2^3 \int_0^1 |\frac{1}{8} ln(1 + |x(t)|) - \frac{1}{8} ln(1 + |y(t)|)|^2 dt \\
&= 2^{-3} \int_0^1 |ln(\frac{1 + |x(t)|}{1 + |y(t)|})|^2 dt = 2^{-3} \int_0^1 |ln(1 + \frac{|x(t)| - |y(t)|}{1 + |y(t)|})|^2 dt \\
&\leq 2^{-3} \int_0^1 |ln(1 + |x(t) - y(t)|)|^2 dt \leq 2^{-3} \int_0^1 |x(t) - y(t)|^2 dt \\
&= 2^{-3} d(x,y) \leq \frac{d(x,y)^2 + 1}{4d(x,y)^2 + 8} d(x,y) = \beta(d(x,y)d(x,y).
\end{aligned}$$

From above inequalities, remark that

$$d(Tx,Ty) \leq \frac{1}{2^6} d(x,y),$$

that is, it is obvious that T is continuous in (X,d). Thus, by corollary 3.2 (with $L=0$), we see that T has a fixed point.

4. Application

Let $X = C([0,1],\mathbb{R})$ be the set of real continuous functions defined on $[0,1]$ and let $d : X \times X \to [0,\infty)$ be given by

$$d(x,y) = \| (x-y)^2 \|_\infty = \sup_{t\in[0,1]} (x(t)-y(t))^2,$$

for all $x,y \in X$. Then, (X,d) is a complete b-metric space with $s=2$. We consider the following integral equation

$$x(t) = P(t) + \int_0^1 S(t,u) f(u,x(u))du, t \in [0,1], \tag{23}$$

where $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ and $P : [0,1] \to \mathbb{R}$ are two continuous functions and $S : [0,1]\times[0,1] \to [0,\infty)$ is a function such that $S(t,.) \in L^1([0,1])$ for all $t \in [0,1]$.
Consider the operator $T : X \to X$ defined by

$$T(x)(t) = P(t) + \int_0^1 S(t,u) f(u,x(u))du, t \in [0,1]. \tag{24}$$

Theorem 4.1. *Let $X = C([0,1],\mathbb{R})$. Suppose there exist $\eta : X\times X \to [0,\infty)$, $\alpha : X \times X \to [0,\infty)$ and $\beta : [0,\infty) \to [0,\frac{1}{4})$ such that the following conditions are satisfied:*
(i) for all $u \in [0,1]$ and for all $x,y \in X$;

$$0 \leq |f(u,x(u)) - f(u,y(u))| \leq \eta(x,y)|x(u)-y(u)|,$$

and

$$\| \int_0^1 S(t,u)\eta(x,y)du \|_\infty^2 \leq \frac{\beta(\| (x-y)^2 \|_\infty)}{\alpha(x,y)};$$

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;
(iii) $\alpha(x,Tx) \geq 1 \Rightarrow \alpha(Tx,T^2x) \geq 1$;
(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all k.
Then, the integral equation (23) has a solution in X.

Proof. Clearly, any fixed point of (24) is a solution of (23). By condition (i), we obtain

$$\begin{aligned}
\alpha(x,y)|T(x)(t)-T(y)(t)|^2 &= \alpha(x,y)[|\int_0^1 S(t,u)[f(u,x(u))-f(u,y(u))]du|]^2\\
&\leq \alpha(x,y)[\int_0^1 S(t,u)|f(u,x(u))-f(u,y(u))|du]^2\\
&\leq \alpha(x,y)[\int_0^1 S(t,u)\eta(x,y)\sqrt{|x(u)-y(u)|^2}du]^2\\
&\leq \alpha(x,y)[\int_0^1 S(t,u)\eta(x,y)\sqrt{\parallel (x-y)^2 \parallel_\infty}^2 du]^2\\
&= \alpha(x,y)\parallel (x-y)^2 \parallel_\infty [\int_0^1 S(t,u)\eta(x,y)du]^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\alpha(x,y) \parallel (T(x)-T(y))^2 \parallel_\infty &\leq \alpha(x,y)\parallel (x-y)^2 \parallel_\infty \parallel \int_0^1 S(t,u)\eta(x,y)du \parallel_\infty^2\\
&\leq \beta(\parallel (x-y)^2 \parallel_\infty) \parallel (x-y)^2 \parallel_\infty .
\end{aligned}$$

Thus, for all $x, y \in X$, we obtain

$$\alpha(x,y)d(T(x),T(y)) \leq \beta(d(x,y))d(x,y).$$

This implies that corollary 3.2 holds with $\psi(t) = t$ and $L = 0$. Hence, the operator T has a fixed point, that is, the integral equation (24) has a solution in X. □

The following example illustrates Theorem 4.1.

Example 4.2. *Take $X = C([0,1],\mathbb{R})$. Consider the following functional integral equation*

$$x(t) = \frac{t^2}{1+t^2} + \frac{1}{27}\int_0^1 \frac{ucost}{54(1+t)}\frac{|x(u)|}{1+|x(u)|}du, \tag{25}$$

for $t \in [0,1]$. Observe that the equation (25) is a spatial case of (23) with

$$\begin{aligned}
P(t) &= \frac{t^2}{1+t^2},\\
S(t,u) &= \frac{u}{3(1+t)},\\
f(t,x) &= \frac{cost}{18}\frac{|x|}{(1+|x|)}.
\end{aligned}$$

Consider the operator $T : X \to X$ defined by

$$T(x)(t) = P(t) + \int_0^1 S(t,u)f(u,x(u))du, t \in [0,1]$$

Define the mapping $\alpha : X \times X \to [0, \infty)$ *as*

$$\alpha(x(t), y(t)) = \begin{cases} 1 \text{ if } x(t) \geq y(t), \\ 0 \text{ otherwise.} \end{cases}$$

Take $\beta : [0, \infty) \to [0, \frac{1}{4})$ *as*

$$\beta(t) = \frac{t^2 + 1}{4t^2 + 8}.$$

Let $\eta(x, y) = 1$*. For arbitrary fixed* $x, y \in \mathbb{R}$ *such that* $x \geq y$*, we obtain*

$$\begin{aligned} |f(t, x) - f(t, y)| &= |\frac{cost}{18}\frac{|x|}{(1 + |x|)} - \frac{cost}{18}\frac{|y|}{(1 + |y|)}| \\ &\leq \frac{1}{18}|x - y| \leq \nu(x, y)\,|x - y| \end{aligned}$$

and

$$\begin{aligned} &\| \int_0^1 S(t, u)\eta(x, y)du \|_\infty^2 = \frac{1}{36} \\ &\leq \frac{(\| (x - y)^2 \|_\infty)^2 + 1}{4(\| (x - y)^2 \|_\infty)^2 + 8} = \beta(\| (x - y)^2 \|_\infty). \end{aligned}$$

Again, by definition of $\alpha(x, y)$*, it follows that:*
(i) $\alpha(1, T1) \geq 1$*;*
(ii) $\alpha(x, Tx) \geq 1$ *implies that* $\alpha(Tx, T^2x) \geq 1$*;*
(iii) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k*.*
Hence, by using Theorem 4.1, the integral equation (25) has a solution in X*.*

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