J. Korean Math. Soc. **53** (2016), No. 3, pp. 623–643
http://dx.doi.org/10.4134/JKMS.j150213
pISSN: 0304-9914 / eISSN: 2234-3008

ON PIECEWISE NOETHERIAN DOMAINS

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Abstract. In this paper, we study piecewise Noetherian (resp., piecewise w-Noetherian) properties in several settings including flat (resp., t-flat) overrings, Nagata rings, integral domains of finite character (resp., w-finite character), pullbacks of a certain type, polynomial rings, and $D + XK[X]$ constructions.

0. Introduction

The concept of piecewise Noetherian rings was first introduced and studied by Beachy and Weakley in [4]. They originally defined piecewise Noetherian rings as follows: A commutative ring R is *piecewise Noetherian* if it satisfies the ascending chain condition (ACC) on prime ideals and has finite ideal lengths (a concept that goes back to the work of Krull). As pointed out in [25], this notion of finite ideal length was used before localization techniques were developed.

In [4], the authors showed that being piecewise Noetherian is inherited by factor rings and localizations and extends to integral extensions and polynomial rings, and they distinguished these rings from rings with Krull dimension. In a later paper [5], Beachy and Weakley proved the existence of an associated prime ideal for any module over a commutative piecewise Noetherian ring. They also showed that for a piecewise Noetherian ring the smallest test set (for injectivity) of prime ideals is the set of essential prime ideals. In [25], Pearson extended and studied the notion of piecewise Noetherian rings to the noncommutative setting.

Recall that a commutative ring R is said to have *Noetherian spectrum* if it satisfies the ACC on radical ideals. This is equivalent to the condition that R satisfies the ACC on prime ideals and each ideal has only finitely many prime ideals minimal over it. It is remarked just before [5, Lemma 1] that a commutative ring R is piecewise Noetherian if and only if R has Noetherian spectrum and satisfies the ACC on P-primary ideals, for each prime ideal P of

Received April 1, 2015.
2010 *Mathematics Subject Classification.* 13A15, 13E05.
Key words and phrases. piecewise Noetherian ring, piecewise w-Noetherian domain, Noetherian spectrum, strong Mori spectrum.

R. This characterization shows clearly that every Noetherian ring is piecewise Noetherian, and is used to define the notion of piecewise w-Noetherian rings. In [12], the authors introduced the notion of piecewise w-Noetherian domains and showed that an integral domain R is a generalized Krull domain if and only if R is a piecewise w-Noetherian PvMD.

The purpose of this paper is to study piecewise Noetherian (resp., piecewise w-Noetherian) properties in several settings including flat (resp., t-flat) overrings, Nagata rings, integral domains of finite character (resp., w-finite character), pullback of type ($\square$), polynomial rings, and $D + XK[X]$ constructions. Precisely, in Section 1, we show that (i) a flat overring of a piecewise Noetherian ring is piecewise Noetherian; (ii) if R is an integral domain of finite character, then R is piecewise Noetherian if and only if R_M is piecewise Noetherian for all maximal ideals M of R; (iii) if R is a pullback of type ($\square$) such that Q is the quotient field of D, then R is piecewise Noetherian if and only if T and D are piecewise Noetherian; and (iv) if $R \bowtie J$ is the amalgamation of R along an ideal J of R, R is piecewise Noetherian if and only if $R \bowtie J$ is piecewise Noetherian. Let R be an integral domain. In Section 2, we prove that (v) a t-flat overring of a piecewise w-Noetherian ring is piecewise w-Noetherian; (vi) if R is of w-finite character, then R is piecewise w-Noetherian if and only if R_M is piecewise Noetherian for all maximal w-ideals M of R; (vii) R is piecewise w-Noetherian if and only if $R[X]$ is piecewise w-Noetherian, if and only if $R[X]_{N_v}$ is piecewise Noetherian; (viii) if K is the quotient field of an integral domain D and $R = D + XK[X]$ or $D + XK[\![X]\!]$, then R is piecewise Noetherian (resp., piecewise w-Noetherian) if and only if D is piecewise Noetherian (resp., piecewise w-Noetherian); and (ix) if R is a pullback of type ($\square$) in which T is t-local and Q is the quotient field of D, then R is piecewise w-Noetherian if and only if T and D are piecewise w-Noetherian. We use the result (viii) to construct an easy example of piecewise Noetherian (resp., piecewise w-Noetherian) domains that are not Noetherian (resp., SM domains). We finally give an example of a piecewise w-Noetherian domain R such that $R[\![X]\!]$ is not piecewise w-Noetherian.

Let R be an integral domain with quotient field K, and let $F(R)$ (resp., $f(R)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of R. For $I \in F(R)$, let $I^{-1} = \{x \in K \mid xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{J_v \mid J \subseteq I$ and $J \in f(R)\}$, and $I_w = \{x \in K \mid xJ \subseteq I$ for some $J \in f(R)$ with $J_v = R\}$. It is well known and easy to show that if $* = v, t$, or w, then $(aR)_* = aR$, $(aI)_* = aI_*$, $I \subseteq I_*$, $I \subseteq J$ implies $I_* \subseteq J_*$, and $(I_*)_* = I_*$ for all $0 \neq a \in K$ and $I, J \in F(R)$. The definition of v, t, and w establishes that $I \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in F(R)$, and if I is finitely generated, $I_t = I_v$. Let $* = v$, t, or w. An $I \in F(R)$ is called a *$*$-ideal* if $I_* = I$, and a $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(R) denote the set of maximal $*$-ideals of R. While v-Max(R) can be empty as in the case of rank-one discrete valuation domains, it is known that a maximal $*$-ideal is a prime ideal; each prime ideal minimal over a $*$-ideal is a $*$-ideal; $*$-Max$(R) \neq \emptyset$

when R is not a field; and w-Max$(R) = t$-Max(R). Also, if $I \in F(R)$, then $I_w = \bigcap_{P\in t\text{-Max}(R)} IR_P$, and hence either $Q_w = Q$ or $Q_w = R$ for a nonzero prime ideal Q of R. A finitely generated ideal J of R is called a *GV-ideal* if $J^{-1} = R$. Denote by GV(R) the set of GV-ideals of R. An $I \in F(R)$ is said to be of *w-finite type* if there exists a finitely generated subideal J of I such that $J_w = I_w$. An $I \in F(R)$ is said to be *$*$-invertible* if $(II^{-1})_* = R$, and R is called a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of R is t-invertible. Clearly, if $* = t$ or w, then an $I \in F(R)$ is $*$-invertible if and only if $II^{-1} \nsubseteq P$ for all $P \in *$-Max(R). Hence, w-Max$(R) = t$-Max(R) implies that R is a PvMD if and only if each nonzero finitely generated ideal of R is w-invertible.

An integral domain R is called a *strong Mori domain* (SM domain) if R satisfies the ACC on integral w-ideals. Clearly, Noetherian domains are SM domains. An integral domain R is said to be *of w-finite character* if each nonzero nonunit of R is contained in only finitely many maximal w-ideals. Integral domains of w-finite character include Noetherian domains and SM domains. Let X be an indeterminate over an integral domain R with quotient filed K. For any $f \in K[X]$, we denote by $c(f)$ the fractional ideal of R generated by the coefficients of f. Let $N_v = \{f \in R[X] \mid c(f)_v = R\}$. Then N_v is a saturated multiplicative set of $R[X]$ and the quotient ring $R[X]_{N_v}$ is called the *t-Nagata ring* of R.

1. Piecewise Noetherian rings

A commutative ring R with identity is said to be *piecewise Noetherian* if (i) the set of prime ideals of R satisfies the ACC; (ii) R has the ACC on P-primary ideals for each prime ideal P; and (iii) each ideal has only finitely many prime ideals minimal over it.

The following three results are special cases of results in [4] (with different proofs) or easily deduced from them.

Lemma 1.1. *Let (R, M) be a piecewise Noetherian quasi-local ring.*

(1) *If* $\dim(R) = 0$ *or if R is an integral domain with* $\dim(R) = 1$*, then R is Noetherian.*
(2) *If J is an ideal of R, then R/J is also a piecewise Noetherian ring.*

Proof. This is clear. □

Proposition 1.2. *Let (R, M) be a piecewise Noetherian quasi-local ring with $M \neq (0)$. Then $M^2 \neq M$.*

Proof. If $\dim R = 0$, then R is Noetherian by Lemma 1.1, and hence $M^2 \neq M$. Next, assume that $\dim R \geq 1$. Since R satisfies the ACC on prime ideals, there is a prime ideal P of R such that $\dim(R/P) = 1$. Hence, by Lemma 1.1, R/P is Noetherian, and so $(M/P)^2 \neq M/P$. Thus, $M^2 \neq M$. □

Theorem 1.3. *Let (R, M) be a piecewise Noetherian quasi-local ring. Then M is finitely generated.*

Proof. If $M = (0)$, then M is finitely generated. So we assume that $M \neq (0)$, and hence by Proposition 1.2, $M^2 \neq M$. Pick $x_1 \in M \setminus M^2$ and set $A_1 = Rx_1 + M^2$. If $A_1 \neq M$, pick $x_2 \in M \setminus A_1$ and set $A_2 = A_1 + Rx_2$. If $A_2 \neq M$, repeat this process. Thus $A_1 \subset A_2 \subset \cdots$ is a chain of M-primary ideals of R. By hypothesis, there exists a positive integer n such that $A_n = Rx_1 + \cdots + Rx_n + M^2 = M$. Set $I = Rx_1 + \cdots + Rx_n$. Thus $I + M^2 = M$. Hence $M/I = (M/I)^2$. Since R/I is a piecewise Noetherian ring, we have $M/I = 0$ by Proposition 1.2. Hence $M = I$ is finitely generated. □

Theorem 1.4. *If R is a piecewise Noetherian ring, then a flat overring of R is also piecewise Noetherian.*

Proof. Let T be a flat overring of R. Then $T = R_{\mathscr{S}}$ for $\mathscr{S}$ a generalized multiplicative system of R and $IT = T$ for all $I \in \mathscr{S}$ [3, Theorem 1.3]. Hence T has a Noetherian spectrum [6, Proposition 1.2]. Also, by hypothesis and [3, Theorems 1.1 and 1.2], T has the ACC on P-primary ideals for each prime ideal P of T. Therefore T is piecewise Noetherian. □

It is shown that any localization of a piecewise Noetherian ring is again piecewise Noetherian [4, p. 2689].

Corollary 1.5. *Let R be an integral domain and let S be a multiplicative set of R. If R is piecewise Noetherian, then R_S is also piecewise Noetherian. In particular, R_P is piecewise Noetherian for all prime ideals P of R.*

Let R be an integral domain, $R[X]$ be the polynomial ring over R, and $S = \{f \in R[X] \mid c(f) = R\}$. Then $R(X) = R[X]_S$, called the *Nagata ring* of R, is an overring of $R[X]$. The next result is a piecewise Noetherian domain analogue of the well-known fact that R is Noetherian if and only if $R[X]$ is Noetherian, if and only if $R(X)$ is Noetherian.

Corollary 1.6. *The following are equivalent for an integral domain R.*

(1) *R is piecewise Noetherian.*
(2) *$R[X]$ is piecewise Noetherian.*
(3) *$R(X)$ is piecewise Noetherian.*

Proof. (1) $\Leftrightarrow$ (2) [4, Theorem 3.4]. (2) $\Rightarrow$ (3) This follows directly from Corollary 1.5. (3) $\Rightarrow$ (1) This is an immediate consequence of the following facts: (i) If I is a P-primary ideal in R, then $IR(X)$ is $PR(X)$-primary and $IR(X) \cap R = I$ [15, Proposition 33.1]. (ii) P is a prime ideal of R minimal over an ideal J of R if and only if $PR(X)$ is minimal over $JR(X)$. □

Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of overrings of an integral domain R such that $R = \bigcap_{\alpha \in \Lambda} R_\alpha$. We say that the intersection $R = \bigcap_{\alpha \in \Lambda} R_\alpha$ is of finite character if each nonzero element of R is a unit in R_α for all but a finite number of R_α.

Clearly, $R = \bigcap_{M \in \mathrm{Max}(R)} R_M$, where $\mathrm{Max}(R)$ is the set of maximal ideals of R. We say that R is of *finite character* if $R = \bigcap_{M \in \mathrm{Max}(R)} R_M$ is of finite character. Hence, R is of finite character if and only if every nonzero nonunit of R is contained in only a finite number of maximal ideals of R.

Theorem 1.7. *Let R be an integral domain and let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of flat overrings of R such that $R = \bigcap_{\alpha \in \Lambda} R_\alpha$ is of finite character. Assume that we have $I = \bigcap_{\alpha \in \Lambda} IR_\alpha$ for all ideals I of R. Then each R_α is piecewise Noetherian if and only if R is piecewise Noetherian.*

Proof. ($\Leftarrow$) Theorem 1.4. ($\Rightarrow$) A particular case of [6, Proposition 1.5] shows that if each R_α has Noetherian spectrum, then so does R. (Here, we need the condition that $I = \bigcap_{\alpha \in \Lambda} IR_\alpha$ for all ideals I of R.)

Now let P be a nonzero prime ideal of R and let

$$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$$

be an ascending chain on P-primary ideals. Then there exists $\alpha_1, \ldots, \alpha_s \in \Lambda$ such that for all $\alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_s\}$, $Q_0 R_\alpha = R_\alpha$, and thus for all integers $n \geq 0$, $Q_n R_\alpha = R_\alpha$. Note that for each $\alpha \in \Lambda$, since R_α is a flat overring of R, there exists a generalized multiplicative system $\mathscr{S}_\alpha$ such that $R_\alpha = R_{\mathscr{S}_\alpha}$ and $IR_{\mathscr{S}_\alpha} = R_{\mathscr{S}_\alpha}$ for each $I \in \mathscr{S}_\alpha$ [3, Theorem 1.3]. Note also that for all $n \geq 0$ and for all $1 \leq i \leq s$, $(Q_n)_{\mathscr{S}_{\alpha_i}}$ is a $P_{\mathscr{S}_{\alpha_i}}$-primary ideal of $R_{\mathscr{S}_{\alpha_i}}$ and $(Q_n)_{\mathscr{S}_{\alpha_i}} \cap R = Q_n$ [3, Theorems 1.1, 1.2, and 1.3]. Thus by hypothesis, there exists a positive integer n_0 such that for all $n \geq n_0$ and for all $1 \leq i \leq s$, we have that $(Q_n)_{\mathscr{S}_{\alpha_i}} = (Q_{n_0})_{\mathscr{S}_{\alpha_i}}$. Therefore for all $n \geq n_0$, we have that $Q_n = \bigcap_{\alpha \in \Lambda} (Q_n)_{\mathscr{S}_\alpha} = \bigcap_{\alpha \in \Lambda} (Q_{n_0})_{\mathscr{S}_\alpha} = Q_{n_0}$. □

An *almost Dedekind domain* R is an integral domain such that R_M is a principal ideal domain for all maximal ideals M of R. Clearly, an almost Dedekind domain D is Noetherian if and only if D is of finite character, if and only if D is piecewise Noetherin (cf. Lemma 1.1). Hence, a non-Noetherian almost Dedekind domain is locally piecewise Noetherian but not piecewise Noetherian.

Corollary 1.8. *Let R be an integral domain of finite character. Then R is piecewise Noetherian if and only if R_M is piecewise Noetherian for all maximal ideals M of R.*

Proof. This follows from Theorem 1.7 by taking $\{R_M \mid M \in \mathrm{Max}(R)\}$ as a family of flat overrings of R. □

Recall that a valuation domain is *strongly discrete* if it has no non-zero idempotent prime ideal; a strongly discrete Prüfer domain is an integral domain whose localization at any nonzero prime ideal is a strongly discrete valuation domain; an integral domain R is a generalized Dedekind domain if it is a strongly discrete Prüfer domain and every prime ideal of R is the radical of a finitely generated ideal. It is well known that Dedekind domains are precisely the Noetherian Prüfer domains. Facchini [13, Proposition 3.1] has given the corresponding result for generalized Dedekind domains, namely, that generalized Dedekind domains can be characterized as piecewise Noetherian Prüfer domains. Beachy and Weakley proved that a valuation domain is piecewise Noetherian if and only if it is strongly discrete [4, Theorem 2.9]. Thus, by Corollary 1.8, we have the following:

Corollary 1.9. *If R is a strongly discrete Prüfer domain of finite character, then R is piecewise Noetherian, and hence a generalized Dedekind domain.*

Let T be a quasi-local integral domain with maximal ideal M, $Q = T/M$, $\phi : T \to Q$ the canonical ring homomorphism, D a proper subring of Q, and $R = \phi^{-1}(D)$ the pullback.

$$(\Box) \qquad \begin{array}{ccc} R = \phi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & Q = T/M \end{array}$$

We shall refer to R as a pullback of type $(\Box)$. Then R is a subring of T, isomorphic to a fiber product $T \times_Q D$. It is well known that M is a prime ideal of R, therefore comparable to the prime ideals of R; any prime ideal of R contained in M is a prime ideal of T; M is a t-ideal of R since $M = (R : T)$; and $D = R/M$.

Theorem 1.10. *Consider a pullback of type $(\Box)$. If Q is the quotient field of D, then R is piecewise Noetherian if and only if T and D are piecewise Noetherian.*

Proof. It was shown in [6, Proposition 1.8] that R has Noetherian spectrum if and only if T and D have Noetherian spectrum. Hence, it suffices to show that the condition (ii) of piecewise Noetherian rings holds true.

($\Rightarrow$) Let P be a prime ideal of T and let $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$ be an ascending chain of P-primary ideals of T. Since T is quasi-local, $Q_i \subseteq P \subseteq M$ for each i, and hence $Q_i \cap R = Q_i$ and Q_i is a P-primary ideal of R. Thus by hypothesis, there exists a positive integer n_0 such that for all integers $n \geq n_0$, $Q_n = Q_{n_0}$. Next let P' be a prime ideal of D, and let $\{Q'_i\}$ be an ascending chain of P'-primary ideals of D. Put $P_0 = \phi^{-1}(P')$ and $Q_{0i} = \phi^{-1}(Q'_i)$ for all i. Clearly, P_0 is a prime ideal of R and $\{Q_{0i}\}$ is an ascending chain of P_0-primary ideals. Hence, by assumption, $\{Q_{0i}\}$ is finite, and since $\phi(\phi^{-1}(Q_{0i})) = Q_{0i}$, we have that $\{Q'_i\}$ is finite. Thus, T and D are both piecewise Noetherian.

($\Leftarrow$) Let P be a prime ideal of R, and let $\{Q_\alpha\}$ be an ascending chain of P-primary ideals. If $M \nsubseteq P$, there is a unique prime ideal P' of T such that $P' \cap R = P$ and $R_P = T_{P'}$ [14, page 805]. Note that $T_{P'} = R_P$ is piecewise Noetherian by Corollary 1.5, and $\{Q_\alpha R_P\}$ is an ascending chain of PR_P-primary ideals; hence $\{Q_\alpha R_P\}$ is finite by assumption. Thus, $\{Q_\alpha\}$ is finite because $Q_\alpha R_P \cap R = Q_\alpha$. Next, assume $M \subseteq P$. Then $Q_\alpha T = T$, and hence $M \subsetneq Q_\alpha \subseteq P$ [14, Proposition 1.1]. Hence, $\phi(P)$ is a prime ideal of D and $\phi(Q_\alpha)$ is a $\phi(P)$-primary ideal of D, and since D is piecewise Noetherian, $\{\phi(Q_\alpha)\}$ is finite. Note that $\phi^{-1}(\phi(P)) = P$ and $\phi^{-1}(\phi(Q_\alpha)) = Q_\alpha$ for all α because $M \subsetneq Q_\alpha \subseteq P$. Thus, $\{Q_\alpha\}$ is finite. Therefore, R is piecewise Noetherian. □

Let $A \subseteq B$ be an extension of integral domains, and let $B[\![X]\!]$ be the power series ring over B. Then $R := A + XB[\![X]\!] = \{f \in B[\![X]\!] \mid f(0) \in A\}$ is a subring of $B[\![X]\!]$. Note that if B is the quotient field of A, then $XB[\![X]\!] \subsetneq aR$ for all $0 \neq a \in A$. Moreover, if P is a prime ideal of A minimal over aA, then $PR = P + XB[\![X]\!]$ is a prime ideal of R that is minimal over aR and $\mathrm{ht}PR \geq 2$. Hence, if A is not a field, then R does not satisfy the principal ideal theorem, and thus R is not Noetherian. The next corollary gives an easy example of piecewise Noetherian domains that are not Noetherian. Note that an example is given of a (one-dimensional) piecewise Noetherian ring, but not Noetherian in [5, Example 5].

Corollary 1.11. *Let K be the quotient field of an integral domain D and $R = D + XK[\![X]\!]$. Then R is a piecewise Noetherian ring if and only if D is a piecewise Noetherian ring.*

Proof. Note that $K[\![X]\!]$ is a rank-one DVR with $XK[\![X]\!]$ maximal; hence $K[\![X]\!]$ is piecewise Noetherian. Thus, the result follows directly from Theorem 1.10. □

The proof of ($\Rightarrow$) of Theorem 1.10 shows that if R is piecewise Noetherian, then D and T are piecewise Noetherian even though Q is not the quotient field of D. However, the next example shows that Theorem 1.10 (resp., Corollary 1.11) does not hold when Q (resp., K) is not the quotient field of D.

Example 1.12. Let $T = \mathbb{R}[\![X]\!]$ be the power series ring over the field $\mathbb{R}$ of real numbers, and let $R = \mathbb{Q} + X\mathbb{R}[X]$. Clearly, both T and $\mathbb{Q}$ are piecewise Noetherian. However, if we let $I_n = XR + \pi XR + \pi^2 XR + \cdots + \pi^n XR$ for each integer $n \geq 0$, then $\sqrt{I_n} = M$ in R, where $M = X\mathbb{R}[X]$, and since M is a maximal ideal of R, I_n is an M-primary ideal of R. Note that $\pi^{n+1}X \notin I_n$ for all $n \geq 0$, and hence $I_n \subsetneq I_{n+1}$. Hence, $\{I_n\}$ is a strictly increasing infinite chain of M-primary ideals of R, and thus R is not piecewise Noetherian.

Now we consider piecewise Noetherian property in amalgamated algebras along an ideal. To do so, we need the following:

Lemma 1.13. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then B satisfies the ACC on primary submodules if and only if both A and C satisfy the ACC on primary submodules.*

Proof. The necessity follows immediately from the fact that an ascending chain of primary submodules of A (resp., C) gives rise to a chain in B, hence is stationary.

For sufficiency, we may assume that A is a submodule of B and $C = B/A$. Let

$$Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$$

be an ascending chain on primary submodules of B. Clearly, each $Q_i \cap A$ is a primary submodule of A and $Q_1 \cap A \subseteq Q_2 \cap A \subseteq \cdots$. Also, each $(Q_i + A)/A$ is a primary submodule of C and $(Q_1 + A)/A \subseteq (Q_2 + A)/A \subseteq \cdots$. Thus, by hypothesis, there is a positive integer n such that $Q_n \cap A = Q_{n+i} \cap A$ and $(Q_n + A)/A = (Q_{n+i} + A)/A$ for all $i \geq 1$. A simple computation shows that $Q_n = Q_{n+i}$ for all $i \geq 1$. Therefore B satisfies the ACC on primary submodules. $\square$

Let R and T be two rings, let J be an ideal of T and let $f : R \to T$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times T$:

$$R \bowtie^f J := \{(a, f(a) + j) \mid a \in R, j \in J\},$$

which is called the amalgamation of R with T along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [9]).

Let $\alpha : A \to C$ and $\beta : B \to C$ be ring homomorphisms. Then the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β. As mentioned in [9], the fact that D is a pullback can also be described by saying that the triplet (D, p_A, p_B) is a solution of the universal problem of rendering commutative the diagram built on α and β,

$$\begin{array}{ccc} D & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

where p_A (resp., p_B) is the restriction to $\alpha \times_C \beta$ of the projection of $A \times B$ onto A (resp., B). It was shown in [9, Proposition 4.2] that $R \bowtie^f J$ can be represented as a pullback $\breve{f} \times_{T/J} \pi$ of $\breve{f}$ and π, where $\pi : T \to T/J$ is the canonical projection and $\breve{f} := \pi \circ f$. The following is an analogue of [9, Proposition 4.10] and its proof is the same as that of [9, Proposition 4.10], equipped with Lemma 1.13.

Proposition 1.14. *With the above notation, the following conditions are equivalent*:

(1) $D(= \alpha \times_C \beta)$ *satisfies the ACC on primary ideals.*
(2) $\ker(\beta)$ *satisfies the ACC on primary D-submodules* (*with the D-module structure naturally induced by p_B*) *and $p_A(D)$ satisfies the ACC on primary ideals.*

It was also shown in [26, Proposition 2.11] that the ring $R \bowtie^f J$ has Noetherian spectrum if and only if R and $f(R) + J$ have Noetherian spectrum. In particular, if T has Noetherian spectrum, then $R \bowtie^f J$ has Noetherian spectrum if and only R has Noetherian spectrum. Among other things, it is shown in [9, Proposition 5.1] that $p_R : R \bowtie^f J \to R$ is surjective and the following canonical isomorphisms hold:

$$\frac{R \bowtie^f J}{\{0\} \times J} \cong R \qquad \text{and} \qquad \frac{R \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(R) + J.$$

Theorem 1.15. *The ring $R \bowtie^f J$ is piecewise Noetherian if and only if R and $f(R) + J$ are piecewise Noetherian. In particular, if T is piecewise Noetherian, then $R \bowtie^f J$ is piecewise Noetherian if and only R is piecewise Noetherian.*

Proof. If $R \bowtie^f J$ is piecewise Noetherian, then R and $f(R) + J$ are piecewise Noetherian, since the piecewise Noetherian property is stable under factor ring by the remark just after [4, Definition 2.1]. Now assume that R and $f(R) + J$ are piecewise Noetherian. By the remark above, it suffices to show that $R \bowtie^f J$ satisfies the ACC on P-primary ideals for each prime ideal P of $R \bowtie^f J$. This can be done by the same argument of [9, Proposion 5.6] as follows. Since the projection $p_R : R \bowtie^f J \to R$ is surjective and R is a piecewise Noetherian ring, by Proposition 1.14, it suffices to show that $J(= \ker \pi)$, with the structure of $R \bowtie^f J$-module induced by p_B, satisfies the ACC on P-primary ideals. But this fact is easy, since every $R \bowtie^f J$-submodule of J is an ideal of the piecewise Noetherian ring $f(R) + J$. □

The ring $R \bowtie^f J$ is simply denoted by $R \bowtie J$ when $T = R$ and $f : R \to T$ is the identity function. Hence, by Theorem 1.15, we have

Corollary 1.16. *R is piecewise Noetherian if and only if $R \bowtie J$ is piecewise Noetherian.*

2. Piecewise w-Noetherian domains

Let R be an integral domain. We say that R is a *piecewise w-Noetherian domain* if (i) R satisfies the ACC on prime w-ideals; (ii) R has the ACC on P-primary ideals for each prime w-ideal P; and (iii) each w-ideal has only finitely many prime ideals minimal over it. By definition, piecewise Noetherian domains and SM domains are piecewise w-Noetherian domains. The notion of a piecewise w-Noetherian domain was introduced in [12], where the authors called such an integral domain a piecewise strong Mori domain.

Lemma 2.1 (cf. [12, Lemma 2.5]). *Let R be a piecewise w-Noetherian domain and let P be a prime w-ideal of R. Then R_P is a piecewise Noetherian domain.*

Proof. We will verify the three conditions in the definition of piecewise Noetherian domain. (i) Let $\{Q_\alpha\}$ be a chain of prime ideals of R_P. Then $Q_\alpha = P_\alpha R_P$ for some prime ideal P_α of R. Clearly, $P_\alpha \subseteq P$, and so P_α is a w-ideal. Hence, $\{P_\alpha\}$, and thus $\{Q_\alpha\}$ is finite. (ii) This follows because $QR_P \cap R = Q$ for each primary ideal Q of R with $Q \subseteq P$. (iii) Clear. □

Lemma 2.2. *Let P be a prime w-ideal of R, and let Q be a P-primary ideal of R. Then Q is a w-ideal.*

Proof. Let $x \in Q_w$. Then there is $I \in \mathrm{GV}(R)$ such that $xI \subseteq Q$. Note that $I \nsubseteq P$; so $x \in Q$. Thus, $Q_w \subseteq Q$, and hence $Q_w = Q$. □

As in [20], we say that an overring T of an integral domain R is *t-flat* over R if $T_M = R_{M\cap R}$ for all maximal w-ideals M of T. Clearly, a flat overring is t-flat. Also, if Q is a prime w-ideal of a t-flat overring T of R, then $Q\cap R = (Q\cap R)_w \subsetneq R$ because $(Q\cap R)_w = \bigcap_{P\in w\text{-Max}(R)}(Q\cap R)R_P \subseteq \bigcap_{M\in w\text{-Max}(T)}(Q\cap R)R_{M\cap R} = \bigcap_{M\in w\text{-Max}(T)} QT_M = Q_w = Q$.

Theorem 2.3. *If T is a t-flat overring of a piecewise w-Noetherian domain R, then T is a piecewise w-Noetherian domain.*

Proof. We will verify the three conditions in the definition of a piecewise w-Noetherian domain. (i) Let $\{P_\alpha\}$ be a chain of prime w-ideals of T. Then $(\bigcup_\alpha P_\alpha)_w \subsetneq T$, and hence $\bigcup_\alpha P_\alpha \subseteq M$ for some maximal w-ideal M of T. Since T is t-flat over R, $T_M = R_{M\cap R}$, and hence $\{P_\alpha T_M\}$ is an ascending chain of prime ideals of $R_{M\cap R}$. Note that $(M \cap R)_w \subsetneq R$; hence $R_{M\cap R}$ is piecewise Noetherian by Lemma 2.1. Thus, $\{P_\alpha T_M\}$, and hence $\{P_\alpha\}$ is finite.

(ii) Let P be a prime w-ideal of T, and let Q be a P-primary ideal of T. Then $Q_w = Q$ by Lemma 2.2. Since $T_P = R_{P\cap R}$ and $R_{P\cap R}$ is piecewise Noetherian, T satisfies the ACC on P-primary ideals.

(iii) Let A be a w-ideal of T, and let $\{P_\lambda\}$ be the set of minimal prime ideals of A. Then $(P_\lambda)_w \subsetneq T$, and since T is t-flat over R, $(P_\lambda \cap R)_w \subsetneq R$. Note that if $P_{\lambda_1} \neq P_{\lambda_2}$, then $P_{\lambda_1} \cap R \neq P_{\lambda_2} \cap R$ because $T_{P_{\lambda_i}} = R_{P_{\lambda_i}\cap R}$. Also, $P_\lambda \cap R$ is minimal over $A\cap R$. Thus, as R is a piecewise w-Noetherian domain, $\{P_\lambda \cap R\}$, and hence $\{P_\lambda\}$ is finite. □

Corollary 2.4. *If T is a flat overring of a piecewise w-Noetherian domain R (e.g., T is a quotient ring of R), then T is a piecewise w-Noetherian domain.*

Proof. This follows from Theorem 2.3 because a flat overring is t-flat. □

Theorem 2.5. *Let R be a piecewise w-Noetherian domain of w-finite character and let M be a maximal w-ideal of R. Then M is of w-finite type.*

Proof. By Lemma 2.1 and Theorem 1.3 we have $MR_M = BR_M$ for some finitely generated subideal B of M. Since R has w-finite character, there are only finitely many maximal w-ideals of R containing B, say $P_1, \ldots, P_n, M$. Thus $M \nsubseteq P_1 \cup \cdots \cup P_n$. Take $a \in M \setminus (P_1 \cup \cdots \cup P_n)$ and set $A = B + Ra$. Let P be a maximal w-ideal of R. If $P = M$, then $MR_P = AR_P$. If $P \neq M$, then $MR_P = R_P = AR_P$. Therefore, $M = A_w$ is of w-finite type. □

An integral domain R is called a *weakly Krull domain* if $R = \bigcap_{P\in X^1(R)} R_P$, where $X^1(R)$ is the set of height-one prime ideals of R, and each nonzero nonunit of R is contained in only finitely many height-one prime ideals of R.

It is easy to show that R is weakly Krull if and only if $X^1(R)$ is the set of maximal w-ideals of R and R is of w-finite character. Noetherian domains of dimension one and Krull domains are the most well known examples of weakly Krull domains.

Corollary 2.6. *If R is a weakly Krull domain, then R is a piecewise w-Noetherian domain* (*if and*) *only if R is an SM domain.*

Proof. Since R is weakly Krull, R is of w-finite character and each prime w-ideal of R is a maximal w-ideal. Hence, each prime w-ideal of R is of w-finite type by Theorem 2.5, and thus R is an SM domain [27, Theorem 4.3]. □

Theorem 2.7. *If R is of w-finite character, then R is a piecewise w-Noetherian domain if and only if R_M is a piecewise Noetherian domain for each maximal w-ideal M of R.*

Proof. (⇒) Lemma 2.1. (⇐) We will verify the three conditions in the definition of a piecewise w-Noetherian domain. (i) This follows because R satisfies the ACC on prime w-ideals if and only if R_P satisfies the ACC on prime ideals for each maximal w-ideal P of R. (ii) This follows because $QR_S \cap R = Q$ for any primary ideal Q of R, where S is a multiplicative set of R. (iii) Let A be a w-ideal of R. Since R is of w-finite character, there are only finitely many maximal w-ideals of R that contain A. Let $M_1, \ldots, M_k$ be such maximal w-ideals of R. Note that if P is a prime ideal minimal over A, then either $PR_{M_i} = R_{M_i}$ or PR_{M_i} is minimal over AR_{M_i} and $PR_M = R_M$ for all maximal w-ideals M of R with $M \neq M_i$. Thus, R_{M_i} being piecewise Noetherian implies that A has only finitely many minimal prime ideals. □

Recall that a strongly discrete PvMD is an integral domain whose localization at any nonzero prime t-ideal is a strongly discrete valuation domain. El Baghdadi introduced the concept of generalized Krull domains as the t-operation version of generalized Dedekind domains as follows: An integral domain R is a *generalized Krull domain* if it is a strongly discrete PvMD and every prime t-ideal of R is the radical of a finite type t-ideal. Thus an integral domain is a generalized Krull domain if and only if it is a strongly discrete PvMD with strong Mori spectrum (cf., [12, Theorem 2.2]). (See below for the definition of strong Mori spectrum.)

Corollary 2.8 (cf. [11, Lemma 2.5])**.** *If R is a strongly discrete PvMD of w-finite character, then R is a piecewise w-Noetherian domain, and hence a generalized Krull domain.*

Let w-Spec(R) be the set of prime w-ideals of an integral domain R. In [19], Kim *et al* defined R to have *strong Mori spectrum* if it satisfies the descending chain condition on the sets of the form $W(I) := \{P \in w\text{-Spec}(R) \mid I \subseteq P\}$, where I runs over w-ideals of R (or equivalently, the induced topology on w-Spec(R) by the Zariski topology on Spec(R) is Noetherian). Note that for every nonzero ideal I of R, $V(I) \cap w\text{-Spec}(R) = W(I_w)$, where $V(I) := \{P \in \operatorname{Spec}(R) \mid I \subseteq P\}$ is a closed set in $\operatorname{Spec}(R)$. For all w-ideals I, J of R, we have $W(I) \cup W(J) = W(I \cap J)$ and $W(I) \cap W(J) = W((I + J)_w)$. This concept extends that of Noetherian spectrum and certainly an integral domain with Noetherian spectrum has strong Mori spectrum. In the case when every ideal of R is a w-ideal (for example, Prüfer domains or one-dimensional domains), the notions of Noetherian spectrum and strong Mori spectrum coincide. In [19, Theorem 2.6], Kim *et al* characterized integral domains with strong Mori spectrum as follows.

Theorem 2.9. *The following conditions are equivalent for an integral domain R.*

(1) *R has strong Mori spectrum.*
(2) *R satisfies the ascending chain condition on radical w-ideals.*
(3) *For every nonzero ideal I of R, $\sqrt{I_w} = \sqrt{J_w}$ for some finitely generated subideal J of I.*
(4) *Each radical w-ideal of R is the radical of a w-finite type ideal.*
(5) *Each prime w-ideal of R is the radical of a w-finite type ideal.*
(6) *R satisfies the ascending chain condition on prime w-ideals and each proper w-ideal has only finitely many minimal prime (w-)ideals.*
(7) *R satisfies the ascending chain condition on radicals of w-ideals.*

Thus an integral domain R is piecewise w-Noetherian if and only if R has strong Mori spectrum and R has the ACC on P-primary ideals for each prime w-ideal P.

We next give the w-theoretic analogue of [23, Proposition 2.3]. The proof of this result closely follows that of [23, Proposition 2.3]. Yet for the sake of completeness we include a proof. Note that $(\sqrt{I})_w = \sqrt{I_w}$ for each nonzero ideal I of R ([27, Proposition 2.4]).

Lemma 2.10. *Let $\mathscr{F}$ be the family of radical w-ideals in an integral domain R which are not the radicals of w-finite type ideals. If $\mathscr{F} \neq \emptyset$, then $\mathscr{F}$ contains maximal elements, and any such maximal element is a prime w-ideal.*

Proof. Let $\{A_i\}$ be a chain in $\mathscr{F}$. Then $A := \bigcup_i A_i$ is a w-ideal of R. Thus A is in $\mathscr{F}$, and so an upper bound of $\{A_i\}$. Indeed, if not, then there is a finitely generated subideal I of A such that $A = \sqrt{I_w}$. So there is k such that $I \subseteq A_k$. Thus $A_k = \sqrt{I_w}$, which contradicts to the fact that $A_k \in \mathscr{F}$. Hence, by Zorn's lemma, there is a maximal element M of $\mathscr{F}$. Suppose to the contrary that M is not a prime ideal of R. Then there are ideals B and C with $M \subsetneq B$, $M \subsetneq C$, and $BC \subseteq M \subseteq B \cap C$. Hence, $\sqrt{B_w} = \sqrt{J_w}$ and $\sqrt{C_w} = \sqrt{J'_w}$ for some finitely generated ideals J and J' of R, and $\sqrt{B_w} \cap \sqrt{C_w} = \sqrt{J_w} \cap \sqrt{J'_w} = \sqrt{J_w J'_w} = \sqrt{(J_w J'_w)_w} = \sqrt{(JJ')_w}$, where the third equality follows from the fact that $\sqrt{B_w} \cap \sqrt{C_w}$ is a w-ideal. Note that $\sqrt{B_w} \cap \sqrt{C_w} = \sqrt{B_w C_w} \subseteq \sqrt{(BC)_w} \subseteq \sqrt{M} = M \subseteq \sqrt{B_w} \cap \sqrt{C_w}$; hence $M = \sqrt{B_w} \cap \sqrt{C_w} = \sqrt{(JJ')_w}$. Therefore M is the radical of a w-finite type ideal, which is a contradiction. $\square$

Lemma 2.11. *If I is the radical of a w-finite type w-ideal of an integral domain R, then*

(1) *there is a finite subset $T \subseteq I$ such that $I = \sqrt{(T)_w}$,*
(2) *$I[X]$ is the radical of a w-finite type w-ideal of $R[X]$.*

Proof. (1) Clear. (2) This follows from $(J[X])_w = J_w[X]$ and $\sqrt{J_w[X]} = \sqrt{J_w}[X]$ for any nonzero ideal J of R. □

Theorem 2.12. *If an integral domain R satisfies strong Mori spectrum, then $R[X]$ satisfies strong Mori spectrum.*

Proof. Suppose to the contrary that $R[X]$ does not satisfy strong Mori spectrum. Then the set $\mathscr{F}$ of all radical w-ideals of $R[X]$ that is not the radical of a w-finite type ideal is nonempty. Thus, by Lemma 2.10, $\mathscr{F}$ has a maximal element M, which is a prime w-ideal of $R[X]$. By hypothesis, $M \cap R = (0)$ or $M \cap R \neq (0)$ such that $M \cap R$ is the radical of a w-finite type w-ideal of R. Thus if we set $N = (M \cap R)[X]$, then $N = (0)$ or N is the radical of a w-finite type ideal of $R[X]$ by Lemma 2.11(2). Hence, by Lemma 2.11(1), there is a w-finite type w-ideal $F \subseteq N$ of $R[X]$ (take $F = (0)$ when $N = (0)$) such that $N = \sqrt{F}$. Note that $N \subsetneq M$; so one can choose a polynomial $f \in M \setminus N$ of minimal degree m and denote by a its leading coefficient. If $a \in M$, then $aX^m \in (M \cap R)[X] = N$. So $f - aX^m \in M$ and $\deg(f - aX^m) < m$. Thus $f - aX^m \in N$, and hence $f = (f - aX^m) + aX^m \in N$, which is absurd. Whence $a \notin M$ and by the maximality of M in $\mathscr{F}$, the ideal $(M + aR[X])_w$ is the radical of a w-finite type w-ideal of $R[X]$. Again by Lemma 2.11(1), there exists a w-finite type w-ideal $T(\subseteq M)$ of $R[X]$ such that $\sqrt{(M + aR[X])_w} = \sqrt{(T + aR[X])_w}$.

For any $g \in M$, there exist a nonnegative integer n, and $q, r \in R[X]$ such that $a^n g = fq + r$, with $r = 0$ or $\deg(r) < \deg f$. Since $r = a^n g - fq \in M$, we have $r \in N$ by the choice of f. Then $(ag)^n = g^{n-1}qf + g^{n-1}r \in fR[X] + N$. So $ag \in \sqrt{fR[X] + N}$. Therefore $aM \subseteq \sqrt{fR[X] + N}$.

By using arithmetic on radicals and the fact that $(\sqrt{I})_w = \sqrt{I_w}$ for any ideal I of R, we have $\sqrt{fR[X] + N} = \sqrt{fR[X] + \sqrt{F}} = \sqrt{fR[X] + F}$ and

$$\begin{aligned}
(M^2)_w &\subseteq (M\sqrt{(M + aR[X])_w})_w \\
&= (M\sqrt{(T + aR[X])_w})_w \\
&\subseteq (\sqrt{MT + aM})_w \\
&\subseteq (\sqrt{T + \sqrt{fR[X] + F}})_w \\
&= \sqrt{(T + fR[X] + F)_w}.
\end{aligned}$$

Hence $M \subseteq \sqrt{(T + fR[X] + F)_w}$. Thus we have $M = \sqrt{(T + fR[X] + F)_w}$, which implies that M is the radical of a w-finite type w-ideal of $R[X]$, which is absurd. □

The t-Nagata ring $R[X]_{N_v}$ is very useful when we study ring-theoretic properties via the w-operation because $IR[X]_{N_v} \cap K = I_w$ and $I_wR[X]_{N_v} = IR[X]_{N_v}$ for all $I \in F(R)$ [7, Lemma 2.1], where $N_v = \{f \in R[X] \mid c(f)_v = R\}$. For example, R is a PvMD (resp., an SM domain) if and only if $R[X]_{N_v}$ is a Prüfer domain (resp., Noetherian domain) [17, Theorem 3.7] (resp., [7, Theorem 2.2]). The next result is concerned about the piecewise Noetherian domain property of $R[X]_{N_v}$.

Corollary 2.13. *The following are equivalent for an integral domain R.*

(1) *R is a piecewise w-Noetherian domain.*
(2) *$R[X]$ is a piecewise w-Noetherian domain.*
(3) *$R[X]_{N_v}$ is a piecewise w-Noetherian domain.*
(4) *$R[X]_{N_v}$ is a piecewise Noetherian domain.*

Proof. (1) $\Rightarrow$ (2) By Theorem 2.12, it suffices to show that $R[X]$ satisfies the ACC on Q-primary ideals for each prime w-ideal Q of $R[X]$. Let Q be a prime w-ideal of $R[X]$, and let M be a maximal w-ideal of $R[X]$ with $Q \subseteq M$. Set $M \cap R = P$. If $P = (0)$, then $Q = M$ and $R[X]_M$ is a rank-one DVR. Hence, $R[X]_Q$, and thus $R[X]$ satisfies the ACC on Q-primary ideals. Next, if $P \neq (0)$, then P is a maximal w-ideal of R and $M = P[X]$ [16, Proposition 1.1]. Hence, R_P is piecewise Noetherian by Lemma 2.1; so $R_P[X]$ is piecewise Noetherian by Corollary 1.6. Note that $Q_{R\setminus P}$ is a prime ideal of $R_P[X]$ and each $(Q_{R\setminus P})$-primary ideal is extended from a distinct Q-primary ideal. Thus, $R_P[X]$, and so $R[X]$ satisfies the ACC on Q-primary ideals.

(2) $\Rightarrow$ (3) Corollary 2.4.

(3) $\Leftrightarrow$ (4) This follows because each maximal ideal of $R[X]_{N_v}$ is a w-ideal [17, Propositions 2.1 and 2.2], and hence each nonzero ideal of $R[X]_{N_v}$ is a w-ideal.

(3) $\Rightarrow$ (1) Let I be a nonzero ideal of R with $I_w \subsetneq R$. Then $IR[X]_{N_v} = I_wR[X]_{N_v}$, I is a prime (resp., primary) ideal if and only if $IR[X]_{N_v}$ is a prime (resp., primary) ideal, and P is a prime ideal of R minimal over I if and only if $PR[X]_{N_v}$ is minimal over $IR[X]_{N_v}$. Thus, we have the result. $\square$

It is well known that an integral domain R is a PvMD if and only if $R[X]_{N_v}$ is a Prüfer domain and that a Prüfer domain (resp., PvMD) is a generalized Dedekind domain (resp., generalized Krull domain) if and only if it is piecewise Noetherian (resp., piecewise w-Noetherian domain)[13, Proposition 3.1] (resp., [12, Theorem 2.6]). Thus we can recover [11, Theorem 2.12(4)].

Corollary 2.14. *An integral domain R is a generalized Krull domain if and only if $R[X]_{N_v}$ is a generalized Dedekind domain.*

Let K be the quotient field of an integral domain D and X be an indeterminate over D. The $D + XK[X]$ construction has been very useful when we construct an easy example with prescribed properties. For example, $D+XK[X]$ is a GCD domain (resp., Bezout doman, Prüfer domain) if and only if D is [8, Corollaries 1.3, 4.14, and 4.15]. We next study the piecewise Noetherian and piecewise w-Noetherian domain properties of $D + XK[X]$.

Theorem 2.15. *Let $R = D + XK[X]$. Then R is a piecewise Noetherian domain (resp., piecewise w-Noetherian domain) if and only if D is a piecewise Noetherian domain (resp., piecewise w-Noetherian domain).*

Proof. Let A be a nonzero ideal of R. Then $A \cap D = (0)$ or $A \cap D \neq (0)$ and $A = A \cap D + XK[X]$, and in this case, A is a prime ideal (resp., w-ideal) if and only if $A \cap D$ is [8, Theorem 2.1] (resp., [1, Lemma 2.1]). Hence, (i) R satisfies the ACC on prime ideals (resp., w-ideals) if and only if D does. (ii) Let M be a nonzero prime ideal (resp., w-ideal) of R, and let Q be an M-primary ideal. If $M \cap D = (0)$, then R_M is a rank-one DVR, and hence R satisfies the ACC on M-primary ideals. Next, assume $M \cap D \neq (0)$. Then $Q \cap D \neq (0)$, and hence $Q = Q \cap D + XK[X]$, and clearly, $Q \cap D$ is an $(M \cap D)$-primary ideal. Conversely, if P is a prime ideal of D and Q' is a P-primary ideal, then $P + XK[X]$ is a prime ideal of R and $Q' + XK[X]$ is $(P + XK[X])$-primary. Hence, D satisfies the ACC on $(M \cap D)$-primary ideals if and only if R satisfies the ACC on M-primary ideals. (iii) Clearly, if J is a nonzero ideal (resp., w-ideal) of D, then $J + XK[X]$ is a nonzero ideal (resp., w-ideal) of R. Also, if Q is a prime ideal of R minimal over $J + XK[X]$, then $Q \cap D$ is minimal over J. Hence if R satisfies (iii), then J has a finite number of minimal prime ideals. For the converse, let A be a nonzero ideal (resp., w-ideal) of R. If $A \cap D \neq (0)$, then $A = A \cap D + XK[X]$, and hence each minimal prime ideal of A is of the form $P + XK[X]$, where P is a prime ideal of D minimal over $A \cap D$. Next, suppose $A \cap D = (0)$, and let $I = \{f(0) \mid f \in A\}$. Then if M is a minimal prime ideal of A, $M \cap D = (0)$ or $I \neq (0)$ and $M \cap D$ is minimal over I. Thus, A has a finite number of minimal prime ideals when D satisfies (iii). $\square$

Example 2.16. Let D be a Noetherian domain (resp., SM domain) with quotient field K and $R = D + XK[X]$. Then R is a piecewise Noetherian domain (resp., piecewise w-Noetherian domain) by Theorem 2.15, but if $D \neq K$, then R is not an SM domain (hence not a Noetherian domain). For if a is a nonzero nonunit of D and P is a minimal prime ideal of aD, then $P + XK[X]$ is a minimal prime ideal of aR but $\mathrm{ht}(P + XK[X]) \geq 2$, so R does not satisfy the principal ideal theorem.

Following [22], a commutative ring R is said to *satisfy* (*accr*) if the ascending chain of the form $N : B \subseteq N : B^2 \subseteq N : B^3 \subseteq \cdots$ terminates for every ideal N of R and every finitely generated ideal B of R. Analogously, a commutative ring R is said to *satisfy* (*accr$_w$*) if the ascending chain of (w-)residuals of the form $N : B \subseteq N : B^2 \subseteq N : B^3 \subseteq \cdots$ terminates for every w-ideal N of R and every finitely generated ideal B of R. In [22, Theorem 1], it was shown that if R is of finite character, then R is Noetherian if and only if R is a piecewise Noetherian ring satisfying (accr). It follows from [21, Theorem 4] that for a quasi-local ring (R, M) whose maximal ideal M is finitely generated, R is

Noetherian if and only if the ascending chain of ideals $\{A : r^k\}_{k\in\mathbb{Z}^+}$ terminates for every $r \in R$ and every finitely generated ideal A of R.

Theorem 2.17. *If R is of w-finite character, then R is an SM domain if (and only if) R is a piecewise w-Noetherian domain satisfying $(accr_w)$.*

Proof. Suppose that R is a piecewise w-Noetherian domain satisfying (accr_w). Then every maximal w-ideal of R is of finite type by Theorem 2.5. Let M be a maximal w-ideal of R. Then MR_M is finitely generated and it is routine to verify that R_M satisfies (accr). Thus by the above remark, R_M is Noetherian, i.e., R is w-locally Noetherian. Now we can conclude that R is an SM domain by [7, Theorem 2.2]. □

It is shown in [2, Theorem 4.1] that if R is a quasi-local ring whose maximal ideal M is finitely generated, then R is Noetherian if and only if every finitely generated ideal of R has a primary decomposition. Following [19], we say that a commutative ring R is *w-Laskerian* if each proper w-ideal of R may be expressed as a finite intersection of primary w-ideals of R. Then it was shown in [18, Theorem 2.7] that R is an SM domain if and only if R is a w-locally Noetherian and w-Laskerian domain.

Theorem 2.18. *A piecewise w-Noetherian domain R is an SM domain if and only if R is a w-Laskerian domain.*

Proof. ($\Rightarrow$) This is clear. ($\Leftarrow$) Suppose that R is a w-Laskerian domain. Let M be a maximal w-ideal of R. Then MR_M is finitely generated by Lemma 2.1 and Theorem 1.3 and every finitely generated ideal of R_M has a primary decomposition. Thus by [2, Theorem 4.1], R_M is Noetherian, i.e., R is w-locally Noetherian. Therefore by [18, Theorem 2.7], R is an SM domain. □

We next study a piecewise w-Noetherian domain analog of Theorem 1.10. We say that an integral domain is t-local if it is a quasi-local domain whose maximal ideal is a t-ideal. An overring R of an integral domain D is said to be *t-linked over D* if $I \in GV(D)$ implies $IR \in GV(R)$. It is known that R is t-linked over D if and only if $(Q\cap D)_t \subsetneq D$ for all maximal t-ideals Q of R [10, Proposition 2.1] and t-flat overrings are t-linked.

Lemma 2.19. *Consider a pullback of type (□) in which T is t-local.*

1. *T is a DW-domain, i.e., each nonzero ideal of T is a w-ideal.*
2. *T is a t-linked extension of R.*
3. *$(\phi^{-1}(I))^{-1} = \phi^{-1}(I^{-1})$ and $(\phi^{-1}(I))_v = \phi^{-1}(I_v)$ for $I \in F(D)$.*
4. *If $I \in F(D)$ with $I^{-1} = D$, then $(\phi^{-1}(I))^{-1} = R$. Therefore, if $I \in \mathrm{GV}(D)$, then $\phi^{-1}(I) \in \mathrm{GV}(R)$.*
5. *If $J \in \mathrm{GV}(R)$, then $\phi(J) \in \mathrm{GV}(D)$.*
6. *If I is a nonzero fractional ideal of D, then $(\phi^{-1}(I))_w = \phi^{-1}(I_w)$.*
7. *If A is an ideal of R with $M \subsetneq A$, then $\phi(A_w) = \phi(A)_w$.*

Proof. (1) This follows from the fact that the unique maximal ideal M of T is a t-ideal.

(2) Let $P \in w\text{-Spec}(T)$. Then $P \subseteq M \subsetneq R$ since T is local with maximal ideal M. Thus $P \in w\text{-Spec}(R)$. In fact, this follows from the fact that $M^{-1} = (R : M) \supseteq T$ implies that $M^{-1} \supsetneq R$, and hence $M_v \subsetneq R$.

(3) $\phi^{-1}(I^{-1}) = \phi^{-1}([D : I]) = [R : \phi^{-1}(I)] = (\phi^{-1}(I))^{-1}$.

(4) By (3) $(\phi^{-1}(I))^{-1} = \phi^{-1}(I^{-1}) = \phi^{-1}(D) = R$.

(5) Since $J^{-1} = R$, we have $(J + M)^{-1} = R$. Note that $J + M$ is finitely generated. Since $J + M = \phi^{-1}(\phi(J))$, $\phi^{-1}(\phi(J)_v) = (J + M)_v = R$ by (3). Therefore, $\phi(J)_v = D$, and thus $\phi(J) \in \text{GV}(D)$.

(6) Let $J \in \text{GV}(R)$ and $x \in T$ with $Jx \subseteq \phi^{-1}(I)$. Then $\phi(J)\phi(x) \subseteq I$, and since $\phi(J) \in \text{GV}(D)$ by (5), $\phi(x) \in I_w$. It follows that $(\phi^{-1}(I))_w \subseteq \phi^{-1}(I_w)$. On the other hand, if $J_0 \in \text{GV}(D)$ and $x \in Q$ with $J_0 x \subseteq I$. Then $\phi^{-1}(J_0)\phi^{-1}(xD) \subseteq \phi^{-1}(I)$, and since $\phi^{-1}(J_0) \in \text{GV}(R)$ by (4), we have $\phi^{-1}(xD) \subseteq (\phi^{-1}(I))_w$. Hence $\phi^{-1}(I_w) \subseteq (\phi^{-1}(I))_w$.

(7) If $y \in \phi(A_w)$, then there exists $x \in A_w$ such that $\phi(x) = y$. Thus $Jx \subseteq A$ for some $J \in \text{GV}(R)$. Hence $\phi(J)y = \phi(J)\phi(x) = \phi(Jx) \subseteq \phi(A)$. By (5), $\phi(J) \in \text{GV}(D)$. Thus we have $y \in \phi(A)_w$. Therefore $\phi(A_w) \subseteq \phi(A)_w$. For the reverse, if $z \in \phi(A)_w$, then there is $J_0 \in \text{GV}(D)$ such that $J_0 z \subseteq \phi(A)$. Note that by (4), $\phi^{-1}(J_0) \in \text{GV}(R)$. Thus $\phi^{-1}(J_0)\phi^{-1}(zD) = \phi^{-1}(J_0 z) \subseteq \phi^{-1}(\phi(A)) = A$, and hence $\phi^{-1}(zD) \subseteq A_w$. So $z \in \phi(A_w)$. Therefore $\phi(A)_w \subseteq \phi(A_w)$. $\square$

Theorem 2.20. *Consider a pullback of type* ($\square$) *in which T is t-local. If Q is the quotient field of D, then R is piecewise w-Noetherian if and only if D and T are piecewise w-Noetherian.*

Proof. The proof of the second condition of the definition of piecewise w-Noetherian domains is similar to that of Theorem 1.10. Thus it suffices to show that R has strong Mori spectrum if and only if D and T has strong Mori spectrum.

($\Rightarrow$) Suppose that R has strong Mori spectrum. We show that D and T have strong Mori spectrum. Now let $P \in w\text{-Spec}(D)$. Then, by Lemma 2.19, $\phi^{-1}(P)$ is a prime w-ideal of R. By hypothesis, there exist $x_1, \dots, x_n \in R$ such that $\phi^{-1}(P) = \sqrt{((x_1, \dots, x_n)R))_w}$. Thus

$$\begin{aligned} P &= \phi(\sqrt{((x_1, \dots, x_n)R))_w}) \\ &= \sqrt{\phi(((x_1, \dots, x_n)R)_w)} \\ &= \sqrt{(\phi((x_1, \dots, x_n)R))_w} \\ &= \sqrt{((\phi(x_1), \dots, \phi(x_n))D)_w}. \end{aligned}$$

Therefore P is the radical of a w-finite type w-ideal of D.

Let $Q \in w\text{-Spec}(T)$. Then $Q \cap R = Q$, and by Lemma 2.19(2), $Q \in w\text{-Spec}(R)$. Thus $Q = \sqrt{(z_1 R + \cdots + z_l R)_w}$ for some $z_1, \dots, z_l \in R$. So if $x \in Q$, then there exists an integer $k \geq 1$ such that $x^k J \subseteq z_1R+\cdots+z_lR$ for some $J \in \mathrm{GV}(R)$. So we have $x^k JT \subseteq z_1T+\cdots+z_lT$. Note that $JT \in \mathrm{GV}(T)$ by Lemma 2.19(2). Thus $x^k \in (z_1T+\cdots+z_lT)_w$, and so $Q = \sqrt{(z_1T+\cdots+z_lT)_w}$. Therefore Q is the radical of a w-finite type w-ideal of T.

($\Leftarrow$) Suppose that D and T have strong Mori spectrum. Let $P \in w\text{-Spec}(R)$. Then P and M are comparable.

(i) $P \subseteq M$ case: Since T is t-local, $P \in w\text{-Spec}(T)$. Thus there exist $x_1,\dots,x_l \in P$ such that $P = \sqrt{(x_1T+\cdots+x_lT)_w}$. We show that $P = \sqrt{x_1R+\cdots+x_lR}$. Let $x \in P$. Then there exists an integer $k \geq 1$ such that $x^kJ \subseteq x_1T+\cdots+x_lT$ for some $J \in \mathrm{GV}(T)$. But $\mathrm{GV}(T) = \{T\}$ because T is t-local. Thus $x^k = a_1x_1+\cdots+a_lx_l$ for some $a_1,\dots,a_l \in T$. So $x^{2k} = (\sum a_ix_i)^2 = \sum_{i=1}^l x_i(a_i^2x_i) + \sum_{i<j} x_i(2a_ia_jx_j)$ with each $a_i^2x_i, 2a_ia_jx_j \in x_1T+\cdots+x_lT \subseteq P \subset R$. Hence $x^{2k} \in x_1R+\cdots+x_lR$, and so we have $x \in \sqrt{x_1R+\cdots+x_lR}$, as desired.

(ii) $M \subsetneq P$ case: Note that by (i) $P \subseteq M$ case, $M = \sqrt{y_1R+\cdots+y_sR}$ for some $y_1,\dots,y_s \in M$. By Lemma 2.19(7), $\phi(P) \in w\text{-Spec}(D)$. By hypothesis, there exist $z_1,\dots,z_m \in P$ such that $\phi(P) = \sqrt{(\phi(z_1)D+\cdots+\phi(z_m)D)_w}$. We show that

$$P = \sqrt{(z_1R+\cdots+z_mR+y_1R+\cdots+y_sR)_w}.$$

Let $x \in P$. Then there is an integer $u \geq 1$ such that $\phi(x)^u \in (\phi(z_1)D+\cdots+\phi(z_m)D)_w$, and so $\phi(x)J_1 \subseteq \sqrt{\phi(z_1)D+\cdots+\phi(z_m)D}$ for some $J_1 \in \mathrm{GV}(D)$. Note that

$$\phi(x\phi^{-1}(J_1)) = \phi(x)\phi(\phi^{-1}(J_1)) = \phi(x)J_1 \quad \text{and}$$
$$\sqrt{\phi(z_1)D+\cdots+\phi(z_m)D} = \phi(\sqrt{z_1R+\cdots+z_mR});$$

so

$$x\phi^{-1}(J_1) \subseteq \sqrt{z_1R+\cdots+z_mR} + M \subseteq \sqrt{z_1R+\cdots+z_mR+y_1R+\cdots+y_sR}.$$

Since $\phi^{-1}(J_1) \in \mathrm{GV}(R)$ by Lemma 2.19(4), we have

$$x \in (\sqrt{z_1R+\cdots+z_mR+y_1R+\cdots+y_sR})_w$$
$$= \sqrt{(z_1R+\cdots+z_mR+y_1R+\cdots+y_sR)_w}.$$

Therefore $P = \sqrt{(\sum y_iR + \sum x_jR)_w}$, as desired. □

Corollary 2.21. *Let K be the quotient field of an integral domain D and $R = D + XK[\![X]\!]$. Then R is a piecewise w-Noetherian domain if and only if D is a piecewise w-Noetherian domain.*

Proof. This follows directly from Theorem 2.20 because $K[\![X]\!]$ is t-local with maximal ideal $XK[\![X]\!]$. □

By utilizing an example due to M. H. Park [24], we give a piecewise w-Noetherian domain R such that $R[\![X]\!]$ is not piecewise w-Noetherian.

Example 2.22. Let G be a nonfinitely generated Abelian group of torsion-free rank 2 such that each rank 1 subgroup of G is cyclic. Let K be a field of characteristic 0. Then the group ring $T := K[G]$ is a 2-dimensional non-Noetherian UFD which is locally Noetherian. In [24, Lemma 9], Park showed that there exists a maximal ideal M of T such that $T[\![X]\!]_{M[\![X]\!]}$ is not Noetherian. In fact, Park's proof demonstrated that $T[\![X]\!]_{M[\![X]\!]}$ does not satisfy the ACC on its prime ideals, and hence is not piecewise Noetherian. Let R be the ring as in [24, Theorem 10]. That is, $R = \varphi^{-1}(k)$, where k is a proper subfield of T/M such that $[T/M : k] < \infty$ and $\varphi : T \to T/M$ is the natural projection. Then it is shown that R is an SM domain, and hence a piecewise w-Noetherian domain. Now consider the following pullback diagram:

$$\begin{array}{ccc} R[\![X]\!]_{M[\![X]\!]} & \longrightarrow & R[\![X]\!]_{M[\![X]\!]}/M[\![X]\!]_{M[\![X]\!]} \\ \downarrow & & \downarrow \\ T[\![X]\!]_{M[\![X]\!]} & \xrightarrow{\phi} & T[\![X]\!]_{M[\![X]\!]}/M[\![X]\!]_{M[\![X]\!]} \end{array}$$

Since $T[\![X]\!]_{M[\![X]\!]}$ is not piecewise Noetherian, the proof ($\Rightarrow$) of Theorem 1.10 shows that $R[\![X]\!]_{M[\![X]\!]}$ is not piecewise Noetherian. In [24, Theorem 10], it is also shown that $M[\![X]\!]$ is a maximal w-ideal of $R[\![X]\!]$. But since $R[\![X]\!]_{M[\![X]\!]}$ is not piecewise Noetherian, by Lemma 2.1 we get that $R[\![X]\!]$ is not a piecewise w-Noetherian domain.

Unlike the "piecewise w-Noetherian domain" case, we do not know the answer to the following natural question.

Question 2.23. Is the power series ring $R[\![X]\!]$ a piecewise Noetherian ring if R is a piecewise Noetherian ring?

Acknowledgements. This work was supported by the Incheon National University Research Fund in 2014 (Grant No. 20140939).

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