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ACCELERATION OF ONE-PARAMETER RELAXATION METHODS FOR SINGULAR SADDLE POINT PROBLEMS

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Abstract. In this paper, we first introduce two one-parameter relaxation (OPR) iterative methods for solving singular saddle point problems whose semi-convergence rate can be accelerated by using scaled preconditioners. Next we present formulas for finding their optimal parameters which yield the best semi-convergence rate. Lastly, numerical experiments are provided to examine the efficiency of the OPR methods with scaled preconditioners by comparing their performance with the parameterized Uzawa method with optimal parameters.

1. Introduction

We consider convergence acceleration of one-parameter relaxation iterative methods for solving the following saddle point problem

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, and $B \in \mathbb{R}^{m\times n}$ is a matrix with $m \geq n$. The saddle point problem (1) is important since this problem occurs very often in many different applications of scientific computing and engineering, such as the mixed finite element methods for Navier-Stokes equations [8, 9], computational fluid dynamics, constrained optimization [14], linear elasticity, the constrained least squares problems and generalized least squares problems [1, 17]. So many iterative methods for solving the saddle point problem (1) have been proposed by many researchers.

When B has a full column rank, the coefficient matrix of (1) is nonsingular and so the problem (1) is called a nonsingular saddle point problem. Many relaxation iterative methods for solving the nonsingular saddle point problem

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have been proposed. For example, Golub et al. [10] proposed the SOR-like method and presented an incomplete formula for finding one optimal parameter, Bai et al. [3] proposed the GSOR (Generalized SOR) method and presented a formula for finding two optimal parameters for the GSOR and a complete formula for finding one optimal parameter for SOR-like method, Wu et al. [15] proposed the MSSOR (Modified symmetric SOR) method, Zhang and Lu [20] studied the GSSOR (Generalized symmetric SOR) method and Chao et al. [6] presented a formula for finding two optimal parameters for the GSSOR, Yun studied several variants of Uzawa method [18, 19], and so on.

In case of B being a rank-deficient matrix, the coefficient matrix of (1) is singular and so the problem (1) is called a singular saddle point problem. Several authors have presented *semi-convergence analysis* of relaxation iterative methods for solving the singular saddle point problem (1). Zheng et al. [24] studied semi-convergence of the PU (Parameterized Uzawa) method, Li and Huang [12] examined semi-convergence of the GSSOR method, Zhang and Wang [21] studied semi-convergence of the GPIU method, Chao and Chen [5] provided semi-convergence analysis of the Uzawa-SOR method, and so on.

The purpose of this paper is to propose two one-parameter relaxation (OPR) iterative methods for solving the singular saddle point problems whose semi-convergence rate can be accelerated. This paper is organized as follows. In Section 2, we provide preliminary results for semi-convergence of the basic iterative methods. In Section 3, we first introduce two OPR methods for solving the singular saddle point problems, and then we show that their semi-convergence rate can be accelerated by using scaled preconditioners. We also present formulas for finding their optimal parameters which yield the best semi-convergence rate. In Section 4, numerical experiments are provided to examine the effectiveness of the OPR methods with scaled preconditioners by comparing their performance with the parameterized Uzawa (PU) method with optimal parameters. Lastly, some conclusions are drawn.

2. Preliminaries for semi-convergence analysis

For a square matrix G, G^* denotes the complex conjugate transpose of the matrix G, $\sigma(G)$ denotes the set of all eigenvalues of G and $\rho(G)$ denotes the spectral radius of G. We first introduce a useful lemma which will be used later.

Lemma 2.1 ([16]). *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Let us recall some useful results on iterative methods for solving singular linear systems based on matrix splitting. Let $S = M - N$ be a splitting of a singular matrix S, where M is nonsingular. Then an iterative method corresponding to this splitting for solving a consistent singular linear system

$Sx = b$ is given by

$$x_{i+1} = M^{-1}Nx_i + M^{-1}b \quad \text{for } i = 0, 1, \ldots. \tag{2}$$

It is well-known that if S is nonsingular, then the iterative method (2) is convergent if and only if $\rho(M^{-1}N) < 1$. Since S is singular, the iteration matrix $M^{-1}N$ has an eigenvalue 1 and thus $\rho(M^{-1}N)$ can not be less than 1. Thus, we need to introduce its *pseudo-spectral radius* $\nu(M^{-1}N)$

$$\nu(M^{-1}N) = \max\{|\lambda| \mid \lambda \in \sigma(M^{-1}N) - \{1\}\}.$$

For a matrix $E \in \mathbb{R}^{n\times n}$, the smallest nonnegative integer k such that $rank(E^k) = rank(E^{k+1})$ is called the *index* of E, and denoted by $k = index(E)$. Notice that a matrix T is called *semi-convergent* if $\lim_{k\to\infty} T^k$ exists, or equivalently $index(I - T) = 1$ and $\nu(T) < 1$ [4].

Theorem 2.2 ([4]). *The iterative method* (2) *is semi-convergent if and only if* $index(I - M^{-1}N) = 1$ *and* $\nu(M^{-1}N) < 1$, *i.e.,* $M^{-1}N$ *is semi-convergent.*

3. Semi-convergence acceleration of OPR methods for singular saddle point problem

In this section, we consider the saddle point problem (1) whose coefficient matrix has the following splitting

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U,$$

where

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -B \\ 0 & Q \end{pmatrix},$$

and $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix which approximates $B^T A^{-1} B$. Let

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad c = \begin{pmatrix} b \\ -q \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega I_m & 0 \\ 0 & \tau I_n \end{pmatrix},$$

where $\omega > 0$ and $\tau > 0$ are relaxation parameters, $I_m \in \mathbb{R}^{m\times m}$ and $I_n \in \mathbb{R}^{n\times n}$ denote the identity matrices of order m and n, respectively. Then the GSOR method [4] for solving the saddle point problem (1) is defined by

$$z_{k+1} = T_{\omega,\tau}\, z_k + g_{\omega,\tau}, \ \ k = 0, 1, 2, \ldots,$$

where $T_{\omega,\tau} = (D - \Omega L)^{-1}((I - \Omega)D + \Omega U)$ is an iteration matrix for the GSOR method, $g_{\omega,\tau} = (D - \Omega L)^{-1}\Omega\, c$, and I is an identity matrix of order $m + n$.

Now we introduce two one-parameter relaxation (OPR) iterative methods whose semi-convergence rate can be accelerated. One is the GSOR with $\tau = \frac{1}{\omega}$ which is called OPR-A method, and the other is the GSOR with $\tau = 1$ which is called OPR-B method in this paper. That is, the OPR-A method is defined by

$$x_{k+1} = (1 - \omega)x_k + \omega A^{-1}(b - By_k),$$

$$y_{k+1} = y_k + (\omega\, Q)^{-1}(B^T x_{k+1} - q),$$

and the OPR-B method is defined by

$$\begin{aligned} x_{k+1} &= (1-\omega)x_k + \omega A^{-1}(b - By_k), \\ y_{k+1} &= y_k + Q^{-1}(B^T x_{k+1} - q). \end{aligned}$$

We first provide convergence analysis of the OPR methods for the nonsingular saddle point problem which is required for semi-convergence analysis of the OPR methods for the singular saddle point problem. In order to study convergence of the OPR-A method for the nonsingular saddle point problem, let λ be an eigenvalue of $T_{\omega,\omega^{-1}}$ and $\binom{u}{v}$ be the corresponding eigenvector. Then we have

$$\begin{aligned} &(1-\lambda-\omega)Au = \omega Bv, \\ &\frac{\lambda}{\omega} B^T u = (\lambda - 1)Qv. \end{aligned} \tag{3}$$

The following lemma provides the convergence result for the OPR-A method.

Lemma 3.1. *Let $\mu_{\max}$ be the spectral radius of $Q^{-1}B^T A^{-1}B$. If $\mu_{\max} < 4$, then the OPR-A method converges for all $0 < \omega < 2 - \frac{\mu_{\max}}{2}$.*

Proof. Let μ be an eigenvalue of $Q^{-1}B^T A^{-1}B$ and λ be an eigenvalue of $T_{\omega,\omega^{-1}}$. Then $\mu > 0$. From equation (3), one can obtain the following quadratic equation for λ

$$\lambda^2 + (\omega + \mu - 2)\lambda + 1 - \omega = 0. \tag{4}$$

Applying Lemma 2.1 to (4), one easily obtains $0 < \omega < 2 - \frac{\mu}{2}$. If $0 < \omega < 2 - \frac{\mu_{\max}}{2}$, then $\rho(T_{\omega,\omega^{-1}}) < 1$, which completes the proof. □

Notice that if $\mu_{\max} \geq 4$ in Lemma 3.1, then the convergence region for which the OPR-A method converges may be an empty set. Next theorem provides an optimal parameter ω for which the OPR-A method performs best.

Theorem 3.2. *Let $\mu_{\min}$ and $\mu_{\max}$ be the minimum and maximum eigenvalues of $Q^{-1}B^T A^{-1}B$, respectively. Assume that $\mu_{\max} < 4$. Then the optimal parameter ω for the OPR-A method is given by $\omega = \omega_o$, where*

$$\omega_o = \min\{2\sqrt{\mu_{\min}} - \mu_{\min},\ 2\sqrt{\mu_{\max}} - \mu_{\max}\}.$$

Moreover $\rho(T_{\omega_o,\omega_o^{-1}}) = \sqrt{1-\omega_o}$. That is,

$$\rho(T_{\omega_o,\omega_o^{-1}}) = \begin{cases} |1 - \sqrt{\mu_{\min}}| & \textit{if } \omega_o = 2\sqrt{\mu_{\min}} - \mu_{\min} \\ |1 - \sqrt{\mu_{\max}}| & \textit{if } \omega_o = 2\sqrt{\mu_{\max}} - \mu_{\max}. \end{cases}$$

Proof. Let μ be an eigenvalue of $Q^{-1}B^T A^{-1}B$ and λ be an eigenvalue of $T_{\omega,\omega^{-1}}$. From the quadratic equation (4) for λ, one obtains two roots

$$\lambda = \frac{1}{2}\left((2 - \omega - \mu) \pm \sqrt{(\omega+\mu)^2 - 4\mu}\right).$$

Let $f(\omega) = 2 - \omega - \mu$ and $g(\omega) = (\omega + \mu)^2 - 4\mu$. The necessary and sufficient condition for the roots λ to be real is $g(\omega) \geq 0$, which is equivalent to $\omega \geq 2\sqrt{\mu} - \mu$. Since $\mu_{\max} < 4$, $2\sqrt{\mu} - \mu < 2 - \frac{\mu}{2}$. Hence one obtains

$$|\lambda| = \begin{cases} \frac{1}{2}\left(|f(\omega)| + \sqrt{g(\omega)}\right) & \text{if } 2\sqrt{\mu} - \mu \leq \omega < 2 - \frac{\mu}{2} \\ \sqrt{1-\omega} & \text{if } 0 < \omega \leq 2\sqrt{\mu} - \mu. \end{cases} \tag{5}$$

Notice that $(2\sqrt{\mu} - \mu) \in (0, 1]$ for $\mu \in (0, 4)$ and it has the maximum value 1 at $\mu = 1$. Since $\frac{\partial}{\partial\omega}(|f| + \sqrt{g}) = -\text{sign}(f) + \frac{\omega+\mu}{\sqrt{g}} > 0$ for $\omega \geq 2\sqrt{\mu} - \mu$, $\frac{1}{2}(|f| + \sqrt{g})$ is an increasing function for $\omega \geq 2\sqrt{\mu} - \mu$. Clearly $\sqrt{1-\omega}$ is a decreasing function for $0 < \omega \leq 2\sqrt{\mu} - \mu$. Thus, (5) implies that given μ, $|\lambda|$ takes the minimum $\sqrt{1-\omega} = |1 - \sqrt{\mu}|$ when $\omega = 2\sqrt{\mu} - \mu$. If S is a set containing all eigenvalues of $Q^{-1}B^TA^{-1}B$, then

$$\min_{\omega} \rho(T_{\omega,\omega^{-1}}) = \max_{\mu} \min_{\omega} |\lambda| = \max_{\omega = 2\sqrt{\mu}-\mu,\ \mu\in S} \sqrt{1-\omega} = \sqrt{1-\omega_o},$$

where $\omega_o = \min\{2\sqrt{\mu_{\min}} - \mu_{\min},\ 2\sqrt{\mu_{\max}} - \mu_{\max}\}$. Hence the theorem follows. □

We now study convergence of the OPR-B method for the nonsingular saddle point problem. Let λ be an eigenvalue of $T_{\omega,1}$ and $\binom{u}{v}$ be the corresponding eigenvector. Then we have

$$\begin{aligned} (1 - \lambda - \omega)Au &= \omega Bv, \\ \lambda B^T u &= (\lambda - 1)Qv. \end{aligned} \tag{6}$$

The following lemma provides the convergence result for the OPR-B method.

Lemma 3.3. *Let $\mu_{\max}$ be the spectral radius of $Q^{-1}B^TA^{-1}B$. If $0 < \omega < \frac{4}{2+\mu_{\max}}$, then the OPR-B method converges.*

Proof. Let μ be an eigenvalue of $Q^{-1}B^TA^{-1}B$ and λ be an eigenvalue of $T_{\omega,1}$. From equation (6), one can obtain the following quadratic equation for λ

$$\lambda^2 + (\omega\mu + \omega - 2)\lambda + 1 - \omega = 0. \tag{7}$$

Applying Lemma 2.1 to (7), one easily obtains $0 < \omega < \frac{4}{2+\mu}$. If $0 < \omega < \frac{4}{2+\mu_{\max}}$, then $\rho(T_{\omega,1}) < 1$, which completes the proof. □

Next theorem provides an optimal parameter ω for which the OPR-B method performs best.

Theorem 3.4. *Let $\mu_{\min}$ and $\mu_{\max}$ be the minimum and maximum eigenvalues of $Q^{-1}B^TA^{-1}B$, respectively. Then the optimal parameter ω for the OPR-B method is given by $\omega = \omega_o$, where*

$$\omega_o = \min\left\{\frac{4\,\mu_{\min}}{(1+\mu_{\min})^2},\ \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}\right\}.$$

Moreover $\rho(T_{\omega_o,1}) = \sqrt{1-\omega_o}$. *That is,*

$$\rho(T_{\omega_o,1}) = \begin{cases} \frac{|1-\mu_{\min}|}{1+\mu_{\min}} & \text{if } \omega_o = \frac{4\,\mu_{\min}}{(1+\mu_{\min})^2} \\ \frac{|1-\mu_{\max}|}{1+\mu_{\max}} & \text{if } \omega_o = \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}. \end{cases}$$

Proof. Let μ be an eigenvalue of $Q^{-1}B^TA^{-1}B$ and λ be an eigenvalue of $T_{\omega,1}$. From the quadratic equation (7) for λ, one obtains two roots

$$\lambda = \frac{1}{2}\left((2-\omega-\omega\mu) \pm \sqrt{\omega^2(1+\mu)^2 - 4\omega\mu}\right).$$

Let $f(\omega) = 2-\omega-\omega\mu$ and $g(\omega) = \omega^2(1+\mu)^2 - 4\omega\mu$. The necessary and sufficient condition for the roots λ to be real is $g(\omega) \geq 0$, which is equivalent to $\omega \geq \frac{4\,\mu}{(1+\mu)^2}$. Hence one obtains

$$|\lambda| = \begin{cases} \frac{1}{2}\left(\,|f(\omega)| + \sqrt{g(\omega)}\,\right) & \text{if } \frac{4\,\mu}{(1+\mu)^2} \leq \omega < \frac{4}{2+\mu_{\max}} \\ \sqrt{1-\omega} & \text{if } 0 < \omega \leq \frac{4\,\mu}{(1+\mu)^2}. \end{cases} \tag{8}$$

Notice that $\frac{4\,\mu}{(1+\mu)^2} \in (0,1]$ for $\mu > 0$ and it has the maximum value 1 at $\mu = 1$. Also note that $\frac{\partial}{\partial\omega}(|f| + \sqrt{g}) = -\text{sign}(f)(1+\mu) + \frac{\omega(\mu+1)^2-2\mu}{\sqrt{g}}$. Since $\frac{\omega(\mu+1)^2-2\mu}{\sqrt{g}} > 1+\mu$, $\frac{\partial}{\partial\omega}(|f| + \sqrt{g}) > 0$ for $\omega \geq \frac{4\,\mu}{(1+\mu)^2}$. Hence $\frac{1}{2}(|f| + \sqrt{g})$ is an increasing function for $\omega \geq \frac{4\,\mu}{(1+\mu)^2}$. Clearly $\sqrt{1-\omega}$ is a decreasing function for $0 < \omega \leq \frac{4\,\mu}{(1+\mu)^2}$. Thus, (8) implies that given μ, $|\lambda|$ takes the minimum $\sqrt{1-\omega} = \frac{|1-\mu|}{1+\mu}$ when $\omega = \frac{4\,\mu}{(1+\mu)^2}$. If S is a set containing all eigenvalues of $Q^{-1}B^TA^{-1}B$, then

$$\min_{\omega} \rho(T_{\omega,1}) = \max_{\mu} \min_{\omega} |\lambda| = \max_{\omega=\frac{4\,\mu}{(1+\mu)^2},\ \mu\in S} \sqrt{1-\omega} = \sqrt{1-\omega_o},$$

where $\omega_o = \min\left\{\frac{4\,\mu_{\min}}{(1+\mu_{\min})^2},\ \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}\right\}$. Hence the theorem follows. □

We next consider **semi-convergence** of the OPR methods for the consistent singular saddle point problem (1), where B is rank-deficient with $rank(B) < n \leq m$. We first provide semi-convergence analysis for the OPR-A method. Let λ be an eigenvalue of $T_{\omega,\omega^{-1}}$ and $\binom{u}{v}$ be the corresponding eigenvector.

Lemma 3.5. *Suppose that* $\omega > 0$. *Then* $\lambda = 1$ *if and only if* $u = 0$.

Proof. This lemma can be proved similarly as was done in Lemma 3.7 in [22] □

Lemma 3.6 ([23])**.** *Let* $S = M - N$ *be a splitting of* S *with* $T = M^{-1}N$. *Then* $index(I-T) = 1$ *if and only if for any* $y \in R(S) - \{0\}$, $y \notin N(SM^{-1})$, *where* $R(S)$ *and* $N(S)$ *denote the range space and the null space of* S, *respectively.*

Lemma 3.7. *If* $\omega > 0$, *then* $index(I - T_{\omega,\omega^{-1}}) = 1$.

Proof. Notice that $\frac{1}{\omega}A$ and Q are symmetric positive definite. Using Lemma 3.6, the proof of this theorem can be done similarly to that of Theorem 3.6 in [22]. □

Lemma 3.8. *If $0 \neq u \in N(B^T)$ and $0 < \omega < 2$, then $|\lambda| < 1$.*

Proof. Since $u \neq 0$ and $u \in N(B^T)$, from (3) and Lemma 3.5 $v = 0$. From the first equation of (3), $(1 - \lambda - \omega)Au = 0$. It follows that $\lambda = 1 - \omega$. Since $0 < \omega < 2$, $|\lambda| < 1$ is obtained. □

Lemma 3.9. *If $u \notin N(B^T)$ and $\mu = \frac{u^* BQ^{-1}B^T u}{u^* Au}$, then*

$$|\lambda| < 1 \quad \textit{if and only if} \quad 0 < \omega < 2 - \frac{\mu}{2}.$$

Proof. Since $u \notin N(B^T)$, $\lambda \neq 1$ from Lemma 3.5. From (3), one obtains

$$(1 - \lambda - \omega) = \frac{\lambda}{\lambda - 1} \frac{u^* BQ^{-1}B^T u}{u^* Au} = \frac{\lambda}{\lambda - 1}\mu. \tag{9}$$

Notice that $\mu > 0$ since $u \notin N(B^T)$. Rearranging (9), one has the following real quadratic equation

$$\lambda^2 + (\omega + \mu - 2)\lambda + 1 - \omega = 0. \tag{10}$$

Applying Lemma 2.1 to equation (10), one easily obtains

$$0 < \omega < 2 - \frac{\mu}{2},$$

which completes the proof. □

The following theorem provides semi-convergence result of the OPR-A method.

Theorem 3.10. *Let $\mu_{\max}$ be the largest eigenvalue of $Q^{-1}B^T A^{-1} B$. If $0 < \omega < 2 - \frac{\mu_{\max}}{2}$, then the OPR-A method is semi-convergent.*

Proof. Let $\mu_{\max} = \max_{z \neq 0} \frac{z^* BQ^{-1}B^T z}{z^* Az}$. Then it is easy to show that $\mu_{\max}$ is equal to the largest eigenvalue of $Q^{-1}B^T A^{-1} B$. From Lemmas 3.8 and 3.9, $\nu(T_{\omega,\omega^{-1}}) < 1$ is obtained when $0 < \omega < 2 - \frac{\mu_{\max}}{2}$. Since $index(I - T_{\omega,\omega^{-1}}) = 1$ from Lemma 3.7, Theorem 2.2 implies that the OPR-A method is semi-convergent. □

Next theorem provides an optimal parameter ω and the corresponding optimal semi-convergence factor for the OPR-A method.

Theorem 3.11. *Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest* **nonzero** *eigenvalues of $Q^{-1}B^T A^{-1} B$, respectively. Assume that $\mu_{\max} < 4$. Then the optimal parameter ω for the OPR-A method is given by $\omega = \omega_o$, where*

$$\omega_o = \min\{2\sqrt{\mu_{\min}} - \mu_{\min},\ 2\sqrt{\mu_{\max}} - \mu_{\max}\}. \tag{11}$$

Moreover $\nu(T_{\omega_o,\omega_o^{-1}}) = \sqrt{1-\omega_o}$. *That is,*

$$\nu(T_{\omega_o,\omega_o^{-1}}) = \begin{cases} |1-\sqrt{\mu_{\min}}| & \text{if } \omega_o = 2\sqrt{\mu_{\min}} - \mu_{\min} \\ |1-\sqrt{\mu_{\max}}| & \text{if } \omega_o = 2\sqrt{\mu_{\max}} - \mu_{\max}. \end{cases} \tag{12}$$

Proof. Notice that

$$T_{\omega,\omega^{-1}} = \begin{pmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ (\frac{1}{\omega}-1)Q^{-1}B^T & I_n - Q^{-1}B^TA^{-1}B \end{pmatrix}.$$

Assume that the rank of B is r, i.e., $r = rank(B) < n$. Let

$$B = W\Sigma V^*,\ \Sigma = (B_1\ 0) \in \mathbb{R}^{m\times n} \quad \text{and} \quad B_1 = \begin{pmatrix} \Sigma_r \\ 0 \end{pmatrix} \in \mathbb{R}^{m\times r}$$

be the singular value decomposition of B, where W and V are unitary matrices, $\Sigma_r = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ and σ_i's are positive singular values of B. Let us define an $(m+n)\times(m+n)$ unitary matrix $\mathcal{P}$ as

$$\mathcal{P} = \begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix}.$$

If we let $\hat{T}_{\omega,\omega^{-1}} = \mathcal{P}^* T_{\omega,\omega^{-1}} \mathcal{P}$, $\hat{A} = W^*AW$ and $\hat{Q} = V^*QV$, then by simple calculation one obtains

$$\hat{T}_{\omega,\omega^{-1}} = \begin{pmatrix} (1-\omega)I_m & -\omega\hat{A}^{-1}\Sigma \\ (\frac{1}{\omega}-1)\hat{Q}^{-1}\Sigma^T & I_n - \hat{Q}^{-1}\Sigma^T\hat{A}^{-1}\Sigma \end{pmatrix}. \tag{13}$$

Assume that the unitary matrix V is partitioned into the block form $V = (V_1\ V_2)$ with $V_1 \in \mathbb{R}^{n\times r}$. Then (13) can be rewritten as

$$\hat{T}_{\omega,\omega^{-1}} = \begin{pmatrix} \hat{H}_{\omega,\omega^{-1}} & 0 \\ \hat{L}_{\omega,\omega^{-1}} & I_{n-r} \end{pmatrix}, \tag{14}$$

where

$$\hat{H}_{\omega,\omega^{-1}} = \begin{pmatrix} (1-\omega)I_m & -\omega\hat{A}^{-1}B_1 \\ (\frac{1}{\omega}-1)V_1^*Q^{-1}V_1B_1^T & I_r - V_1^*Q^{-1}V_1B_1^T\hat{A}^{-1}B_1 \end{pmatrix}$$

and

$$\hat{L}_{\omega,\omega^{-1}} = \left((\tfrac{1}{\omega}-1)V_2^*Q^{-1}V_1B_1^T \quad -V_2^*Q^{-1}V_1B_1^T\hat{A}^{-1}B_1 \right).$$

Since B_1 is of full column rank, $\hat{H}_{\omega,\omega^{-1}}$ is the iteration matrix of the OPR-A method applied to the following nonsingular saddle point problem

$$\begin{pmatrix} \hat{A} & B_1 \\ -B_1^T & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{b} \\ -\hat{q} \end{pmatrix} \tag{15}$$

with the preconditioning matrix $\hat{Q}_1 = (V_1^*Q^{-1}V_1)^{-1}$. Hence, from (14) and (15), one obtains

$$\nu(T_{\omega,\omega^{-1}}) = \nu(\hat{T}_{\omega,\omega^{-1}}) = \nu(\hat{H}_{\omega,\omega^{-1}}) = \rho(\hat{H}_{\omega,\omega^{-1}}). \tag{16}$$

From (16), it can be seen that finding an optimal parameter ω which minimizes $\nu(T_{\omega,\omega^{-1}})$ is equivalent to finding an optimal parameter ω which minimizes $\rho(\hat{H}_{\omega,\omega^{-1}})$. Applying Theorem 3.2 to (15), (11) and (12) are obtained, where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest eigenvalues of $\hat{Q}_1^{-1}B_1^T\hat{A}^{-1}B_1$ respectively. On the other hand

$$\begin{aligned} V^*(Q^{-1}B^TA^{-1}B)V &= \hat{Q}^{-1}\Sigma^T\hat{A}^{-1}\Sigma \\ &= \begin{pmatrix} \hat{Q}_1^{-1}B_1^T\hat{A}^{-1}B_1 & 0 \\ V_2^*Q^{-1}V_1B_1^T\hat{A}^{-1}B_1 & 0 \end{pmatrix}. \end{aligned}$$

Hence $\mu_{\min}$ and $\mu_{\max}$ are also the smallest and largest **nonzero** eigenvalues of the matrix $Q^{-1}B^TA^{-1}B$, respectively. Therefore, the proof is complete. $\square$

As can be seen from Theorems 3.10 and 3.11, one drawback of the OPR-A method is that it may require a rather strong condition $\mu_{\max} < 4$ which is not generally true for some types of preconditioners Q. To remedy this problem, we need to scale the preconditioner Q so that $0 < \mu_{\min},\ \mu_{\max} < 4$. From Theorem 3.11, it can be also seen that in order to minimize $\nu(T_{\omega_o,\omega_o^{-1}})$, Q needs to be scaled so that $2\sqrt{\mu_{\min}} - \mu_{\min} = 2\sqrt{\mu_{\max}} - \mu_{\max}$. Next lemma shows how to scale the preconditioner Q so that $\nu(T_{\omega_o,\omega_o^{-1}})$ can be minimized.

Lemma 3.12. *Let $Q_s = s\,Q$ be a scaled preconditioner, where $s > 0$ is a scaling factor, and let $\nu_{\min}$ and $\nu_{\max}$ be the smallest and largest* **nonzero** *eigenvalues of $Q_s^{-1}B^TA^{-1}B$, respectively. Then $2\sqrt{\nu_{\min}} - \nu_{\min} = 2\sqrt{\nu_{\max}} - \nu_{\max}$ if and only if $s = \left(\frac{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}{2}\right)^2$, where $\mu_{\min}$ and $\mu_{\max}$ denote the smallest and largest* **nonzero** *eigenvalues of $Q^{-1}B^TA^{-1}B$, respectively.*

Proof. Since $Q_s^{-1}B^TA^{-1}B = \frac{1}{s}Q^{-1}B^TA^{-1}B$, $\nu_{\min} = \frac{\mu_{\min}}{s}$ and $\nu_{\max} = \frac{\mu_{\max}}{s}$. Using these relations, one obtains the following equivalent equations

$$\begin{aligned} 2\sqrt{\nu_{\min}} - \nu_{\min} &= 2\sqrt{\nu_{\max}} - \nu_{\max}, \\ 2\sqrt{\frac{\mu_{\min}}{s}} - \frac{\mu_{\min}}{s} &= 2\sqrt{\frac{\mu_{\max}}{s}} - \frac{\mu_{\max}}{s}, \\ 2(\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}) &= \frac{\mu_{\max} - \mu_{\min}}{\sqrt{s}}. \end{aligned} \tag{17}$$

Solving the third equation of (17) for s, $s = \left(\frac{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}{2}\right)^2$ is obtained. $\square$

Next theorem provides an optimal parameter and an optimal semi-convergence factor for the OPR-A method with the scaled preconditioner Q_s which is chosen by Lemma 3.12, and it also shows that $0 < \nu_{\min},\ \nu_{\max} < 4$.

Theorem 3.13. *Let $Q_s = s\,Q$ be a scaled preconditioner, where $s > 0$ is a scaling factor, and let $\nu_{\min}$ and $\nu_{\max}$ be the smallest and largest* **nonzero** *eigenvalues of $Q_s^{-1}B^TA^{-1}B$, respectively. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest* *and largest* **nonzero** *eigenvalues of* $Q^{-1}B^TA^{-1}B$*, respectively. If* s *is chosen such that* $s = \left(\frac{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}{2}\right)^2$*, then* $0 < \nu_{\min},\ \nu_{\max} < 4$ *and*

$$\tilde{\omega}_o = 2\sqrt{\nu_{\min}} - \nu_{\min} = 2\sqrt{\nu_{\max}} - \nu_{\max} = \frac{4\sqrt{\mu_{\min}}\sqrt{\mu_{\max}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}.$$

Moreover, the following holds

$$\nu(\tilde{T}_{\tilde{\omega}_o,\tilde{\omega}_o^{-1}}) = |1 - \sqrt{\nu_{\min}}\,| = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}},$$

where $\tilde{\omega}_o$ *and* $\nu(\tilde{T}_{\tilde{\omega}_o,\tilde{\omega}_o^{-1}})$ *refer to the optimal parameter and the optimal semi-convergence factor for the OPR-A with the scaled preconditioner* Q_s*, respectively.*

Proof. From Lemma 3.12, $2\sqrt{\nu_{\min}} - \nu_{\min} = 2\sqrt{\nu_{\max}} - \nu_{\max}$.

Since $s = \left(\frac{\sqrt{\mu_{\min}}+\sqrt{\mu_{\max}}}{2}\right)^2$,

$$\begin{aligned} \nu_{\min} &= \frac{\mu_{\min}}{s} = \frac{4\mu_{\min}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2} < 4, \\ \nu_{\max} &= \frac{\mu_{\max}}{s} = \frac{4\mu_{\max}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2} < 4. \end{aligned} \tag{18}$$

Using (18) and Theorem 3.11, one obtains the remaining relations

$$\tilde{\omega}_o = 2\sqrt{\nu_{\min}} - \nu_{\min} = \frac{4\sqrt{\mu_{\min}}\sqrt{\mu_{\max}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2},$$

$$\rho(\tilde{T}_{\tilde{\omega}_o,\tilde{\omega}_o^{-1}}) = |1 - \sqrt{\nu_{\min}}\,| = |1 - \sqrt{\nu_{\max}}\,| = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}. \qquad \square$$

From Theorem 3.13, it can be seen that the optimal semi-convergence factor of the OPR-A method with the scaled preconditioner Q_s is the same as that of the PU method [24] with the preconditioner Q. Notice that the scaling factor s in Theorem 3.13 can be easily computed using MATLAB by computing only the largest and smallest nonzero eigenvalues of $Q^{-1}B^TA^{-1}B$.

In a similar manner as was done in Theorems 3.10 and 3.11 for the OPR-A method, we can obtain the following semi-convergence results for the OPR-B method by using equation (6) and Theorem 3.4.

Theorem 3.14. *Let* $\mu_{\max}$ *be the spectral radius of* $Q^{-1}B^TA^{-1}B$*. If* $0 < \omega < \frac{4}{2+\mu_{\max}}$*, then the OPR-B method is semi-convergent.*

Theorem 3.15. *Let* $\mu_{\min}$ *and* $\mu_{\max}$ *be the smallest and largest* **nonzero** *eigenvalues of* $Q^{-1}B^TA^{-1}B$*, respectively. Then the optimal parameter* ω *for the OPR-B method is given by* $\omega = \omega_o$*, where*

$$\omega_o = \min\left\{\frac{4\,\mu_{\min}}{(1+\mu_{\min})^2}, \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}\right\}.$$

Moreover $\nu(T_{\omega_o,1}) = \sqrt{1-\omega_o}$. *That is,*

$$\nu(T_{\omega_o,1}) = \begin{cases} \frac{|1-\mu_{\min}|}{1+\mu_{\min}} & \textit{if } \omega_o = \frac{4\,\mu_{\min}}{(1+\mu_{\min})^2} \\ \frac{|1-\mu_{\max}|}{1+\mu_{\max}} & \textit{if } \omega_o = \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}. \end{cases}$$

From Theorem 3.15, it can be seen that in order to minimize $\nu(T_{\omega_o,1})$, the preconditioner Q needs to be scaled so that

$$\frac{4\,\mu_{\min}}{(1+\mu_{\min})^2} = \frac{4\,\mu_{\max}}{(1+\mu_{\max})^2}.$$

Next theorem shows how to choose a scaled preconditioner $Q_s = s\,Q$ such that $\nu(T_{\omega_o,1})$ can be minimized, and it provides an optimal parameter and an optimal semi-convergence factor for the OPR-B method with the scaled preconditioner Q_s.

Theorem 3.16. *Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest* **nonzero** *eigenvalues of $Q^{-1}B^TA^{-1}B$, respectively. Let $Q_s = s\,Q$ be a scaled preconditioner, where $s > 0$ is a scaling factor, and let $\nu_{\min}$ and $\nu_{\max}$ be the smallest and largest* **nonzero** *eigenvalues of $Q_s^{-1}B^TA^{-1}B$, respectively. If $s = \sqrt{\mu_{\min}\,\mu_{\max}}$, then $\frac{4\,\nu_{\min}}{(1+\nu_{\min})^2} = \frac{4\,\nu_{\max}}{(1+\nu_{\max})^2}$. Moreover*

$$\tilde{\omega}_o = \frac{4\,\nu_{\min}}{(1+\nu_{\min})^2} = \frac{4\sqrt{\mu_{\min}}\sqrt{\mu_{\max}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2}$$

and

$$\nu(\tilde{T}_{\tilde{\omega}_o,1}) = \frac{|1-\nu_{\min}|}{1+\nu_{\min}} = \frac{|1-\nu_{\max}|}{1+\nu_{\max}} = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}},$$

where $\tilde{\omega}_o$ and $\nu(\tilde{T}_{\tilde{\omega}_o,1})$ refer to the optimal parameter and the optimal semi-convergence factor for the OPR-B method with the scaled preconditioner Q_s, respectively.

Proof. Since $Q_s^{-1}B^TA^{-1}B = \frac{1}{s}Q^{-1}B^TA^{-1}B$, $\nu_{\min} = \frac{\mu_{\min}}{s}$ and $\nu_{\max} = \frac{\mu_{\max}}{s}$. Since $s = \sqrt{\mu_{\min}\,\mu_{\max}}$, one obtains

$$\nu_{\min} = \sqrt{\frac{\mu_{\min}}{\mu_{\max}}} \quad \text{and} \quad \nu_{\max} = \sqrt{\frac{\mu_{\max}}{\mu_{\min}}} \tag{19}$$

Using (19) and Theorem 3.15, it can be easily shown that

$$\tilde{\omega}_o = \frac{4\,\nu_{\min}}{(1+\nu_{\min})^2} = \frac{4\,\nu_{\max}}{(1+\nu_{\max})^2} = \frac{4\sqrt{\mu_{\min}}\sqrt{\mu_{\max}}}{(\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}})^2},$$

$$\nu(\tilde{T}_{\tilde{\omega}_o,1}) = \frac{|1-\nu_{\min}|}{1+\nu_{\min}} = \frac{|1-\nu_{\max}|}{1+\nu_{\max}} = \frac{\sqrt{\mu_{\max}}-\sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}}+\sqrt{\mu_{\min}}}.$$

Hence the proof is complete. □

From Theorem 3.16, it can be also seen that optimal convergence factor of the OPR-B method with the scaled preconditioner Q_s is the same as that of the PU method with the preconditioner Q.

4. Numerical results

In this section, we provide numerical experiments to examine the effectiveness of the OPR methods by comparing their performance with the PU method with optimal parameters. To see how much semi-convergence rate of the OPR methods can be accelerated, we provide performance results of both the OPR methods and the OPR methods with scaled preconditioners $Q_s = s\,Q$ and $Q_{s+\epsilon} = (s+\epsilon)Q$, where s is the scaling factor defined in Theorems 3.13 or 3.16, and ϵ is a positive number which is chosen appropriately small as compared with s. In Tables 2 to 5, *Iter* denotes the number of iteration steps and *CPU* denotes the elapsed CPU time in seconds. In all experiments, the right hand side vector $(b^T, -q^T)^T \in \mathbb{R}^{m+n}$ was chosen such that the exact solution of the saddle point problem (1) is $(x_*^T, y_*^T)^T = (1, 1, \ldots, 1)^T \in \mathbb{R}^{m+n}$, and the initial vector was set to the zero vector. All iterations for the singular saddle point problem are terminated if the current iteration satisfies RES $< 10^{-6}$, where *RES* is defined by

$$\text{RES} = \frac{\sqrt{\|b - Ax_k - By_k\|^2 + \|q - B^T x_k\|^2}}{\sqrt{\|b\|^2 + \|q\|^2}},$$

where $\|\cdot\|$ denotes the L_2-norm.

Example 4.1. Consider the Stokes equations of the following form: find $\mathbf{u}$ and v such that

$$\begin{cases} -\triangle \mathbf{u} + \nabla w = \mathbf{f} & \text{in } \Omega \\ \quad -\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \end{cases} \tag{20}$$

where $\Omega = (0,1)\times(0,1)$, $\mathbf{u}$ is a vector-valued function representing the velocity, and w is a scalar function representing the pressure. The boundary conditions are $\mathbf{u} = (0,0)^T$ on the three fixed walls ($x = 0,\ y = 0,\ x = 1$) and $\mathbf{u} = (1,0)^T$ on the moving wall ($y = 1$). Dividing Ω into a uniform grid with mesh size $h = \frac{1}{p}$ and discretizing (20) by using MAC (marker and cell) finite difference scheme [7, 11], the singular saddle point problem (1) is obtained, where $A \in \mathbb{R}^{2p(p-1)\times 2p(p-1)}$ is a symmetric positive definite matrix and $B = \begin{pmatrix}\hat{B} & \tilde{B}\end{pmatrix} \in \mathbb{R}^{2p(p-1)\times p^2}$ is a rank-deficient matrix of $rank(B) = p^2 - 1$ with $\hat{B} \in \mathbb{R}^{2p(p-1)\times(p^2-1)}$ and $\tilde{B} \in \mathbb{R}^{2p(p-1)}$. For this example, $m = 2p(p-1)$ and $n = p^2$. Thus the total number of variables is $3p^2 - 2p$. Numerical results for this example are listed in Tables 2 and 3. In Table 2, numerical results for the OPR-B method are not listed since it converge so slowly ($Iter > 2000$). In Table 3, numerical results for the OPR methods are not listed since they do not converge because of $\mu_{\max} > 4$. See Table 1 for the values of $\mu_{\max} = \rho(Q^{-1}B^T A^{-1}B)$.

Example 4.2. We consider the singular saddle point problem (1) used in [24], in which

$$A = \begin{pmatrix} I\otimes T + T\otimes I & 0 \\ 0 & I\otimes T + T\otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times 2p^2},$$

$$B = \begin{pmatrix}\hat{B} & \tilde{B}\end{pmatrix} = \begin{pmatrix}\hat{B} & b_1 & b_2\end{pmatrix} \in \mathbb{R}^{2p^2\times(p^2+2)}, \quad \hat{B} = \begin{pmatrix} I\otimes F \\ F\otimes I\end{pmatrix} \in \mathbb{R}^{2p^2\times p^2},$$

$$b_1 = \hat{B}\begin{pmatrix} e_{p^2/2} \\ 0\end{pmatrix}, \quad b_2 = \hat{B}\begin{pmatrix} 0 \\ e_{p^2/2}\end{pmatrix}, \quad e_{p^2/2} = (1,1,\ldots,1)^T \in \mathbb{R}^{p^2/2},$$

$$T = \frac{1}{h^2}\cdot \text{tridiag}(-1,2,-1) \in \mathbb{R}^{p\times p},\ F = \frac{1}{h}\cdot\text{tridiag}(-1,1,0) \in \mathbb{R}^{p\times p},$$

with $\otimes$ denoting the Kronecker product and $h = \frac{1}{p+1}$ the discretization mesh size. For this example, $m = 2p^2$ and $n = p^2 + 2$. Thus the total number of variables is $3p^2 + 2$. Clearly B is a rank-deficient matrix of $rank(B) = p^2 < n$. Numerical results for this example are listed in Tables 4 and 5. In Table 5, numerical results for the OPR methods are not listed since they do not converge because of $\mu_{\max} > 4$ (see Table 1).

We choose the preconditioning matrices Q as an approximation to the matrix $B^TA^{-1}B$, according to two cases listed in Table 1, where $\hat{Q}$ denotes a block diagonal matrix consisting of two submatrices $\hat{B}^T\hat{A}^{-1}\hat{B}$ and $\tilde{B}^T\tilde{B}$. All numerical tests are carried out on a PC equipped with Intel Core i5-4570 3.2GHz CPU and 8GB RAM using MATLAB R2015a. For test runs of the OPR methods with the scaled preconditioner $Q_{s+\epsilon}$, we have tried the values of $\epsilon \in [1\times10^{-d}, 5\times10^{-d}]$ in Tables 2 to 5, where d is chosen to be 2, 3 or 4 depending upon the size of s. For all of these values of ϵ, the OPR methods with $Q_{s+\epsilon}$ performs at least as well as the PU method, and the value of ϵ reported in Tables 2 to 5 is the best one out of test runs for five different values of ϵ, i.e., $\epsilon = k\times10^{-d}$ $(k = 1,2,3,4,5)$. According to numerical experiments, it is recommended that a near optimal value of ϵ may be chosen by the following rule:

when $0.01 \le s < 1$, choose ϵ such that $\epsilon \in [\,1\times10^{-4},\ 5\times10^{-4}\,]$
when $1 \le s < 10$, choose ϵ such that $\epsilon \in [\,1\times10^{-3},\ 4\times10^{-3}\,]$
when $10 \le s < 100$, choose ϵ such that $\epsilon \in [\,1\times10^{-2},\ 3\times10^{-2}\,]$.

As can be expected from the theorems described in Section 3, the OPR methods with the scaled preconditioner Q_s perform as well as the PU method. The OPR methods with the scaled preconditioner $Q_{s+\epsilon}$ perform better than the PU method. More specifically, they perform significantly better than PU method for Example 4.1 and Case II of Example 4.2 where $\mu_{\max} > 4$ (see Tables 2 to 5).

TABLE 1. Choices of the matrix Q with $\hat{Q} = \text{Diag}(\hat{B}^T\hat{A}^{-1}\hat{B}, \tilde{B}^T\tilde{B})$ and the values of $\mu_{\max} = \rho(Q^{-1}B^TA^{-1}B)$ for Examples 4.1 and 4.2

		Example 4.1($\mu_{\max}$)		Example 4.2($\mu_{\max}$)		
Case Number	Q	$n = 576$	$n = 1024$	$n = 578$	$n = 1026$	Description
I	tridiag($\hat{Q}$)	1.785	1.821	1.668	1.696	$\hat{A} = \text{tridiag}(A)$
II	$\hat{Q}$	102.8	181.9	98.40	169.7	$\hat{A} = \text{diag}(A)$

TABLE 2. Numerical results for Example 4.1 with Case I of Q.

	m	1104	1984
	n	576	1024
PU	ω_o	0.0949	0.0707
	τ_o	22.49	29.94
	$Iter$	452	630
	CPU	0.468	1.142
OPR-A	ω_o	0.0655	0.0489
	$Iter$	473	637
	CPU	0.488	1.147
OPR-A with Q_s	ω_o	0.0946	0.0707
	s	0.4687	0.4721
	$Iter$	453	630
	CPU	0.468	1.142
OPR-A with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.0948	0.0707
	ϵ	0.0003	0.0002
	$Iter$	340	464
	CPU	0.352	0.843
OPR-B with Q_s	ω_o	0.0947	0.0705
	s	0.0444	0.0333
	$Iter$	452	632
	CPU	0.468	1.143
OPR-B with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.0942	0.0703
	ϵ	0.0004	0.0003
	$Iter$	332	456
	CPU	0.344	0.829

TABLE 3. Numerical results for Example 4.1 with Case II of Q.

	m	1104	1984
	n	576	1024
PU	ω_o	0.2442	0.1895
	τ_o	0.1392	0.1047
	$Iter$	132	177
	CPU	0.176	0.420
OPR-A with Q_s	ω_o	0.2440	0.1895
	s	29.42	50.38
	$Iter$	132	177
	CPU	0.176	0.420
OPR-A with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.2442	0.1895
	ϵ	0.01	0.03
	$Iter$	100	127
	CPU	0.131	0.300
OPR-B with Q_s	ω_o	0.2442	0.1895
	s	7.185	9.549
	$Iter$	132	173
	CPU	0.176	0.407
OPR-B with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.2441	0.1895
	ϵ	0.004	0.002
	$Iter$	100	145
	CPU	0.131	0.342

5. Conclusions

We introduced two one-parameter relaxation (OPR) iterative methods for solving the singular saddle point problems whose semi-convergence rate can be accelerated by using scaled preconditioners. Both theoretical and computational results show that the OPR methods with the scaled preconditioner Q_s performs as well as the PU method with optimal parameters. In addition, the

TABLE 4. Numerical results for Example 4.2 with Case I of Q.

	m	1152	2048
	n	578	1026
PU	ω_o	0.5622	0.5115
	τ_o	2.9447	3.3270
	$Iter$	44	52
	CPU	0.048	0.097
OPR-A	ω_o	0.4568	0.4083
	$Iter$	51	59
	CPU	0.055	0.109
OPR-A with Q_s	ω_o	0.5622	0.5115
	s	0.6040	0.5877
	$Iter$	44	51
	CPU	0.048	0.094
OPR-A with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.5621	0.5113
	ϵ	0.0004	0.0005
	$Iter$	41	45
	CPU	0.045	0.083
OPR-B	ω_o	0.2420	0.1920
	$Iter$	111	144
	CPU	0.119	0.263
OPR-B with Q_s	ω_o	0.5622	0.5114
	s	0.3396	0.3006
	$Iter$	44	51
	CPU	0.048	0.094
OPR-B with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.5619	0.5112
	ϵ	0.0003	0.0002
	$Iter$	38	46
	CPU	0.041	0.085

TABLE 5. Numerical results for Example 4.2 with Case II of Q.

	m	1152	2048
	n	578	1026
PU	ω_o	0.2489	0.1956
	τ_o	0.1423	0.1084
	$Iter$	131	174
	CPU	0.174	0.410
OPR-A with Q_s	ω_o	0.2489	0.1954
	s	28.24	47.15
	$Iter$	131	174
	CPU	0.174	0.410
OPR-A with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.2488	0.1955
	ϵ	0.02	0.03
	$Iter$	110	131
	CPU	0.146	0.313
OPR-B with Q_s	ω_o	0.2489	0.1955
	s	7.028	9.221
	$Iter$	131	174
	CPU	0.174	0.410
OPR-B with $Q_{s+\epsilon}$ (s is the same as above)	ω_o	0.2488	0.1955
	ϵ	0.004	0.001
	$Iter$	98	128
	CPU	0.129	0.305

OPR methods with the scaled preconditioner $Q_{s+\epsilon}$ perform better than the PU method. More specifically, the OPR methods with $Q_{s+\epsilon}$ perform significantly better than the PU method for Example 4.1 and Case II of Example 4.2 where $\mu_{\max} > 4$ (see Tables 2 to 5). Hence, it may be concluded that the OPR methods with the scaled preconditioner $Q_{s+\epsilon}$ are recommended for use when solving the singular saddle point problems. Also we provided how to choose a near optimal value of ϵ (see Section 4). Also notice that computations of $\mu_{\max}$ and $\mu_{\min}$ which are needed in order to find optimal parameters can be easily computed using the powerful Computer Algebra System such as MATLAB.

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