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RELATIONS BETWEEN QUATERNIONIC DIFFERENTIAL AND THE CORRESPONDING CAUCHY RIEMANN SYSTEM

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Abstract. In this paper, we investigate several properties of quaternionic functions. We research some differentials of quaternionic functions, and relations between the differentials and the corresponding Cauchy Riemann system in Clifford analysis.

1. Introduction

The quaternion field $\mathcal{T}$ is a non-commutative four dimensional skew field of real numbers. The field $\mathcal{T}$ is identified with $\mathbb{C}^2$ and $\mathbb{R}^4$, where $\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ denotes the field of complex numbers.

Naser [9] have studied some properties of hyperholomorphic functions over the field $\mathcal{T}$ and quaternionic conjugate harmonic functions in 1971. In 1995, Nôno [10] has shown several properties of regular hypercomplex functions on two approaches. And Nôno [10] has searched properties of hyperholomorphic functions by using partial differential equation.

In 2011, Luna-Elizarrarás and Shapiro [1] have shown that some properties for regular functions in one complex analysis are feasible in Clifford analysis. Also Luna-Elizarrarás and Shapiro [1] have explained the derivatives of functions in one complex variable theory and the quaternionic analysis. In 2011, Koriyama et al. [8] have given regularities on quaternionic functions for several differential operators in Clifford analysis. Also Koriyama et al. [8] have shown the corresponding Cauchy Riemann system for each operator D_j^* $(j = 1, 2, 3, 4)$ on the field $\mathcal{T}$.

Jung et al. [3] have researched properties of the corresponding Cauchy theorem on the dual quaternion field $\mathcal{T} \times \mathcal{T}$. Also Jung et al. [3] investigated regularities of dual quaternionic functions in an open set of product complex spaces. Jung and Shon [2] have shown some properties of hyperholomorphic

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functions in the dual ternary number system in 2013. Kim et al. [6, 7] obtained several properties of regular functions on three dimensional field, the ternary number system and the reduced quaternion field, in the sense of Clifford analysis.

In 2015, Kang and Shon [4] have shown the corresponding Cauchy Riemann system for several differential operators and some properties of L-regular functions on the generalized quaternion field. Kang et al. [5] investigated quaternionic regular functions and have studied properties of Jacobian matrix on the field $\mathcal{T}$.

In this paper, we investigate some differentials of quaternioninc functions, and relations between the definition of differential and the corresponding Cauchy Riemann system. We provide the notation and direct computation of the derivative for functions valued with quaternion in Clifford analysis.

2. Preliminaries

Let $\mathcal{T}$ be the quaternion field generated by a basis $\{e_0, e_1, e_2, e_3\}$ over the real field $\mathbb{R}$,

$$\mathcal{T} = \{z \mid z = \sum_{j=0}^{3} e_j x_j,\ x_j \in \mathbb{R}\ (j = 0,\ 1,\ 2,\ 3)\}.$$

Each basis of $\mathcal{T}$ can be expressed by

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

as matrices, where $i = \sqrt{-1}$. Then, e_0, e_1, e_2 and e_3 satisfy the followings:

$$e_0 = id,\ e_j^2 = -1,\ \text{and}\ e_j e_k + e_k e_j = -2\delta_{jk}\ (j, k = 1, 2, 3), \tag{1}$$

where δ_{jk} is Kronecker delta.

The quaternion z is an element of $\mathcal{T}$ denoted by $z = z_1 + z_2 e_2$, where $z_1 = x_0 + e_1 x_1$ and $z_2 = x_2 + e_1 x_3$. The quaternionic conjugate z^* and the absolute value $|z|$ of z are defined by

$$z^* = x_0 - \sum_{j=0}^{3} e_j x_j = \overline{z_1} - z_2 e_2,\ |z|^2 = zz^* = z^* z = \sum_{j=0}^{3} x_j^2.$$

And every non-zero quaternion z has a unique inverse $z^{-1} = \dfrac{z^*}{|z|^2}$ $(z \neq 0)$.

Let Ω be a bounded open set in $\mathbb{C}^2$ and a function $f : \Omega \to \mathcal{T}$ is defined by

$$f(z) = \sum_{j=1}^{3} e_j u_j(x_0, x_1, x_2, x_3) = e_0 u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3,$$

where u_j $(j = 0,\ 1,\ 2,\ 3)$ are real valued functions.

We consider the following quaternionic differential operators:

$$\begin{aligned} D &:= \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}\right), \\ D^* &= \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3}\right), \end{aligned}$$

where $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z_j}}$ $(j = 1,\ 2)$ are complex differential operators. Since each basis of $\mathcal{T}$ satisfies (1), we have $\frac{\partial}{\partial z_j} e_2 = e_2 \frac{\partial}{\partial \overline{z_j}}$ and $z_j e_2 = e_2 \overline{z_j}$ for $j = 1,\ 2$.

3. Differential of quaternionic functions

Let Ω be a bounded open set in $\mathbb{C}^2$, $z \in \Omega$ and let $f : \Omega \to \mathcal{T}$ defined by $f(z) = e_0 u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3$ be a quaternionic function. For increment of the argument at the point z, we put a quaternion $h \neq 0$ satisfying $z + h \in \Omega$. Then we can consider

$$f(z+h) - f(z) = \sum_{j=0}^{3} e_j u_j(z+h) - \sum_{j=0}^{3} e_j u_j(z) = \sum_{j=0}^{3} e_j (u_j(z+h) - u_j(z)).$$

And let

$$u_j(z+h) - u_j(z) := \Delta u_j(h) := \Delta u_j \ (j = 0,\ 1,\ 2,\ 3)$$

for convenience.

Since h is an arbitrary element of $\mathcal{T}$, h can be expressed by $h = a + e_1 b + e_2 c + e_3 d$ for $a,\ b,\ c,\ d \in \mathbb{R}$. Then the inverse of h is $h^{-1} = \dfrac{a - e_1 b - e_2 c - e_3 d}{a^2 + b^2 + c^2 + d^2}$.

Definition 1. Let Ω be a bounded open set in $\mathcal{T}$ and $z \in \Omega$.

If $h^{-1}\{f(z+h) - f(z)\}$ $(\{f(z+h) - f(z)\}h^{-1})$ has a limit as $h \to 0$, then we call that f is *L(R)-differentiable at* $z \in \Omega$. And the limit is called *L(R)-derivative of* f at z denoted by

$$'f(z) := \lim_{h \to 0} h^{-1}\{f(z+h) - f(z)\} \quad (f'(z) := \lim_{h \to 0}\{f(z+h) - f(z)\}h^{-1}). \tag{2}$$

Proposition 3.1. *Let Ω be a bounded open set in $\mathcal{T}$. If the function f is L(R)-differentiable at $z \in \Omega$, then f satisfies*

$$\begin{aligned} &'f(z) = \frac{\partial f}{\partial x_0} = -e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = -e_3 \frac{\partial f}{\partial x_3} \\ &\left(f'(z) = \frac{\partial f}{\partial x_0} = -\frac{\partial f}{\partial x_1} e_1 = -\frac{\partial f}{\partial x_2} e_2 = -\frac{\partial f}{\partial x_3} e_3\right). \end{aligned} \tag{3}$$

Proof. By direct computation, we have

$$
\begin{aligned}
&h^{-1}\{f(z+h)-f(z)\} = h^{-1}\sum_{j=0}^{3} e_j \Delta u_j \\
&= \frac{a-e_1 b-e_2 c-e_3 d}{a^2+b^2+c^2+d^2}\left(\Delta u_0+e_1\Delta u_1+e_2\Delta u_2+e_3\Delta u_3\right) \\
&= \frac{1}{a^2+b^2+c^2+d^2}\{(a\Delta u_0+b\Delta u_1+c\Delta u_2+d\Delta u_3)+ \\
&e_1\left(a\Delta u_1-b\Delta u_0-c\Delta u_3+d\Delta u_2\right)+e_2\left(a\Delta u_2+b\Delta u_3-c\Delta u_0-d\Delta u_1\right)+ \\
&e_3\left(a\Delta u_3-b\Delta u_2+c\Delta u_1-d\Delta u_0\right)\}.
\end{aligned}
$$

We consider the cases of that h forms $h=a,\ e_1 b,\ e_2 c$ and $e_3 d$. And all the limit of each cases have to be same to clarify (2). At first, if $h=a\in\mathbb{R}$, then

$$
\begin{aligned}
h^{-1}\{f(z+h)-f(z)\} &= h^{-1}\{f(z+a)-f(z)\} \\
&= \frac{1}{a^2}\left(a\Delta u_0+e_1 a\Delta u_1+e_2 a\Delta u_2+e_3 a\Delta u_3\right) \\
&= \frac{1}{a}\left(\Delta u_0+e_1\Delta u_1+e_2\Delta u_2+e_3\Delta u_3\right)=\frac{1}{a}\sum_{j=0}^{3} e_j\Delta u_j .
\end{aligned}
$$

So we have

$$
\lim_{h\to 0} h^{-1}\{f(z+h)-f(z)\} = \lim_{a\to 0}\frac{1}{a}\sum_{j=0}^{3} e_j\Delta u_j = \frac{\partial f}{\partial x_0}. \tag{4}
$$

In the case of $h=e_1 b$,

$$
\begin{aligned}
h^{-1}\{f(z+h)-f(z)\} &= h^{-1}\{f(z+e_1 b)-f(z)\} \\
&= \frac{1}{b}\left(\Delta u_1-e_1\Delta u_0+e_2\Delta u_3-e_3\Delta u_2\right) \\
&= \frac{1}{e_1 b}\left(e_1\Delta u_1-e_1^2\Delta u_0+e_1 e_2\Delta u_3-e_1 e_3\Delta u_2\right) \\
&= \frac{1}{e_1 b}\sum_{j=0}^{3} e_j\Delta u_j
\end{aligned}
$$

by quaternionic multiplications (1). Then we have

$$
\lim_{h\to 0} h^{-1}\{f(z+h)-f(z)\} = \lim_{b\to 0}\frac{1}{e_1 b}\sum_{j=0}^{3} e_j\Delta u_j = -e_1\frac{\partial f}{\partial x_1}. \tag{5}
$$

Similarly, we have

$$
\lim_{h\to 0} h^{-1}\{f(z+h)-f(z)\} = -e_2\frac{\partial f}{\partial x_2}\ (h=e_2 c) \tag{6}
$$

and

$$\lim_{h\to 0} h^{-1}\{f(z+h)-f(z)\} = -e_3\frac{\partial f}{\partial x_3}\ (h=e_3 d). \tag{7}$$

By (4), (5), (6) and (7), we obtain the result (3). Similarly for the R-differentiable functions, we can obtain the result. □

Definition 2. Let Ω be a bounded open set in $\mathcal{T}$. A function f is said to be a L(R)-regular function in Ω if
(a) $f \in C^1(\Omega)$,
(b) $D^* f = 0$ $(fD^* = 0)$ in Ω.

Theorem 3.2. *Let Ω be a bounded open set in $\mathcal{T}$. A function f is L-regular in Ω if and only if the function is L-differentiable at $z \in \Omega$:*

$$D^* f = 0 \quad \text{iff} \quad \frac{\partial f}{\partial x_0} = -e_1\frac{\partial f}{\partial x_1} = -e_2\frac{\partial f}{\partial x_2} = -e_3\frac{\partial f}{\partial x_3}.$$

Proof. We denote that

$$\frac{\partial}{\partial x_0} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_1}}, \quad \frac{\partial}{\partial x_1} = e_1\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial \overline{z_1}}\right),$$
$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \overline{z_2}}, \quad \frac{\partial}{\partial x_3} = e_1\left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial \overline{z_2}}\right).$$

We have the following equation from (3):

$$\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial \overline{z_1}} = -e_1^2\left(\frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial \overline{z_1}}\right) = -e_2\frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial \overline{z_2}} = -e_3 e_1\left(\frac{\partial f}{\partial z_2} - \frac{\partial f}{\partial \overline{z_2}}\right).$$

Then,

$$\frac{\partial f}{\partial \overline{z_1}} = -e_2\frac{\partial f}{\partial \overline{z_2}},$$

$$\frac{\partial f}{\partial \overline{z_1}} + e_2\frac{\partial f}{\partial \overline{z_2}} = 0.$$

Thus, we obtain

$$D^* f = 0.$$

□

Remark 1. For the R-regular functions and the R-differentiable functions, we can obtain a similar result. We consider the following quaternionic differential operators:

$$D_1 := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}, \quad D_1^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial \overline{z_2}},$$
$$D_2 := \frac{\partial}{\partial \bar{z_1}} - e_2 \frac{\partial}{\partial z_2}, \quad D_2^* = \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial \overline{z_2}},$$
$$D_3 := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_3^* = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial z_2},$$
$$D_4 := \frac{\partial}{\partial \overline{z_1}} - e_2 \frac{\partial}{\partial \overline{z_2}}, \quad D_4^* = \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2}.$$

Then, the corresponding L-Cauchy Riemann systems for each differential operators are

$$D_1^* f = 0 \text{ iff } \frac{\partial f}{\partial x_0} = -e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = e_3 \frac{\partial f}{\partial x_3} \text{ for } j = 1,$$
$$D_2^* f = 0 \text{ iff } \frac{\partial f}{\partial x_0} = e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = e_3 \frac{\partial f}{\partial x_3} \text{ for } j = 2,$$
$$D_3^* f = 0 \text{ iff } \frac{\partial f}{\partial x_0} = -e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = -e_3 \frac{\partial f}{\partial x_3} \text{ for } j = 3,$$
$$D_4^* f = 0 \text{ iff } \frac{\partial f}{\partial x_0} = e_1 \frac{\partial f}{\partial x_1} = -e_2 \frac{\partial f}{\partial x_2} = -e_3 \frac{\partial f}{\partial x_3} \text{ for } j = 4.$$

Proposition 3.3. *Let Ω be a bounded open set in $\mathcal{T}$. If a function f is L-regular in Ω, then*

$$'f(z) = Df = \frac{\partial f}{\partial x_0}$$

Proof. Let a function f be L-regular in Ω. Then, f satisfies $D^* f = 0$ and the corresponding L-Cauchy Riemann system. We have

$$'f(z) = Df = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} - e_1 \frac{\partial f}{\partial x_1} - e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3}\right) = \frac{\partial f}{\partial x_0}$$

by (3). Thus, we obtain the result. □

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