J. Appl. Math. & Informatics Vol. **34**(2016), No. 1 - 2, pp. 107 - 114
http://dx.doi.org/10.14317/jami.2016.107

# SYMMETRIC PROPERTIES FOR GENERALIZED TWISTED $q$-EULER ZETA FUNCTIONS AND $q$-EULER POLYNOMIALS

N.S. JUNG AND C.S. RYOO*

Abstract. In this paper we give some symmetric property of the generalized twisted $q$-Euler zeta functions and $q$-Euler polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
*Key words and phrases* : generalized twisted $q$-Euler numbers and polynomials, generalized twisted $q$-Euler zeta function, symmetric property, power sum.

## 1. Introduction

The Euler numbers and polynomials possess many interesting properties in many areas of mathematics and physics. Many mathematicians have studied in the area of various $q$-extensions of Euler polynomials and numbers (see [1-11]). Recently, Y. Hu investigated several identities of symmetry for Carlitz's $q$-Bernoulli numbers and polynomials in complex field (see [3]). D. Kim *et al.* [4] derived some identities of symmetry for Carlitz's $q$-Euler numbers and polynomials in complex field. J.Y. Kang and C.S. Ryoo studied some identities of symmetry for $q$-Genocchi polynomials (see [2]). In [1], we obtained some identities of symmetry for Carlitz's twisted $q$-Euler zeta function in complex field. In this paper, we establish some interesting symmetric identities for generalized twisted $q$-Euler zeta functions and generalized ] twisted $q$-Euler polynomials in complex field. If we take $\chi = 1$ in all equations of this article, then [1] are the special case of our results. Throughout this paper we use the following notations. By $\mathbb{N}$ we denote the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{(see } [1, 2, 3, 4]).$$

---

Received September 20, 2015. Revised November 4, 2015. Accepted November 11, 2015.
*Corresponding author.

Note that $\lim_{q\to 1}[x] = x$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Let $r$ be a positive integer, and let $\varepsilon$ be the $r$-th root of unity. Let $\chi$ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$. Then the generalized twisted $q$-Euler polynomials associated with associated with $\chi$, $E_{n,\chi,q,\varepsilon}$, are defined by the following generating function

$$F_{\chi,q,\varepsilon}(t,x) = [2]_q \sum_{n=0}^{\infty}(-1)^n q^n \varepsilon^n \chi(n) e^{[x+n]_q t} \quad = \sum_{n=0}^{\infty} E_{n,\chi,q,\varepsilon}(x)\frac{t^n}{n!} \tag{1.1}$$

and their values at $x = 0$ are called the generalized twisted $q$-Euler numbers and denoted $E_{n,\chi,q,\varepsilon}$.

By (1.1) and Cauchy product, we obtain

$$E_{n,\chi,q,\varepsilon}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx}[x]_q^{n-l} E_{l,\chi,q,\varepsilon}, \tag{1.2}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon})^n$ by $E_{n,\chi,q,\varepsilon}$.

By using (1.1), we note that

$$\frac{d^k}{dt^k} F_{\chi,q,\varepsilon}(t,x)\bigg|_{t=0} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n \varepsilon^n q^n [n+x]_q^k, (k \in \mathbb{N}). \tag{1.3}$$

By (1.3), we are now ready to define the Hurwitz type of the generalized twisted $q$-Euler zeta functions.

**Definition 1.1.** Let $s \in \mathbb{C}$ and $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \ldots$. We define

$$\zeta_{\chi,q,\varepsilon}(s,x) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n) \varepsilon^n q^n}{[n+x]_q^s}. \tag{1.4}$$

Note that $\zeta_{\chi,q,\varepsilon}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_{\chi,q,\varepsilon}(s,x)$ and $E_{k,\chi,q,\varepsilon}(x)$ is given by the following theorem.

**Theorem 1.2.** *For $k \in \mathbb{N}$, we get*

$$\zeta_{\chi,q,\varepsilon}(-k,x) = E_{k,\chi,q,\varepsilon}(x). \tag{1.5}$$

Observe that $\zeta_{\chi,q,\varepsilon}(-k,x)$ function interpolates $E_{k,\chi,q,\varepsilon}(x)$ polynomials at non-negative integers. If $\chi = 1$, then $\zeta_{\chi,q,\varepsilon}(s,x) = \zeta_{q,\varepsilon}(s,x)$ (see [1]).

## 2. Symmetric property of generalized twisted $q$-Euler zeta functions

In this section, by using the similar method of [1, 2, 3, 4, 9], expect for obvious modifications, we give some symmetric identities for generalized twisted $q$-Euler polynomials and generalized twisted $q$-Euler zeta functions. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$.

**Theorem 2.1.** *Let $\chi$ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$ and $\varepsilon$ be the $r$-th root of unity. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we obtain*

$$\begin{aligned}&\sum_{i=0}^{w_2 d-1}[2]_{q^{w_1}}[w_1]_q^s(-1)^i\chi(i)\varepsilon^{w_1 i}q^{w_1 i}\zeta_{\chi,q^{w_2},\varepsilon^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}i\right)\\ &=\sum_{j=0}^{w_1 d-1}[2]_{q^{w_2}}[w_2]_q^s(-1)^j\chi(j)\varepsilon^{w_2 j}q^{w_2 j}\zeta_{\chi,q^{w_1},\varepsilon^{w_1}}\left(s, w_2x+\frac{w_1}{w_2}j\right).\end{aligned}$$

*Proof.* Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In Definition 1.1, we derive next result by substitute $w_1x + \frac{w_1}{w_2}i$ for $x$ in and replace $q$ and $\varepsilon$ by $q^{w_2}$ and $\varepsilon^{w_2}$, respectively.

$$\begin{aligned}\zeta_{\chi,q^{w_2},\varepsilon^{w_2}}(s, w_1x+\frac{w_1}{w_2}i) &= [2]_{q^{w_2}}\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)\varepsilon^{w_2 n}q^{w_2 n}}{[n+w_1x+\frac{w_1}{w_2}i]_{q^{w_2}}^s}\\ &=[2]_{q^{w_2}}[w_2]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)\varepsilon^{w_2 n}q^{w_2 n}}{[w_1w_2x+w_1i+w_2n]_q^s}.\end{aligned}\tag{2.1}$$

Since for any non-negative integer $n$ and odd positive integer $w_1$, there exist unique non-negative integer $r, j$ such that $m = w_1r + j$ with $0 \le j \le w_1 - 1$. So, the equation (2.1) can be written as

$$\begin{aligned}&\zeta_{\chi,q^{w_2},\varepsilon^{w_2}}(s, w_1x+\frac{w_1}{w_2}i)\\ &=[2]_{q^{w_2}}[w_2]_q^s\sum_{\substack{w_1dr+j=0\\0\le j\le w_1d-1}}^{\infty}\frac{(-1)^{w_1dr+j}\chi(w_1dr+j)\varepsilon^{w_2(w_1dr+j)}q^{w_2(w_1dr+j)}}{[w_1w_2dr+w_1w_2x+w_1i+w_2j]_q^s}\\ &=[2]_{q^{w_2}}[w_2]_q^s\sum_{j=0}^{w_1d-1}\sum_{r=0}^{\infty}\frac{(-1)^j\chi(j)\varepsilon^{w_2(w_1dr+j)}q^{w_2(w_1dr+j)}}{[w_1w_2(dr+x)+w_1i+w_2j]_q^s}.\end{aligned}\tag{2.2}$$

In similarly, we obtain

$$\begin{aligned}\zeta_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x+\frac{w_2}{w_1}j) &= [2]_{q^{w_1}}\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)\varepsilon^{w_1 n}q^{w_1 n}}{[n+w_2x+\frac{w_2}{w_1}j]_{q^{w_1}}^s}\\ &=[2]_{q^{w_1}}[w_1]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)\varepsilon^{w_1 n}q^{w_1 n}}{[w_1w_2x+w_1n+w_2j]_q^s}.\end{aligned}\tag{2.3}$$

Using the method in (2.2), we obtain

$$
\begin{aligned}
&\zeta_{\chi,q^{w_1},\zeta^{w_1}}(s, w_2x + \frac{w_2}{w_1}j) \\
&= [2]_{q^{w_1}}[w_1]_q^s \sum_{\substack{w_2dr+i=0\\ 0\le i\le w_2d-1}}^{\infty} \frac{(-1)^{w_2dr+i}\chi(w_2dr+i)\varepsilon^{w_1(w_2dr+i)}q^{w_1(w_2dr+i)}}{[w_1w_2dr+w_1w_2x+w_1i+w_2j]_q^s} \qquad (2.4)\\
&= [2]_{q^{w_1}}[w_1]_q^s \sum_{i=0}^{w_2d-1}\sum_{r=0}^{\infty} \frac{(-1)^i\chi(i)\varepsilon^{w_1(w_2dr+i)}q^{w_1(w_2dr+i)}}{[w_1w_2(dr+x)+w_1i+w_2j]_q^s}.
\end{aligned}
$$

By (2.2) and (2.4), we obtain

$$
\begin{aligned}
&\sum_{i=0}^{w_2d-1} [2]_{q^{w_1}}[w_1]_q^s(-1)^i\chi(i)\varepsilon^{w_1i}q^{w_1i}\zeta_{\chi,q^{w_2},\varepsilon^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}i\right) \\
&= \sum_{j=0}^{w_1d-1} [2]_{q^{w_2}}[w_2]_q^s(-1)^j\chi(j)\varepsilon^{w_2j}q^{w_2j}\zeta_{\chi,q^{w_1},\varepsilon^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right). \qquad (2.5)
\end{aligned}
$$

□

Next, we obtain the symmetric results by using definition and theorem of the generalized twisted $q$-Euler polynomials.

**Theorem 2.2.** *Let $\chi$ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $\varepsilon$ be the $r$-th root of unity. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $i, j$ and $n$ be non-negative integer, we obtain*

$$
\begin{aligned}
&\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2d-1}(-1)^i\chi(i)\varepsilon^{w_1i}q^{w_1i}E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right)\\
&= \frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\varepsilon^{w_2j}q^{w_2j}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right).
\end{aligned}
$$

*Proof.* By substitute $w_1x + \frac{w_1 i}{w_2}$ for $x$ in Theorem 1.2 and replace $q$ and $\varepsilon$ by $q^{w_2}$ and $\varepsilon^{w_2}$, respectively, we derive

$$
\begin{aligned}
&E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right)\\
&= [2]_{q^{w_2}}\sum_{m=0}^{\infty}(-1)^m\chi(m)\varepsilon^{w_2m}q^{w_2m}\left[w_1x+\frac{w_1}{w_2}i+m\right]_{q^{w_2}}^n \qquad (2.6)\\
&= \frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{m=0}^{\infty}(-1)^m\chi(m)\varepsilon^{w_2m}q^{w_2m}[w_1w_2x+w_1i+w_2m]_q^n.
\end{aligned}
$$

Since for any non-negative integer $m$ and odd positive integer $w_1$, there exist unique non-negative integer $r, j$ such that $m = w_1r + j$ with $0 \le j \le w_1 - 1$.

Hence, the equation (2.6) is written as

$$\begin{aligned}
&E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{\substack{w_1dr+j=0\\0\le j\le w_1d-1}}^{\infty}(-1)^{w_1dr+j}\chi(w_1dr+j)\varepsilon^{w_2(w_1dr+j)}q^{w_2(w_1dr+j)}\\
&\qquad\times[w_1w_2x+w_1i+w_2(w_1dr+j)]_q^n\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{i=0}^{w_1d-1}\sum_{r=0}^{\infty}(-1)^{w_1dr+j}\chi(j)\varepsilon^{w_2(w_1dr+j)}q^{w_2(w_1dr+j)}\\
&\qquad\times[w_1w_2(x+dr)+w_1i+w_2j]_q^n.
\end{aligned}\tag{2.7}$$

In similar, we obtain

$$\begin{aligned}
&E_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\
&=[2]_{q^{w_1}}\sum_{m=0}^{\infty}(-1)^m\chi(m)\varepsilon^{w_1m}q^{w_1m}\left[w_2x+\frac{w_2}{w_1}j+m\right]_{q^{w_1}}^n\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{m=0}^{\infty}(-1)^m\chi(m)\varepsilon^{w_1m}q^{w_1m}[w_1w_2x+w_2j+w_1m]_q^n,
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&E_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{\substack{w_2dr+i=0\\0\le i\le w_2d-1}}^{\infty}(-1)^{w_2dr+i}\chi(w_2dr+i)\varepsilon^{w_1(w_2dr+i)}q^{w_1(w_2dr+i)}\\
&\qquad\times[w_1w_2x+w_2j+w_1(w_2dr+i)]_q^n\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2d-1}\sum_{r=0}^{\infty}(-1)^{w_2dr+i}\chi(i)\varepsilon^{w_1(w_2dr+i)}q^{w_1(w_2dr+i)}\\
&\qquad\times[w_1w_2(x+dr)+w_1i+w_2j]_q^n.
\end{aligned}\tag{2.9}$$

It follows from the above equation that

$$\begin{aligned}
&\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{w_1 i}E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right)\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1d-1}\sum_{i=0}^{w_2d-1}\sum_{r=0}^{\infty}(-1)^{i+j}\chi(i)\chi(j)\varepsilon^{w_1w_2dr+w_1i+w_2j}\\
&\qquad\times q^{w_1w_2dr+w_1i+w_2j}[w_1w_2(x+dr)+w_1i+w_2j)]_q^n \qquad (2.10)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{w_2 j}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right).
\end{aligned}$$

From (2.7), (2.8), (2.9) and (2.10), the proof of the Theorem 2.2 is completed. □

By (1.2) and Theorem 2.2, we have the following theorem.

**Theorem 2.3.** *Let $i, j$ and $n$ be non-negative integers. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned}
&[2]_{q^{w_1}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^k[w_2]_q^{n-k}E_{n-k,\chi,q^{w_2},\varepsilon^{w_2}}(w_1x)\\
&\qquad\times\sum_{i=0}^{w_2d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k\\
&=[2]_{q^{w_2}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^{n-k}[w_2]_q^kE_{n-k,\chi,q^{w_1},\varepsilon^{w_1}}(w_2x)\\
&\qquad\times\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k.
\end{aligned}$$

*Proof.* After some calculations, we have

$$\begin{aligned}
&\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{i=0}^{w_2d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{w_1 i}E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}i\right)\\
&=\frac{[2]_{q^{w_1}}}{[w_1]_q^n}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{w_1}{w_2}\right]_{q^{w_2}}^k E_{n-k,\chi,q^{w_2},\varepsilon^{w_2}}(w_1x) \qquad (2.11)\\
&\qquad\times\sum_{i=0}^{w_2d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{j=0}^{w_1 d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{w_2 j}E_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_2 x+\frac{w_2}{w_1}j\right)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]_q^n}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{w_1}{w_2}\right]_{q^{w_1}}^k E_{n-k,\chi,q^{w_1},\varepsilon^{w_1}}(w_2 x)\\
&\qquad\times\sum_{j=0}^{w_1 d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k.
\end{aligned}\tag{2.12}$$

By (2.11), (2.12) and Theorem 2.2, we obtain that

$$\begin{aligned}
&[2]_{q^{w_1}}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{[w_1]_q^{n-k}[w_2]_q^k}E_{n-k,\chi,q^{w_2},\varepsilon^{w_2}}(w_1 x)\\
&\qquad\times\sum_{i=0}^{w_2 d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k\\
&=[2]_{q^{w_2}}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{[w_1]_q^{k}[w_2]_q^{n-k}}E_{n-k,\chi,q^{w_1},\varepsilon^{w_1}}(w_2 x)\\
&\qquad\times\sum_{j=0}^{w_1 d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k.
\end{aligned}$$

Hence, we have above theorem. □

By Theorem 2.3, we have the interesting symmetric identity for generalized twisted $q$-Euler numbers in complex field.

**Corollary 2.4.** *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned}
&[2]_{q^{w_1}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^k[w_2]_q^{n-k}E_{n-k,\chi,q^{w_2},\varepsilon^{w_2}}\\
&\qquad\times\sum_{i=0}^{w_2 d-1}(-1)^i\chi(i)\varepsilon^{w_1 i}q^{(1+n-k)w_1 i}[i]_{q^{w_1}}^k\\
&=[2]_{q^{w_2}}\sum_{k=0}^{n}\binom{n}{k}[w_1]_q^{n-k}[w_2]_q^{k}E_{n-k,\chi,q^{w_1},\varepsilon^{w_1}}\\
&\qquad\times\sum_{j=0}^{w_1 d-1}(-1)^j\chi(j)\varepsilon^{w_2 j}q^{(1+n-k)w_2 j}[j]_{q^{w_2}}^k.
\end{aligned}$$

## References


1. N.S. Jung, C.S. Ryoo, *Symmetric identities for twisted q-Euler zeta functions*, J. Appl. Math. & Informatics **33** (2015), 649-656.
2. J.Y. Kang, C.S. Ryoo, *On Symmetric Property for q-Genocchi Polynomials and Zeta Function*, Int. Journal of Math. Analysis **8** (2014), 9-16.
3. Yuan He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Advances in Difference Equations **246** (2013), 10 pages.
4. D. Kim, T. Kim, S.-H. Lee, J.-J. Seo, *Symmetric Identities of the q-Euler Polynomials*, Adv. Studies Theor. Phys. **7** (2014), 1149-1155.
5. T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals,* J. Nonlinear Math. Phys. **14** (2007), 15-27.
6. B.A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), 412-422.
7. C.S. Ryoo, *On the Barnes type multiple q-Euler polynomials twisted by ramified roots of unity*, Proc. Jangjeon Math. Soc. **13** (2010), 255-263.
8. C.S. Ryoo, *A note on the weighted q-Euler numbers and polynomials*, Advan. Stud. Contemp. Math. **21** (2011), 47-54.
9. C.S. Ryoo, *Some Identities of Symmetry for Carlitz's Twisted q-Euler Polynomials Associated with p-Adic q-Integral on* $\mathbb{Z}_p$, Int. Journal of Math. Analysis **9** (2015), 1747 - 1753.
10. C.S. Ryoo, *Analytic Continuation of Euler Polynomials and the Euler Zeta Function*, Discrete Dynamics in Nature and Society **2014** (2014), Article ID 568129, 6 pages.
11. C.S. Ryoo, *A Note on the Reflection Symmetries of the Genocchi polynomials*, J. Appl. Math. & Informatics **27** (2009), 1397-1404.



**N.S. Jung** received Ph.D. degree from Hannam University. Her research interests are analytic number theory and $p$-adic functional analysis.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea.
e-mail: jns4235@nate.com

**C.S. Ryoo** received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing and $p$-adic functional analysis.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea.
e-mail: ryoocs@hnu.kr