J. Korean Math. Soc. **53** (2016), No. 3, pp. 597–621
http://dx.doi.org/10.4134/JKMS.j150191
pISSN: 0304-9914 / eISSN: 2234-3008

LOCAL REGULARITY CRITERIA OF THE NAVIER-STOKES EQUATIONS WITH SLIP BOUNDARY CONDITIONS

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Abstract. We present regularity conditions for suitable weak solutions of the Navier-Stokes equations with slip boundary data near the curved boundary. To be more precise, we prove that suitable weak solutions become regular in a neighborhood boundary points, provided the scaled mixed norm $L^{p,q}_{x,t}$ with $3/p + 2/q = 2$, $1 \le q < \infty$ is sufficiently small in the neighborhood.

1. Introduction

We study the regularity problem for suitable weak solutions $(u, p) : \Omega \times I \to \mathbb{R}^3 \times \mathbb{R}$ to the Navier-Stokes equations in three dimensions,

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla \mathrm{p} = f, \quad \operatorname{div} u = 0 \quad \text{in } \; Q_T = \Omega \times I,$$

where u is the velocity field and p is the pressure. Here f is an external force and Ω is a bounded domain with $\mathcal{C}^2$ boundary. After the existence of weak solutions was proved by Leray [18] and Hopf [11], regularity problem has remained open. It has been known that weak solutions become unique and regular in $\Omega \times [0, T)$ if the following additional conditions are imposed on weak solutions:

$$\|v\|_{L^{p,q}_{x,t}(\Omega\times[0,T))} := \Big\| \|v(\cdot,t)\|_{L^p_x(\Omega)} \Big\|_{L^q_t[0,T)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 \le p \le \infty.$$

In this direction, lots of significant contributions have been made so far (refer to e.g. [6, 7, 8, 9, 13, 15, 21, 22, 30, 32, 33, 35, 36]).

For the partial regularity theory, after Scheffer's works in a series of papers [23, 24, 25, 26], Caffarelli, Kohn and Nirenberg [4] proved that the one-dimensional parabolic Hausdorff measure of possible singular set is zero for suitable weak solutions of the Navier-Stokes equations. The extension up to boundary was shown in [28] (see also [29]). In [5], the estimate of size of a

Received March 22, 2015.
2010 *Mathematics Subject Classification.* Primary 35D30, 35Q30.
Key words and phrases. Navier-Stokes equations, slip boundary data, suitable weak solution.
K. Kang was supported by NRF-2012R1A1A2001373. M. Kim was supported by KRF-2008-331-C00024.

possible singular set was improved by a logarithmic factor. The following local regularity criterion was proved in [4] and crucially used for partial regularity: there exists $\epsilon > 0$ such that if suitable weak solution u satisfies

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_{z,r}} |\nabla u(y,s)|^2 \, dyds \le \epsilon,$$

then u is regular in a neighborhood of z (refer to [27] for flat boundary and [29] for curved boundary). This regularity criterion was improved in terms of scaled mixed norm regarding velocity field in [10, Theorem 1.1]. On the other hand, in [9], the following regularity criteria was proved near the flat boundary:

$$\limsup_{r\to 0} \frac{1}{r} \left\| \|u\|_{L^p(B^+_{x,r})} \right\|_{L^q(t-r^2,t)} \le \epsilon, \quad \frac{3}{p}+\frac{2}{q}=2, \quad 2<q<\infty. \tag{1}$$

In [14], the following local regularity criteria was proved near the curved boundary in case of homogeneous boundary conditions:

$$\limsup_{r\to 0} r^{-(\frac{3}{p}+\frac{2}{q}-1)} \left\| \|u\|_{L^p(\Omega_{x,r})} \right\|_{L^q(t-r^2,t)} \le \epsilon,$$

$$1 \le \frac{3}{p}+\frac{2}{q} \le 2, \quad 2<q\le\infty, \quad (p,q) \neq \left(\frac{3}{2},\infty\right).$$

For the case of slip boundary conditions, the existence of the weak or strong solutions was studied by Solonnikov, Ščadilov [34], Maremonti [20] and Itoh, Tani [12]. Some regularity results for weak solutions were showed in [3] for the stationary case. Bae, Choe and Jin [2] proved the following: Suppose (u,p) is a suitable weak solution. There exists a positive constant σ such that if $u \in L^{p,q}(Q^+_r)$ for some (p,q) satisfying $\frac{3}{p}+\frac{2}{q} \le 1$ with $q>3$, or if $u \in L^{3,\infty}(Q^+_r)$ with $\|u\|_{L^{3,\infty}(Q^+_r)} \le \varepsilon_0$ for some small ε_0, then

$$\sup_{Q^+_{\frac{r}{2}}} |u| \le N\left(\int_{Q^+_r} |u|^3 dxdt\right)^{\frac{5+\sigma}{3\sigma}} + N$$

for some positive constant N depending on ε_0.

The main objective of this paper is to establish the regularity criteria (1) for the Navier-Stokes equations with ship boundary conditions near the curved boundary.

To be more precise, we study suitable weak solutions of the following Navier-Stokes equations in three dimensions

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla \mathrm{p} = f, \quad \operatorname{div} \mathrm{u} = 0 & \text{in } \ Q_T = \Omega\times I, \\ u\cdot n = 0, \quad n\cdot T(u,\mathrm{p})\cdot\tau = 0 & \text{on } \ \partial\Omega\times I, \end{cases} \tag{2}$$

where u is the velocity field, p is the pressure, n is the outer unit normal vector, τ is the unit tangent vector and $T(u,\mathrm{p})$ is a stress tensor, which is given as

$$T(u,\mathrm{p}) = \frac{1}{2}\left(\nabla u + (\nabla u)^\top\right) - \mathrm{p}\delta_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)_{i,j=1,2,3} - \mathrm{p}\delta_{ij}.$$

Here f is an external force and Ω is a bounded domain with $\mathcal{C}^2$ boundary. Suitable weak solution will be defined in Definition 2.1 in next section. The existence of suitable weak solutions with slip boundary conditions was proved in [2] for the case of half space. In Appendix, we provide the existence of suitable weak solutions for the bounded domains as in [4].

We prove that suitable weak solution u becomes Hölder continuous near regular curved boundary, provided that the scaled mixed $L^{p,q}$-norm of the velocity field u is sufficiently small (the proof will be given in Section 3). More precisely, our main result reads as follows:

Theorem 1.1. *Let u be a suitable weak solution of the Navier-Stokes equations in Ω with extra force $f \in M_{2,\gamma}$ for some $\gamma > 0$, $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$ and $B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}$. Assume further that Ω is any domain with $\mathcal{C}^2$ boundary satisfying Assumption 2.1. Suppose that $(x,t) \in \partial\Omega \times I$. For every pair p, q satisfying*

$$\frac{3}{p} + \frac{2}{q} = 2, \quad 1 \le q < \infty,$$

there exists a constant $\epsilon > 0$ depending on p, q, γ and $\|f\|_{M_{2,\gamma}}$ such that, if the pair u, p is a suitable weak solution of the Navier-Stokes equations (2) *satisfying Definition 2.1 and*

$$\limsup_{r \to 0} r^{-1} \left\| \|u\|_{L^p(\Omega_{x,r})} \right\|_{L^q(t-r^2,t)} < \epsilon,$$

then u is regular at $z = (x,t)$.

2. Preliminaries

In this section, we introduce notations, define suitable weak solutions, and derive equations (5) changed by flatting the boundary. For notational convenience, we denote for a point $x = (x', x_3) \in \mathbb{R}^3$ with $x' \in \mathbb{R}^2$

$$B_{x,r} = \left\{ y \in \mathbb{R}^3 : |y - x| < r \right\}, \quad D_{x',r} = \left\{ y' \in \mathbb{R}^2 : |y' - x'| < r \right\}.$$

For $x \in \overline{\Omega}$, we use the notation $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$. If $x = 0$, we drop x in the above notations, for instance $\Omega_{x,r}$ is abbreviated to Ω_r. A solution u to (2) is said to be regular at $z = (x,t) \in \overline{\Omega} \times I$ if $u \in L^\infty(\Omega_{x,r} \times (t - r^2, t))$ for some $r > 0$. In such case, z is called a regular point. Otherwise we say that u is singular at z and z is a singular point. We begin with some notations. Let Ω be a bounded domain in $\mathbb{R}^3$. We denote by $N = N(\alpha, \beta, \ldots)$ a constant depending on the prescribed quantities $\alpha, \beta, \ldots$, which may change from line to line. For $1 \le p \le \infty$, $W^{k.p}(\Omega)$ denotes the usual Sobolev space, i.e., $W^{k.p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ 0 \le |\alpha| \le k\}$. We write the average of f on E as $⨍_E f$, that is $⨍_E f = \frac{1}{|E|} \int_E f$. We suppose that f belongs to a parabolic Morrey space $M_{2,\gamma}(Q_T)$ for some $0 < \gamma \le 2$ equipped with the norm

$$\|f\|_{M_{2,\gamma}(Q_T)} = \sup\left\{\left(\frac{1}{r^{1+2\gamma}}\int_{Q_{z,r}} |f|^2\,dx\right)^{\frac{1}{2}} : z=(x,t)\in \overline{Q}_T, r>0\right\},$$

where $Q_{z,r} = \big(\Omega_{x,r}\times(t-r^2,t)\big)\cap Q_T$. We note that $M_{2,\gamma}(Q_T)$ contains $L^{\frac{5}{2-\gamma}}(Q_T)$. We make some assumptions on the boundary of Ω.

Assumption 2.1. *Suppose that Ω be a domain with $\mathcal{C}^2$ boundary such that the following is satisfied: For each point $x=(x',x_3)\in\partial\Omega$, there exist absolute constant N and r_0 independent of x such that we can find a Cartesian coordinate system $\{y_i\}_{i=1}^3$ with the origin at x and a $\mathcal{C}^2$ function $\varphi : D_{r_0}\to\mathbb{R}$ satisfying*

$$\Omega_{r_0} = \Omega\cap B_{x,r_0} = \{y=(y',y_3)\in B_{x,r_1} \,:\, y_3>\varphi(y')\}$$

and

$$\varphi(0)=0,\quad \nabla_y\varphi(0)=0,\quad \sup_{D_{r_0}}\left|\nabla_y^2\varphi\right|\le N.$$

Remark 2.1. The main condition on Assumption 2.1 is the uniform estimate of the $\mathcal{C}^2-$norms of the function φ for each $x\in\partial\Omega$. More precisely, there exists a sufficiently small r_1 with $r_1<r_0$, where r_0 is the number in Assumption 2.1 such that for any $r<r_1$

$$\sup_{x\in\partial\Omega}\|\varphi\|_{\mathcal{C}^2(D_r)}\le N(1+r+r^2). \tag{3}$$

Next lemma is related with Gagliardo-Nirenberg in [1, 17] :

Lemma 2.2. *Let Ω be a domain of $\mathbb{R}^3$ satisfying Assumption 2.1 and $\int_\Omega u=0$. For every fixed number $r\ge 1$ there exists a constant N such that*

$$\|u\|_{L^q_\Omega}\le N\|\nabla u\|^{\theta}_{L^p_\Omega}\|u\|^{1-\theta}_{L^r_\Omega},$$

where $\theta\in[0,1], p,q\ge 1$, are linked by $\theta=(\frac{1}{r}-\frac{1}{q})(\frac{1}{3}-\frac{1}{p}+\frac{1}{r})^{-1}$.

Next we recall suitable weak solutions for the Navier-Stokes equations (2) in three dimensions.

Definition 2.1. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain satisfying Assumption 2.1 and $I=[0,T)$. We denote $Q_T=\Omega\times I$. Suppose that f belongs to the Morrey space $M_{2,\gamma}(Q_T)$ for some $\gamma>0$. A pair of (u,p) is a suitable weak solution to (2) if the following conditions are satisfied:

(a) The functions $u : Q_T\to\mathbb{R}^3$ and $\mathrm{p} : Q_T\to\mathbb{R}$ satisfy

$$u\in L^\infty\left(I;L^2(\Omega)\right)\cap L^2\left(I;W^{1,2}(\Omega)\right),\quad \mathrm{p}\in L^{\frac{3}{2}}(\Omega\times I),$$

$$\nabla^2 u,\nabla\mathrm{p}\in L^{\frac{9}{8},\frac{3}{2}}_{x,t}(\Omega\times I).$$

(b) u and p solve the Navier-Stokes equations in Q_T in the sense of distributions and u satisfies slip boundary conditions on $\partial\Omega\times I$.

(c) u and p satisfy the local energy inequality

$$\int_\Omega |u(x,t)|^2 \phi(x,t)dx + 2\int_{t_0}^t \int_\Omega |\nabla u(x,t')|^2 \phi(x,t')dxdt'$$
$$\leq \int_{t_0}^t \int_\Omega \Big(|u|^2 (\partial_t \phi + \Delta\phi) + \big(|u|^2 + 2\mathrm{p}\big) u \cdot \nabla\phi + 2f \cdot u\phi \Big) dxdt'$$

for all $t \in I = (0,T)$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, vanishing in a neighborhood of the set $\Omega \times \{t = 0\}$.

Let $x_0 \in \partial\Omega$. Under Assumption 2.1, we can represent $\Omega_{x_0,r_0} = \Omega \cap B_{x_0,r_0} = \{y = (y', y_3) \in B_{x_0,r_0} : y_3 > \varphi(y')\}$ where φ is the graph of $\mathcal{C}^2$ in Assumption 2.1. Flatting the boundary near x_0, we introduce new coordinates $x = \psi(y)$ by formulas

$$x = \psi(y) \equiv (y_1,\, y_2,\, y_3 - \varphi(y_1, y_2))\,, \tag{4}$$

where φ is a bijection whose Jacobian is equal to 1. We note that the mapping $y \mapsto x = \psi(y)$ straightens out $\partial\Omega$ near x_0 such that $\Omega_{x_0,\rho}$ is transformed onto a subdomain $\psi(\Omega_{x_0,\rho})$ of $\mathbb{R}^3_+ \equiv \{x \in \mathbb{R}^3 : x_3 > 0\}$. We define $v = u \circ \psi^{-1}$, $\pi = \mathrm{p}\circ\psi^{-1}$ and $g = f\circ\psi^{-1}$ in $\psi(\Omega_{x_0,\rho})$. Then using the change of variables (4), in this case, the outer unit normal vector is $(0,0,-1)$ and unit tangent vectors are $(1,0,0)$, $(0,1,0)$. The equations (2) result in the following equations for v and π:

$$\begin{cases} v_t - \widehat{\Delta} v + (v \cdot \widehat{\nabla})v + \widehat{\nabla}\pi = g, & \\ \qquad\qquad \widehat{\nabla} \cdot v = 0 & \text{in } \ \psi(\Omega_{x_0,\rho}), \\ v_3 = 0, \quad \partial_3 v_1 = \varphi_{x_1}\partial_3 v_3, & \\ \qquad \partial_3 v_2 = \varphi_{x_2}\partial_3 v_3 & \text{on } \ \partial\psi(\Omega_{x_0,\rho}) \cap \{x_3 = 0\}, \end{cases} \tag{5}$$

where $\widehat{\nabla}$ and $\widehat{\Delta}$ are differential operators with variable coefficients defined by

$$\begin{aligned} \widehat{\nabla} &= (\partial_{x_1} - \varphi_{x_1}\partial_{x_3}, \partial_{x_2} - \varphi_{x_2}\partial_{x_3}, \partial_{x_3}), \\ \widehat{\Delta} &= a_{ij}(x)\partial^2_{x_i,x_j} + b_i(x)\partial_{x_i}, \end{aligned} \tag{6}$$

where a_{ij} and b_i are given as

$$a_{ij}(x) = \delta_{ij}, \quad a_{i3}(x) = a_{3i}(x) = -\varphi_{x_i}, \quad b_i(x) = 0, \quad i = 1, 2,$$

and

$$a_{33}(x) = 1 + \sum_{i=1}^2 (\varphi_{x_i})^2, \qquad b_3(x) = -\sum_{i=1}^2 \varphi_{x_i x_i}.$$

As mentioned in Remark 2.1, if we take a sufficiently small r_1 with $r_1 < r_0$, then (3) holds for any $r < r_1$. In addition, the followings are satisfied:

$$\frac{1}{2}|\nabla v(x,t)| \leq |\widehat{\nabla} v(x,t)| \leq 2|\nabla v(x,t)| \quad \text{ for all } x \in \psi(\Omega_{(x_0),2r}), \tag{7}$$

$$\begin{aligned} B^+_{\psi(x_0),\frac{r}{2}} \subset \psi(\Omega_{x_0,r}) \subset B^+_{\psi(x_0),2r}, \\ \psi^{-1}(B^+_{\psi(x_0),\frac{r}{2}}) \subset \Omega_{x_0,r} \subset \psi^{-1}(B^+_{\psi(x_0),2r}). \end{aligned} \tag{8}$$

From now on, we fix $x_0 = 0$ without loss of generality. We suppose that, as above, ψ is a coordinate transformation so that v, π satisfies (5) in $\psi(\Omega_{r_0})$.

Remark 2.3. Due to the suitability of u, p (see Definition 2.1), (v, π) solve (5) in a weak sense and satisfies the following local energy inequality: There exists r_2 with $r_2 < r_0$ where r_0 is the number in Assumption 2.1 such that

$$\int_{\psi(\Omega_{r_0})} |v(x,t)|^2 \xi(x,t)dx + 2\int_{t_0}^t \int_{\psi(\Omega_{r_0})} \left|\widehat{\nabla} v(x,t')\right|^2 \xi(x,t')dxdt' \tag{9}$$
$$\leq \int_{t_0}^t \int_{\psi(\Omega_{r_0})} \left(|v|^2 (\partial_t \xi + \widehat{\Delta}\xi) + \left(|v|^2 + 2\pi\right) v \cdot \widehat{\nabla}\xi + 2g \cdot v\xi\right) dxdt',$$

where $\xi \in C_0^\infty(B_r)$ with $r < r_2$ and $\xi \geq 0$, and $\widehat{\nabla}$ and $\widehat{\Delta}$ are differential operators in (6).

Next we define some scaling invariant functionals, which are useful for our purpose. Let $B_r^+ = B_r \cap \{x \in \mathbb{R}^3 : x_3 > 0\}$ and $Q_r^+ = B_r^+ \times (-r^2, 0)$. As defined earlier, we also denote $\Omega_r = \Omega \cap B_r$ and $Q_r = \Omega_r \times (-r^2, 0)$. Let r_0 and r_1 be the numbers in Assumption 2.1 and Remark 2.1, respectively. For any $r < r_1$ and a suitable weak solution (u, p) of (2) we introduce

$$A(r) := \frac{1}{r^2}\int_{\Omega_r} |u(y,s)|^3 dyds,$$

$$D(r) := \sup_{-r^2\leq t\leq 0} \frac{1}{r}\int_{\Omega_r} |u(y,s)|^2 dy, \quad E(r) := \frac{1}{r}\int_{Q_r} |\nabla u(y,s)|^2 dyds,$$

$$K(r) := \frac{1}{r}\left(\int_{t-r^2}^t \left(\int_{\Omega_r} |u(y,s)|^p dy\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}}, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq q < \infty,$$

$$C(r) := \frac{1}{r^2}\int_{\Omega_r} |\mathrm{p}(y,s)|^{\frac{3}{2}} dyds.$$

For a suitable weak solution (v, π) and $B_r^+ \subset \psi(\Omega_{r_1})$, we introduce

$$\widehat{A}(r) := \frac{1}{r^2}\int_{Q_r^+} |v(y,s)|^3 dyds, \quad \widehat{A}_a(r) := \frac{1}{r^2}\int_{Q_r^+} |v - (v)_r|^3 dyds,$$

$$\widehat{D}(r) := \sup_{-r^2\leq t\leq 0} \frac{1}{r}\int_{B_r^+} |v(y,s)|^2 dy, \quad \widehat{E}(r) := \frac{1}{r}\int_{Q_r^+} |\widehat{\nabla} v(y,s)|^2 dyds,$$

$$\widehat{K}(r) := \frac{1}{r}\left(\int_{t-r^2}^t \left(\int_{B_r^+} |v(y,s)|^p dy\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}},$$

$$\widehat{C}(r) := \frac{1}{r^2}\int_{\Omega_r} |\pi(y,s)|^{\frac{3}{2}} dyds, \quad \widehat{C}_a(r) := \frac{1}{r^2}\int_{\Omega_r} |\pi - (\pi)_r|^{\frac{3}{2}} dyds,$$

where $(v)_r = \fint_{B_r^+} v(y,s)dy$. Next lemma shows relations between scaling invariant quantities above.

Lemma 2.4. *Let Ω be a bounded domain satisfying Assumption 2.1 and $x_0 \in \partial\Omega$. Suppose that (u, p) and (v, π) are suitable weak solutions of (2) in $\Omega \times I$ and (5) in $\psi(\Omega_{x_0}) \times I$, respectively, where ψ is the mapping flatting the boundary in Assumption 2.1. Let $x = \psi(x_0)$. Then there exist sufficiently small r_1 and an absolute constant N such that for any $4r < r_1$ the followings are satisfied:*

$$\frac{1}{N}E(r) \le \widehat{E}(2r) \le NE(4r), \qquad \frac{1}{N}A(r) \le \widehat{A}(2r) \le NA(4r),$$

$$\frac{1}{N}K(r) \le \widehat{K}(2r) \le NK(4r), \qquad \frac{1}{N}C(r) \le \widehat{C}(2r) \le NS(4r),$$

$$\frac{1}{N}D(r) \le \widehat{D}(2r) \le ND(4r).$$

Proof. We just show one of above estimates, since others follows similar arguments. For convenience, we denote $\Pi_r = \psi(\Omega_r) \times (-r^2, 0)$ and $\Pi_r^{-1} = \psi^{-1}(\Omega_r) \times (-r^2, 0)$. As indicated earlier, we take a sufficiently small r_1 such that (3), (7) and (8) hold. Then

$$E(r) \le \frac{N}{r}\int_{\Pi_r} |\nabla v|^2 \le \frac{N}{r}\int_{\Pi_r} \left|\widehat{\nabla} v\right|^2 \le \frac{N}{2r}\int_{Q_{2r}^+} \left|\widehat{\nabla} v\right|^2 = N\widehat{E}(2r).$$

On the other hand,

$$\widehat{E}(2r) \le \frac{1}{2r}\int_{Q_{2r}^+} |\nabla v|^2 \le \frac{N}{2r}\int_{\Pi_{2r}^{-1}} |\nabla u|^2 \le \frac{N}{4r}\int_{Q_{4r}} |\nabla u|^2 = NE(4r).$$

This completes the proof. □

Remark 2.5. We note that f and g have relations as in Lemma 2.4. To be more precise,

$$\int_{Q_r} |f|^2 \le N\int_{\Pi_r} |g|^2 \le N\int_{Q_{2r}^+} |g|^2 \le N\int_{\Pi_{2r}^{-1}} |f|^2 \le N\int_{Q_{4r}} |f|^2 .$$

Therefore, it is direct that $\|g\|_{M_{2,\gamma}(\Pi_r)} \le N\|f\|_{M_{2,\gamma}(Q_r)}$.

In the sequel, for simplicity, we denote $\|f\|_{M_{2,\gamma}} = m_\gamma$.

3. Local regularity near boundary

In this section, we present the proof of Theorem 1.1. We first show a local regularity criterion for v near the boundary.

Lemma 3.1. *Let Ω be a bounded domain satisfying Assumption 2.1 and $x_0 \in \partial\Omega$. Suppose that (v, π) is a suitable weak solution of (5) in $\psi(\Omega_{x_0}) \subset \mathbb{R}^3_+$, where ψ is the mapping flatting the boundary in Assumption 2.1. Let $w = (y, t)$ with $y = \psi(x_0)$. Assume further that $g \in M_{2,\gamma}$ for some $\gamma \in (0, 2]$. Then there exist $\epsilon > 0$ and r_1 depending on $\gamma, \|g\|_{M_{2,\gamma}}$ such that if $\widehat{A}^{\frac{1}{3}}(r) + \widehat{C}^{\frac{2}{3}}(r) < \epsilon$ for some $r < r_1$, then w is a regular point.*

The proof of Lemma 3.1 is based on the following, which shows a decay property of v in a Lebesgue spaces. From now on, we denote $\|g\|_{M_{2,\gamma}} = m_\gamma$, unless any confusion is expected.

Lemma 3.2. *Let $0 < \theta < \frac{1}{2}$ and $\beta \in (0, \gamma)$. Under the same assumption as in Lemma 3.1, there exist $\varepsilon_1 > 0$ and r_1 depending on θ, γ, β and m_γ such that if $\widehat{A}^{\frac{1}{3}}(r) + \widehat{C}^{\frac{2}{3}}(r) + m_\gamma r^\beta < \varepsilon_1$ for some $r \in (0, r_1)$, then*

$$\widehat{A}^{\frac{1}{3}}(\theta r) + \widehat{C}^{\frac{2}{3}}(\theta r) < N\theta^\alpha \left(\widehat{A}^{\frac{1}{3}}(r) + \widehat{C}^{\frac{2}{3}}(r) + m_\gamma r^\beta\right),$$

where $0 < \alpha < 1$ and N is a constant.

Proof. For convenience we denote $\tau(r) := \widehat{A}^{\frac{1}{3}}(r) + \widehat{C}^{\frac{2}{3}}(r) + m_\gamma r^\beta$. Suppose the statement is not true. Then for any $\alpha \in (0, 1)$ and $N > 0$, there exist $z_n = (x_n, t_n)$, $r_n \searrow 0$ and $\varepsilon_n \searrow 0$ such that

$$\tau(r_n) = \varepsilon_n, \qquad \widehat{A}^{\frac{1}{3}}(\theta r_n) + \widehat{C}^{\frac{2}{3}}(\theta r_n) > N\theta^\alpha \varepsilon_n.$$

Let $w = (y, s)$ where $y = \frac{1}{r_n}(x - x_n)$, $s = \frac{1}{r_n^2}(t - t_n)$ and we define $\widehat{v}_n$, $\widehat{\pi}_n$ and $\widehat{g}_n$ by $\widehat{v}_n(w) = \frac{1}{\epsilon_n}(v(z) - (v(z))_{r_n})$, $\widehat{\pi}_n(w) = \frac{1}{\epsilon_n} r_n(\pi(z) - (\pi(z))_{r_n})$ and $\widehat{g}_n(w) = g(z)$, respectively. We also introduce scaling invariant functionals $\widehat{A}_a(\widehat{v}_n, \theta)$ and $\widehat{C}_a(\widehat{\pi}_n, \theta)$ as follows:

$$\widehat{A}_a(\widehat{v}_n, \theta) := \frac{1}{\theta^2} \int_{Q_\theta^+} |\widehat{v}_n - (\widehat{v}_n)_\theta|^3 dw, \quad \widehat{C}_a(\widehat{v}_n, \theta) := \frac{1}{\theta^2} \int_{Q_\theta^+} |\widehat{\pi}_n - (\widehat{\pi}_n)_\theta|^{\frac{3}{2}} dw.$$

The change of variables lead to

$$\varepsilon_n \widehat{\nabla}_y \widehat{v}_n(w) = r_n \widehat{\nabla}_x v(z), \quad \varepsilon_n \widehat{\nabla}_y^2 \widehat{v}_n(w) = {r_n}^2 \widehat{\nabla}_x^2 v(z),$$
$$\varepsilon_n \partial_s \widehat{v}_n(w) = {r_n}^2 \partial_t v(z), \quad \varepsilon_n \widehat{\nabla}_y \widehat{\pi}_n(w) = r_n \widehat{\nabla}_x \pi(z).$$
$$(\widehat{v}_n)_{B_1^+}(s) = 0, \quad (\widehat{\pi}_n)_{B_1^+}(s) = 0, \quad s \in (-1, 0),$$

$$\begin{aligned} \tau_n(1) =& \|\widehat{v}_n\|_{L^3(Q_1^+)} + \|\widehat{\pi}_n\|_{L^{\frac{2}{3}}(Q_1^+)} + m_\gamma^n \frac{r_n^\beta}{\varepsilon_n} = 1, \\ \tau_n(\theta) :=& \widehat{A}^{\frac{1}{3}}(\hat{v}_n, \theta) + \widehat{C}^{\frac{2}{3}}(\hat{\pi}_n, \theta) \geq C\theta^\alpha, \end{aligned} \tag{10}$$

where $m_\gamma^n = \|g_n\|_{M_{2,\gamma}}$. On the other hand, $\widehat{v}_n$, $\widehat{\pi}_n$ solve the following system in a weak sense

$$\left.\begin{aligned} \partial_s \widehat{v}_n - \widehat{\Delta} \widehat{v}_n + \epsilon_n r_n (\widehat{v}_n \cdot \widehat{\nabla}) \widehat{v}_n + (\widehat{v}_n \cdot \widehat{\nabla}) r_n a_n + \widehat{\nabla} \widehat{\pi}_n = \tfrac{{r_n}^2}{\varepsilon_n} \widehat{g}_n, \\ \widehat{\nabla} \cdot \widehat{v}_n = 0 \end{aligned}\right. \quad \text{in } Q_1^+ \tag{11}$$

$$\widehat{v}_{3,n} = 0, \qquad \begin{aligned} \partial_3 \widehat{v}_{1,n} = \varphi_{x_1} \partial_3 \widehat{v}_{3,n} \\ \partial_3 \widehat{v}_{2,n} = \varphi_{x_2} \partial_3 \widehat{v}_{3,n} \end{aligned} \quad \text{on } B_1 \cap \{x_3 = 0\} \times (-1, 0),$$

where $a_n = \big(v(z)\big)_{r_n} = \fint_{B_{r_n}^+} v(y, t) dy$.

Since $\tau_n(1) = 1$, we have following weak convergence:

$$\begin{aligned} \widehat{v}_n \rightharpoonup \widehat{v} \quad \text{in } L^3(Q_1^+), \quad \widehat{\pi}_n \rightharpoonup \widehat{\pi} \quad \text{in } L^{\frac{3}{2}}(Q_1^+), \\ (\widehat{v})_{B_1^+}(s) = 0, \quad (\widehat{\pi})_{B_1^+}(s) = 0. \end{aligned} \tag{12}$$

Then, from (10) and (12),

$$\tau(1) = \widehat{A}^{\frac{1}{3}}(1) + \widehat{C}^{\frac{2}{3}}(1) \le 1.$$

According to the definition of m_γ, we have

$$\begin{aligned} \frac{{r_n}^2}{\varepsilon_n}\|\widehat{g}_n\|_{L^2(Q_1^+)} &\le \frac{{r_n}^2}{\varepsilon_n} m_\gamma {r_n}^{\gamma-2} \\ &= \frac{m_\gamma {r_n}^\beta}{\varepsilon_n} {r_n}^{\gamma-\beta} \le {r_n}^{\gamma-\beta} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{13}$$

Since $|r_n a_n|$ be a bound, without loss of generality it may be assumed that:

$$r_n a_n \to b \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad |b| \le M. \tag{14}$$

Using (10) and (13), we take

$$\begin{aligned} \int_{Q_1^+} (-\widehat{v}_n \cdot \partial_s X) dw = \int_{Q_1^+} \big\{ &\widehat{v}_n \cdot \widehat{\Delta} X + \widehat{v}_n \cdot (\varepsilon_n r_n \widehat{v}_n) \widehat{\nabla} X \\ &+ \widehat{v}_n \cdot (r_n a_n) \widehat{\nabla} X + \widehat{\pi}_n (\widehat{\nabla} \cdot X) + \frac{{r_n}^2}{\varepsilon_n} \widehat{g}_n \cdot X \big\} dw \\ \le N(M) &\|X\|_{L^3(-1,0;W^{2,2}(B_1^+))} \end{aligned}$$

for all $X \in C_0^1(-1,0;W^{2,2}(B_1^+))$.

Therefore, $\partial_s \widehat{v}_n$ is uniformly bounded in $L^{\frac{3}{2}}\big((-1,0);(W^{2,2}(B_1^+))'\big)$ and we also have

$$\partial_s \widehat{v}_n \rightharpoonup \partial_s \widehat{v} \quad \text{in } L^{\frac{3}{2}}\big((-1,0);(W^{2,2}(B_1^+))'\big). \tag{15}$$

From the local energy inequality (9), we obtain for every $\sigma \in (-1,0)$

$$\begin{aligned} &\int_{B_1^+} |\widehat{v}_n(y,\sigma)|^2 \xi(y,\sigma) dy + 2\int_{-1}^{\sigma} \int_{B_1^+} |\widehat{\nabla} \widehat{v}_n|^2 \xi dy ds \\ \le &\int_{-1}^{\sigma} \int_{B_1^+} \Big\{ |\widehat{v}_n|^2 (\partial_s \xi + \widehat{\Delta} \xi) + r_n |\widehat{v}_n|^2 (\varepsilon_n \widehat{v}_n + a_n) \cdot \widehat{\nabla} \xi \\ &\qquad + \widehat{\pi}_n \hat{v}_n \cdot \widehat{\nabla} \xi + \frac{{r_n}^2}{\varepsilon_n} \widehat{g}_n \cdot \widehat{v}_n \xi \Big\} dy ds \end{aligned} \tag{16}$$

for all $\xi \in C_0^\infty(B_r)$. Recalling (10), (13) and (14), we deduce from (16) the bound

$$\operatorname*{ess\,sup}_{s \in (-(3/4)^2, 0)} \|\widehat{v}_n(s)\|^2_{L^2(B_{3/4}^+)} + \|\widehat{\nabla} \widehat{v}_n\|^2_{L^2(Q_{3/4}^+)} \le N(M). \tag{17}$$

The Gagliardo-Nirenberg inequality and (17) yield estimate

$$\|\widehat{v}_n\|_{L^{\frac{10}{3}}(Q_{3/4}^+)} \le N(M). \tag{18}$$

Using the standard compactness arguments and (15), (17) and (18), we conclude following convergence:

$$\widehat{v}_n \rightharpoonup \widehat{v} \qquad \text{in } L^3(Q_{3/4}^+). \tag{19}$$

Next we observe that $\widehat{v}$ and $\widehat{\pi}$ solve the following perturbed Stokes system

$$\partial_s \widehat{v} - \widehat{\Delta}\widehat{v} + \widehat{\nabla}\widehat{\pi} = 0, \quad \text{div } \widehat{v} = 0 \quad \text{in } Q_1^+$$

with

$$\widehat{v}_3 = 0, \qquad \begin{matrix} \partial_3 \widehat{v}_1 = \varphi_{x_1} \partial_3 \widehat{v}_3 \\ \partial_3 \widehat{v}_2 = \varphi_{x_2} \partial_3 \widehat{v}_3 \end{matrix} \qquad \text{on } \ (B_1 \cap \{x_3 = 0\}) \times (-1, 0).$$

Indeed, by the Hölder's inequality, we have

$$\begin{aligned} \left\| (\widehat{v}_n \cdot \widehat{\nabla})\widehat{v}_n \right\|_{L^{\frac{9}{8}}(B^+_{7/8})} &\le N \left\| \widehat{\nabla}\widehat{v}_n \right\|_{L^2(B^+_{7/8})} \|\widehat{v}_n\|_{L^{\frac{18}{7}}(B^+_{7/8})} \\ &\le N \left\| \widehat{\nabla}\widehat{v}_n \right\|_{L^2(B^+_{7/8})} \left\| \widehat{\nabla}\widehat{v}_n \right\|^{\frac{1}{3}}_{L^2(B^+_{7/8})} \|\widehat{v}_n\|^{\frac{2}{3}}_{L^2(B^+_{7/8})} \\ &\le N \left\| \widehat{\nabla}\widehat{v}_n \right\|^{\frac{2}{3}}_{L^2(B^+_{7/8})}. \end{aligned}$$

Therefore,

$$\left\| (\widehat{v}_n \cdot \widehat{\nabla})\widehat{v}_n \right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{7/8})} \le N. \tag{20}$$

Moreover, $\widehat{v}_n$ and $\widehat{\pi}_n$ solves the following problem:

$$\begin{matrix} \partial_s \widehat{v}_n - \widehat{\Delta}\widehat{v}_n + \widehat{\nabla}\widehat{\pi}_n = -\varepsilon_n r_n (\widehat{v}_n \cdot \widehat{\nabla})\widehat{v}_n - (\widehat{v}_n \cdot \widehat{\nabla}) r_n a_n + \frac{{r_n}^2}{\varepsilon_n}\widehat{g}_n \\ \widehat{\nabla} \cdot \widehat{v}_n = 0 \end{matrix} \qquad \text{in } Q^+_{5/6}$$

with

$$\widehat{v}_{3,n} = 0, \qquad \begin{matrix} \partial_3 \widehat{v}_{1,n} = \varphi_{x_1} \partial_3 \widehat{v}_{3,n} \\ \partial_3 \widehat{v}_{2,n} = \varphi_{x_2} \partial_3 \widehat{v}_{3,n} \end{matrix} \qquad \text{on } \big(B_{5/6} \cap \{x_3 = 0\}\big) \times \left(-\left(\frac{5}{6}\right)^2, 0 \right).$$

Due to the local boundary estimate for the Stokes system in Lemma 4.2, we have the following estimate for $\widehat{v}_n$ and $\widehat{\pi}_n$;

$$\begin{aligned} &\|\partial_s \widehat{v}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5})} + \left\| \widehat{\nabla}^2 \widehat{v}_n \right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5})} + \left\| \widehat{\nabla}\widehat{\pi}_n \right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5})} \\ &\le N \Bigg(\epsilon_n r_n \left\| \left(\widehat{v}_n \cdot \widehat{\nabla} \right) \widehat{v}_n \right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{5/6})} + \frac{{r_n}^2}{\varepsilon_n} \|\widehat{g}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{5/6})} \\ &\qquad + \|\widehat{v}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{5/6})} + \left\| \widehat{\nabla}\widehat{v}_n \right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{5/6})} + \|\widehat{\pi}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{5/6})} \Bigg) \\ &\le N \left(1 + \epsilon_n r_n\right), \end{aligned}$$

where we used (10), (13), (17) and (20). Thus, we get

$$\widehat{\Delta}\widehat{v}_n, \widehat{\nabla}\widehat{\pi}_n \in L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5}).$$

According to estimates of the perturbed stokes system near boundary in [29], $\hat{v}$ is Hölder continuous in $Q^+_{1/2}$ with the exponent α. Then, by Hölder continuity of $\widehat{v}$ and strong convergence of the L^3-norm of $\widehat{v}_n$, we obtain

$$\widehat{A}(\widehat{v}_n,\theta)\to\widehat{A}(\widehat{v},\theta),\qquad \widehat{A}^{\frac{1}{3}}(\widehat{v},\theta)\le N_1\theta^{\alpha}, \tag{21}$$

where N_1 is an arbitrary constant.

Let $\overline{B}^+$ be a domain with smooth boundary such that $B^+_{4/5}\subset\overline{B}^+\subset B^+_{5/6}$, and $\overline{Q}^+:=\overline{B}^+\times(-(5/6)^2,0)$. Now we consider the following initial and boundary problem of $\overline{v}_n$, $\overline{\pi}_n$

$$\begin{array}{c}\partial_s\overline{v}_n-\widehat{\Delta}\overline{v}_n+\widehat{\nabla}\overline{\pi}_n=-\varepsilon_n r_n(\widehat{v}_n\cdot\widehat{\nabla})\widehat{v}_n-(\widehat{v}_n\cdot\widehat{\nabla})r_na_n+\widehat{g}_n\\ \widehat{\nabla}\cdot\overline{v}_n=0\end{array}\quad\text{in }\overline{Q}^+,$$

$$(\overline{v}_n)_{\overline{B}^+}(s)=0,\quad(\overline{\pi}_n)_{\overline{B}^+}(s)=0,\quad s\in\left(-\left(\frac{5}{6}\right)^2,0\right),$$

$$\overline{v}_{3,n}=0,\qquad\begin{array}{l}\partial_3\overline{v}_{1,n}=\varphi_{x_1}\partial_3\overline{v}_{3,n}\\ \partial_3\overline{v}_{2,n}=\varphi_{x_2}\partial_3\overline{v}_{3,n}\end{array}\quad\text{on }\partial\overline{B}^+\times\left[-\left(\frac{5}{6}\right)^2,0\right],$$

$$\overline{v}_n=0\quad\text{on }\overline{B}^+\times\left\{s=-\left(\frac{5}{6}\right)^2\right\}.$$

Using the global estimate of perturbed Stokes system (see [29, Lemma 3.1]), we get

$$\begin{aligned}&\|\partial_s\overline{v}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(\overline{Q}^+)}+\|\overline{v}_n\|_{L^{\frac{3}{2}}((-(5/6)^2,0);W_0^{2,\frac{9}{8}}(\overline{B}^+))}\\&+\|\overline{\pi}_n\|_{L^{\frac{3}{2}}((-(5/6)^2,0);W^{1,\frac{9}{8}}(\overline{B}^+))}\\&\le N\varepsilon_nr_n\left\|(v_n\cdot\widehat{\nabla})v_n\right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(\overline{Q}^+)}+N\left\|(v_n\cdot\widehat{\nabla})r_na_n\right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(\overline{Q}^+)}\\&+N\frac{{r_n}^2}{\varepsilon_n}\|\widehat{g}_n\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(\overline{Q}^+_{3/4})}\\&\le N(1+\varepsilon_nr_n+{r_n}^{\gamma-\beta}).\end{aligned}\tag{22}$$

Next, we define $\widetilde{v}_n=\widehat{v}_n-\overline{v}_n$, $\widetilde{\pi}_n=\widehat{\pi}_n-\overline{\pi}_n$. Then it is straightforward that $\widetilde{v}_n$ and $\widetilde{\pi}_n$ solve

$$\partial_s\widetilde{v}_n-\widehat{\Delta}\widetilde{v}_n+\widehat{\nabla}\widetilde{\pi}_n=0,\quad\text{div }\widetilde{v}_n=0\quad\text{in }Q^+_{\frac{4}{5}},$$

$$\widetilde{v}_{3,n}=0,\qquad\begin{array}{l}\partial_3\widetilde{v}_{1,n}=\varphi_{x_1}\partial_3\widetilde{v}_{3,n}\\ \partial_3\widetilde{v}_{2,n}=\varphi_{x_2}\partial_3\widetilde{v}_{3,n}\end{array}\quad\text{on }\big(B^+\cap\{x_3=0\}\big)\times\left[-\left(\frac{4}{5}\right)^2,0\right],$$

$$\left\|\widehat{\nabla}\widetilde{v}_n\right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5})}+\left\|\widehat{\nabla}\widetilde{\pi}_n\right\|_{L^{\frac{9}{8},\frac{3}{2}}_{y,s}(Q^+_{4/5})}\le N(1+\varepsilon_nr_n+{r_n}^{\gamma-\beta}),$$

and we obtain

$$\left\|\widehat{\nabla}\widetilde{\pi}_n\right\|_{L^{9,\frac{3}{2}}_{y,s}(Q^+_{3/4})}\le N(1+\varepsilon_nr_n+{r_n}^{\gamma-\beta}).$$

Next, let $\widehat{C}_1(\widetilde{\pi}_n,\theta)=\frac{1}{\theta}\left(\int_{-\theta^2}^0\left(\int_{B_\theta^+}|\widehat{\nabla}\widetilde{\pi}|^{\frac{9}{8}}dy\right)^{\frac{4}{3}}ds\right)^{\frac{2}{3}}$. By the Poincaré inequality, we have

$$\widehat{C}_a^{\frac{2}{3}}(\widehat{\pi}_n,\theta)\le N_2\left(\widehat{C}_1(\overline{\pi}_n,\theta)+\widehat{C}_1(\widetilde{\pi}_n,\theta)\right).$$

We note that $\widehat{C}_1(\bar{\pi}_n,\theta)$ goes to zero as $n\to\infty$ because of (22). On the other hand, using the Hölder inequality, we have

$$\widehat{C}_1\le\theta^2\left(\int_{-\theta^2}^0\left(\int_{B_\theta^+}\left|\widehat{\nabla}\widetilde{\pi}\right|^9dy\right)^{\frac{1}{6}}ds\right)^{\frac{2}{3}}\le N\theta^\alpha(1+\varepsilon_nr_n+{r_n}^{\gamma-\beta}).$$

Summing up, we obtain

$$\liminf_{n\to\infty}\widehat{C}_a^{\frac{2}{3}}(\widehat{\pi}_n,\theta)\le\lim_{n\to\infty}N_2\theta^\alpha(1+\varepsilon_nr_n+{r_n}^{\gamma-\beta})\le N_2\theta^\alpha. \tag{23}$$

Thus, we obtain from (10) that

$$N\theta^\alpha\le N_1\theta^\alpha+\liminf_{n\to\infty}\widehat{C}_a^{\frac{2}{3}}(\theta).$$

Consequently, if we take a constant N in (10) bigger than $2(N_1+N_2)$ in (21) and (23), this leads to a contradiction, since

$$2(N_1+N_2)\theta^\alpha\le N\theta^\alpha\le\liminf_{n\to\infty}\tau_n(\theta)\le(N_1+N_2)\theta^\alpha.$$

This deduces the lemma. □

Since Lemma 3.2 is the crucial part of the proof of Lemma 3.1, we present only a brief sketch of the streamline of Lemma 3.1.

Proof of Lemma 3.1. We note that due to Lemma 3.2 there exists a positive constant $\alpha<1$ such that

$$\widehat{A}^{\frac{1}{3}}(r)+\widehat{C}^{\frac{2}{3}}(r)<N\theta^\alpha\left(\widehat{A}^{\frac{1}{3}}(\rho)+\widehat{C}^{\frac{2}{3}}(\rho)+m_\gamma r^\beta\right),\qquad r<\rho<r_1,$$

where r_1 is the number in Lemma 3.1. For any $x\in B_{r_1/2}^+$ and for any $r<r_1/4$, let $\widehat{B}(r):=\widehat{A}^{\frac{1}{3}}(r)+\widehat{C}^{\frac{3}{2}}(r)$. By Lemma 3.2, we obtain

$$\widehat{B}(\theta r)\le N\theta^\alpha\widehat{B}(r)\le N\theta^{1+\alpha}\widehat{B}(r).$$

Thus, we have

$$\widehat{B}(\theta^kr)\le N\left(\theta^{1+\alpha}\right)^k\widehat{B}(r).$$

In case of $\rho=\theta^kr$, we get $\widehat{A}_a^{\frac{1}{3}}(\rho)\le\widehat{B}(\rho)\le N\rho^{1+\alpha}$. Next we consider the case that $\theta^kr<\rho<\theta^{k-1}r$. For the scaled L^3- norm of v,

$$\widehat{A}^{\frac{1}{3}}(\theta^kr)=\left(\frac{1}{(\theta^kr)^2}\int_{Q_{\theta^kr}^+}|v|^3\right)^{\frac{1}{3}}\le\theta^{-\frac{2}{3}}\left(\frac{1}{\rho^2}\int_{Q_\rho^+}|v|^3\right)^{\frac{1}{3}}=\theta^{-\frac{2}{3}}\widehat{A}^{\frac{1}{3}}(\rho).$$

In the same way, we get $\widehat{C}^{\frac{2}{3}}(\theta^k r) \le \theta^{-\frac{4}{3}}\widehat{C}^{\frac{2}{3}}(\rho)$ and therefore

$$\widehat{B}(\rho) \le \theta^{\frac{2}{3}}\widehat{B}(\theta^k r) \le N\theta^{\frac{2}{3}}\left(\theta^k\right)^{1+\alpha}\widehat{B}(r) \le N\theta^{\frac{2}{3}}\widehat{B}(r)\left(\frac{\rho}{r}\right)^{1+\alpha} \le N\rho^{1+\alpha}.$$

Thus, we can show that $\widehat{A}_a^{\frac{1}{3}}(r) \le Nr^{1+\alpha}$, where N is an absolute constant independent of v. Hölder continuity of v is a direct consequence of this estimate, which immediately implies that v is also Hölder continuous locally near boundary by the Morrey & Campanato lemma. This completes the proof. □

Next lemma is an estimate of the pressure.

Lemma 3.3. *Suppose* $0 < 2r \le \rho$. *Then*

$$\widehat{C}(r) \le N\left(\frac{\rho}{r}\right)\left(\widehat{A}_a(\rho) + \rho^{\frac{3}{2}(\gamma+1)}m_\gamma^{\frac{3}{2}}\right) + N\left(\frac{r}{\rho}\right)\widehat{C}(\rho). \tag{24}$$

Proof. Define $v^* = (v_1^*, v_2^*, v_3^*)$ by

$$v_1^*(x,t) = \begin{cases} v_1(x,t) & \text{if } x_3 \ge 0, \\ v_1(x^*,t) & \text{if } x_3 < 0, \end{cases}$$

$$v_2^*(x,t) = \begin{cases} v_2(x,t) & \text{if } x_3 \ge 0, \\ v_2(x^*,t) & \text{if } x_3 < 0, \end{cases}$$

$$v_3^*(x,t) = \begin{cases} v_3(x,t) & \text{if } x_3 \ge 0, \\ -v_3(x^*,t) & \text{if } x_3 < 0, \end{cases}$$

where $x^* = (x_1, x_2, -x_3) = (y_1, y_2, -y_3 + \varphi(y_1, y_2))$. We consider π^*, $-(v^* \cdot \widehat{\nabla})v^*$, g^* as the even-even-odd extension. Then, we construct (v^*, π^*) as the solution of the Stokes system in $\mathbb{R}^3 \times (0,T)$:

$$v_t^* - \widehat{\Delta}v^* + \widehat{\nabla}\pi^* = -(v \cdot \widehat{\nabla})v^* + g^* \tag{25}$$

with initial data $v^*(x,0) = v_0^*(x)$.

Let $\phi(x) \ge 0$ be standard cut-off function such that $0 \le \phi \le 1$, $\phi \equiv 1$ in B_ρ, $\phi = 0$ outside on $B_{\frac{\rho}{2}}$. The divergence ($:= \widehat{\nabla}$) of (25) gives in $\mathbb{R}^3 \times (0,T)$

$$-\widehat{\Delta}\pi^* = \widehat{\nabla} \cdot \widehat{\nabla}(v^* \otimes v^*) - \widehat{\nabla} \cdot g^*$$

in the sense of distribution. Let

$$\pi_1(x,t) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}\left\{\widehat{\nabla} \cdot \widehat{\nabla}\left[(v^* - (v^*)_\rho) \otimes (v^* - (v^*)_\rho)\right]\phi - \widehat{\nabla} \cdot (g^*\phi)\right\}(y,t)dy.$$

Then, by Calderon-Zygmund and potential estimates,

$$\frac{r}{\rho^3}\int_{B_\rho} |\pi_1|^{\frac{3}{2}}dx \le \frac{1}{r^2}\int_{B_\rho} |\pi_1|^{\frac{3}{2}}dx$$

$$\leq \frac{N}{r^2}\int_{B_\rho}|v^*-(v^*)_\rho|^3dx+\frac{N}{r^2}\rho^{\frac{9}{4}}\left(\int_{B_\rho}|g^*|^2dx\right)^{\frac{3}{4}}.$$

We set $\pi_2(x,t) := \pi^*(x,t)-\pi_1(x,t)$. It is direct that $\widehat{\Delta}\pi_2 = 0$, $\widehat{\nabla}\cdot v^* = 0$ in $B_{\frac{\rho}{2}}$ and thus we get

$$\begin{aligned}\frac{r}{r^2}\int_{B_r}|\pi_2|^{\frac{3}{2}}dx &\leq N\frac{r}{\rho^3}\int_{B_{\frac{\rho}{2}}}|\pi_2|^{\frac{3}{2}}dx\\ &\leq N\frac{r}{\rho^3}\int_{B_\rho}|\pi^*|^{\frac{3}{2}}dx+N\frac{r}{\rho^3}\int_{B_\rho}|\pi_1|^{\frac{3}{2}}dx.\end{aligned} \tag{26}$$

Integrating the first term of the right side in (26) in time, and using

$$\int_{-r^2}^{0}\frac{\rho^{\frac{9}{4}}}{r^2}\left(\int_{B_\rho}|g^*|^2dx\right)^{\frac{3}{4}}dt \leq Nr^{-\frac{3}{2}}\rho^{3+\frac{3\gamma}{2}}m_\gamma^{\frac{3}{2}},$$

we obtain

$$\begin{aligned}\frac{1}{r^2}\int_{Q_r}|\pi^*|^{\frac{3}{2}}dxdt &\leq \frac{1}{r^2}\int_{Q_r}|\pi_1|^{\frac{3}{2}}+|\pi_2|^{\frac{3}{2}}dxdt\\ &\leq N\left(\frac{\rho}{r}\right)^2\left(\int_{B_\rho}|v^*-(v^*)_\rho|^3dxdt+\rho^{\frac{3}{2}(\gamma+1)}m_\gamma^{\frac{3}{2}}\right)\\ &\quad +N\left(\frac{r}{\rho}\right)\int_{B_\rho}|\pi^*|^{\frac{3}{2}}dxdt.\end{aligned}$$

This completes the proof. □

We estimate the scaled L^3-norm of suitable weak solutions.

Lemma 3.4. *Under the same assumption as in Lemma 3.1. Let p,q be satisfied $\frac{3}{p}+\frac{2}{q}=2$ and $1\leq q<\infty$, there exists r_1 such that for any $r<r_1$*

$$\widehat{A}_a(r) \leq N\left(\widehat{D}(r)+\widehat{E}(r)\right)\widehat{K}(r). \tag{27}$$

Proof. Using the Hölder inequality, we obtain

$$\begin{aligned}&\int_{B_r^+}|v-(v)_r|^3\,dy\\ &\leq N\left(\int_{B_r^+}|v|^2dy\right)^{\frac{1}{q}}\left(\int_{B_r^+}|v-(v)_r|^6dy\right)^{\frac{1}{3}\left(1-\frac{1}{q}\right)}\left(\int_{B_r^+}|v|^pdy\right)^{\frac{1}{p}}\\ &\leq N\left(\int_{B_r^+}|v|^2dy\right)^{\frac{1}{q}}\left[\left(\int_{B_r^+}|\hat{\nabla}v|^2dy\right)^{1-\frac{1}{q}}\left(\int_{B_r^+}|v|^2dy\right)^{1-\frac{1}{q}}\right]\left(\int_{B_r^+}|v|^pdy\right)^{\frac{1}{p}}\\ &= N\left(\int_{B_r^+}|v|^2dy\right)^{\frac{1}{q}}\left(\int_{B_r^+}|\hat{\nabla}v|^2dy\right)^{1-\frac{1}{q}}\left(\int_{B_r^+}|v|^pdy\right)^{\frac{1}{p}}\end{aligned}$$

$$+ N\left(\int_{B_r^+} |v|^2 dy\right)\left(\int_{B_r^+} |v|^p dy\right)^{\frac{1}{p}},$$

where general Sobolev imbedding is used. Integrating in time, we get

$$\begin{aligned}
&\int_{S_r^+} |v-(v)_r|^3 dydt \\
&\le N\Big(\sup_{-r^2\le t\le 0}\int_{B_r^+}|v|^2dy\Big)^{\frac{1}{q}}\int_{-r^2}^0\Big(\int_{B_r^+}|\hat\nabla v|^2dy\Big)^{1-\frac{1}{q}}\Big(\int_{B_r^+}|v|^pdy\Big)^{\frac{1}{p}}dt \\
&\quad + N\Big(\sup_{-r^2\le t\le 0}\int_{B_r^+}|v|^2dy\Big)\int_{-r^2}^0\Big(\int_{B_r^+}|v|^pdy\Big)^{\frac{1}{p}}dt \\
&\le N\Big(\sup_{-r^2\le t\le 0}\int_{B_r^+}|v|^2dy\Big)^{\frac{1}{q}}\Big(\int_{Q_r^+}|\hat\nabla v|^2dydt\Big)^{1-\frac{1}{q}}\left(\int_{-r^2}^0\Big(\int_{B_r^+}|v|^pdy\Big)^{\frac{q}{p}}dt\right)^{\frac{1}{q}} \\
&\quad + N\Big(\sup_{-r^2\le t\le 0}\int_{B_r^+}|v|^2dy\Big)\left(\int_{-r^2}^0\Big(\int_{B_r^+}|v|^pdy\Big)^{\frac{q}{p}}dt\right)^{\frac{1}{q}},
\end{aligned}$$

where Hölder inequality is used. Dividing both sides by r^2, we have

$$\widehat{A}_a(r) \le N\left(\widehat{D}^{\frac{1}{q}}(r)\widehat{E}^{1-\frac{1}{q}}(r)\widehat{K}(r) + \widehat{D}(r)\widehat{K}(r)\right).$$

For the first term, applying Young's inequality, we deduce the lemma. □

Next we observe that for $0 < 2r \le \rho$

$$\widehat{A}(r) \le N\left(\frac{\rho}{r}\right)^2 \widehat{A}_a(\rho) + N\left(\frac{r}{\rho}\right)\widehat{A}(\rho). \tag{28}$$

Indeed, it is straightforward via the Hölder inequality that obtain

$$\widehat{A}(r) \le N\frac{1}{r^2}\int_{Q_r^+} |v-(v)_r|^3 + |(v)_r|^3 dyds \le N\left(\frac{\rho}{r}\right)^2 \widehat{A}_a(\rho) + N\left(\frac{r}{\rho}\right)\widehat{A}(\rho).$$

Remark 3.5. From local energy inequality (9), we obtain

$$\begin{aligned}
\widehat{D}\left(\frac{r}{2}\right) + \widehat{E}\left(\frac{r}{2}\right) &\le N\left(\widehat{A}^{\frac{2}{3}}(r) + \widehat{A}(r) + \widehat{A}^{\frac{1}{3}}(r)\widehat{C}(r) + r\int_{S_r^+}|g|^2dw\right), \\
&\le N\left(\widehat{A}^{\frac{2}{3}}(r) + \widehat{A}(r) + \widehat{A}(r)^{\frac{1}{3}}\widehat{C}(r) + r^{2\gamma+2}m_\gamma^2\right), \\
&\le N\left(1 + \widehat{A}(r) + \widehat{C}(r) + r^{2\gamma+2}m_\gamma^2\right).
\end{aligned} \tag{29}$$

Now we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $4r < \rho$. We consider $\widehat{A}(r)+\widehat{C}(r)$. Due to (28), (24), (27) and (29), we obtain

$$\widehat{A}(r)+\widehat{C}(r) \le N\left(\left(\frac{r}{\rho}\right)+\left(\frac{r}{\rho}\right)^2\widehat{K}(\rho)\right)(\widehat{A}(\rho)+\widehat{C}(\rho)) + N\left(\frac{r}{\rho}\right)^2(1+\rho^{2\gamma+2}m_\gamma^2)\widehat{K}(\rho)+N\left(\frac{r}{\rho}\right)^2\rho^{\frac{3}{2}(\gamma+1)}m_\gamma^{\frac{3}{2}}.$$

We choose $\theta \in (0,1/4)$ such that $C\theta < 1/4$ where N is an absolute constant in the above inequality. Now we fix $r_0 < \min\left\{1, \frac{1}{m_\gamma}, \frac{1}{m_\gamma}(\frac{\varepsilon\theta^2}{8C})^{2/3}\right\}^{-(\gamma+1)}$ such that $\widehat{K}(r) < \frac{\theta^2}{1+8C}\min\{1,\varepsilon\}$ for all $r \le r_0$. By replacing r, ρ by θr and r, respectively, we obtain

$$\widehat{A}(\theta r)+\widehat{C}(\theta r) \le \frac{1}{2}\left(\widehat{A}(r)+\widehat{C}(r)\right)+\frac{\varepsilon}{4}, \quad \forall r \le r_0.$$

By iterating, we have

$$\widehat{A}(\theta^k r)+\widehat{C}(\theta^k r) \le \left(\frac{1}{2}\right)^k\left(\widehat{A}(r)+\widehat{C}(r)\right)+\frac{\varepsilon}{2}, \quad \forall r \le r_0.$$

Thus, for k sufficiently large, $\widehat{A}(\theta^k r)+\widehat{C}(\theta^k r) \le \epsilon$. By Lemma 3.1, this completes the proof. □

4. Appendix

In this section, we provide the existence of suitable weak solutions and Stokes estimates of the Stokes system with slip boundary conditions.

4.1. Existence of suitable weak solutions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $I = (0,T)$. We consider the Stokes system with Slip boundary conditions:

$$\text{(30)} \qquad \begin{cases} u_t - \Delta u + \nabla \mathrm{p} = f - (w\cdot\nabla)v, \quad \operatorname{div} \mathrm{u} = 0 & \text{in } \ Q_T = \Omega\times I, \\ u\cdot n = 0, \quad n\cdot T(u,\mathrm{p})\cdot\tau = 0 & \text{on } \ \partial\Omega\times I, \\ u = u_0 & \text{at } \ t=0, \end{cases}$$

where $w \in C^\infty(Q_T)$, $f \in L^2(Q_T)$ and $u_0 \in H^2(\Omega)$, $v \in W^{2,1}_{2,2}(Q_T) = L^2(I : H^2(\Omega)) \cap H^1(I : L^2(\Omega))$. The Banach space $L^2(\Omega)^3$ admits the Helmholtz decomposition:

$$L^2(\Omega)^3 = J^2(\Omega)\oplus G^2(\Omega),$$

where

$$J^2(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{L^2(\Omega)}, \quad G^2(\Omega) = \{\nabla \mathrm{p} \,|\, \mathrm{p}\in \hat{W}^{1,2}(\Omega)\},$$
$$C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^3 \,|\, \nabla\cdot u = 0 \text{ in } \Omega\},$$
$$\hat{W}^{1,2}(\Omega) = \{\mathrm{p}\in L^2_{loc}(\bar{\Omega}) \,|\, \nabla \mathrm{p} \in L^2(\Omega)^3\}.$$

It should be noted that since boundary is $C^{2,1}$-hypersurface, $J^2(\Omega)$ is characterized as

$$J^2(\Omega) = \{u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Let P be a continuous projection from $L^2(\Omega)^3$ onto $J^2(\Omega)$ along $G^2(\Omega)$. By using P we shall define the Stokes operator with slip boundary conditions A by

$$Au = -P\Delta u \quad \text{for } u \in D(A),$$
$$D(A) = J^2(\Omega) \cap \{u \in W^{2,2}(\Omega)^3 \mid n \cdot T(u, \mathrm{p}) \cdot \tau = 0\}.$$

Now, we consider operator form of system:

$$u_t + Au = P(f - (w \cdot \nabla)v), \quad u(0) = u_0. \tag{31}$$

Since A is the generator of an analytic semigroup in $L^2_\sigma(\Omega)$, solving (31) is equivalent to show that mapping

$$F(v) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(f - (w \cdot \nabla)v)ds$$

has a unique fixed point.

Lemma 4.1. *Let $T \in (0, \infty)$. There exists a unique solution*

$$u \in L^2((0, \infty) : H^2(\Omega)) \cap H^1((0, \infty) : L^2(\Omega))$$

satisfies

$$u_t + Au = P(f - (w \cdot \nabla)u), \quad u(0) = u_0.$$

Proof. Let F is mapping such that $F(v) = u$. Then

$$\begin{aligned}\|u\|_{W^{2,1}_{2,2}(Q_T)} &= \|F(v)\|_{W^{2,1}_{2,2}(Q_T)} \\ &\le N\left\{\|u_0\|_{W^{2,1}_{2,2}(Q_T)} + \|f - (w \cdot \nabla)v\|_{L^2(Q_T)}\right\} \\ &\le N\left\{\|u_0\|_{W^{2,1}_{2,2}(Q_T)} + \|f\|_{L^2(Q_T)} + \|w\|_{L^\infty(Q_T)}\|\nabla v\|_{L^2(Q_T)}\right\}.\end{aligned}$$

Thus, F is well-defined on $W^{2,1}_{2,2}(Q_T)$. For $v_1, v_2 \in W^{2,1}_{2,2}(Q_T)$,

$$\begin{aligned}\|F(v_1) - F(v_2)\|_{H^2(\Omega)} &\le \int_0^t \left\|\nabla e^{-(t-s)A}\nabla P((w \cdot \nabla)(v_2 - v_1))\right\|_{L^2(\Omega)} ds \\ &\le \int_0^t N(t-s)^{-\frac{1}{2}} \|\nabla P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(\Omega)} ds \\ &= Nt^{-\frac{1}{2}} * \|\nabla P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(\Omega)}.\end{aligned}$$

Taking integral on $[0, t]$ for small t,

$$\begin{aligned}&\|F(v_1) - F(v_2)\|_{L^2(0,t:H^2(\Omega))} \\ &\le N\left\|t^{-\frac{1}{2}} * \|\nabla P\left((w \cdot \nabla)(v_2 - v_1)\right)\|_{L^2(\Omega)}\right\|_{L^2(0,t)} \\ &\le N\left\|t^{-\frac{1}{2}}\right\|_{L^1(0,t)} \|\nabla P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(0,t:L^2(\Omega))}\end{aligned}$$

$$\leq N\sqrt{t}\, \|v_2 - v_1\|_{L^2(0,t:H^2(\Omega))}\,.$$

We also note that

$$(F(v_1) - F(v_2))_t = P\left((w\cdot\nabla)(v_2 - v_1)\right) - \int_0^t A^{\frac{1}{2}} e^{-(t-s)A} A^{\frac{1}{2}} P\left((w\cdot\nabla)(v_2 - v_1)\right) ds,$$

and thus, taking L^2-norm, we have

$$\|(F(v_1) - F(v_2))_t\|_{L^2(\Omega)} \leq \|P((w\cdot\nabla)(v_2 - v_1))\|_{L^2(\Omega)} + \int_0^t C(t-s)^{-\frac{1}{2}} \|\nabla P((w\cdot\nabla)(v_2 - v_1))\|_{L^2(\Omega)} ds.$$

Similarly taking integral on $[0,t]$ for small t,

$$\|F(v_1) - F(v_2)\|_{H^1(0,t:L^2(\Omega))} \leq N\sqrt{t}\|v_2 - v_1\|_{L^2(0,t:H^2(\Omega))}.$$

Therefore,

$$\|F(v_1) - F(v_2)\|_{W^{2,1}_{2,2}(Q_T)} \leq N\sqrt{t}\|v_2 - v_1\|_{W^{2,1}_{2,2}(Q_T)}.$$

Hence, the contraction mapping principle then yields a unique solution $u \in W^{2,1}_{2,2}(Q_T)$ for small $T > 0$.

Next, let $T^* < \infty$ be a maximal time. For $T < T^*$, a solution $u \in W^{2,1}_{2,2}(Q_T)$ of

$$u_t + Au = P(f - (w\cdot\nabla)u), \quad u(0) = u_0$$

satisfies the following inequality:

$$\begin{aligned} &\|u_t\|_{L^2(0,T:L^2(\Omega))} + \|\nabla^2 u\|_{L^2(0,T:L^2(\Omega))} \\ &\leq N\left(\|f\|_{L^2(0,T:L^2(\Omega))} + \|(w\cdot\nabla)u\|_{L^2(0,T:L^2(\Omega))} + \|u_0\|_{W^{2,1}_{2,2}(Q_T)}\right). \end{aligned} \tag{32}$$

Let $T \to T^*$. Then, left-hand side of (32) is infinity. But, since $\|(w\cdot\nabla)u\|_{L^2(0,T:L^2(\Omega))} \leq \|w\|_{L^\infty(Q_T)}\|f\|_{L^2(0,T:L^2(\Omega))}$, right-hand side of (32) is uniformly finite. Thus, the contraction mapping principle then yields a unique solution $u \in W^{2,1}_{2,2}(Q_T)$ for all time. □

For fixed $T > 0$, we consider a suitable weak solution u to Navier-Stokes equations:

$$u_t - \Delta u + (u\cdot\nabla)u + \nabla \mathrm{p} = f, \quad \nabla\cdot u = 0 \tag{33}$$

in Q_T with the initial condition $u(x,0) = u_0 \in L^2$ satisfying $\nabla\cdot u_0 = 0$ in a weak sense. For the existence we follow the steps in [4]. For fixed $N > 0$, we set $\delta = T/N$. Then we find a sequences (u_N, p_N) such that

$$u_N \in C(0,T;J^2(\Omega)) \cap L^2(0,T;J(\Omega)),$$

$$\partial_t u_N + \Psi_\delta(u_N)\cdot\nabla u_N - \Delta u_N + \nabla \mathrm{p}_N = f,$$

$$\nabla\cdot u_N = 0, \quad u_N(0) = u_0.$$

Here, the *retarded mollifier* Ψ_δ is defined by

$$\Psi_\delta(v)(x,t) \equiv \delta^{-4} \iint_{\mathbb{R}^4} \psi\left(\frac{y}{\delta}, \frac{\tau}{\delta}\right) v^*(x-y, t-\tau) dydt,$$

where $\psi(x,t) \in C^\infty$ satisfies

$$\psi \geqslant 0, \iint \psi dxdt = 1, \quad \text{and} \quad \text{supp}\psi \subset \left\{(x,t) : |x|^2 < t, 1 < t < 2\right\},$$

and $v^* : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is defined by

$$v^*(x,t) = \begin{cases} v(x,t) & \text{if } (x,t) \in \Omega \times \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

The values of $\Psi_\delta(v)$ at time t clearly depend only on the values of v at times $\tau \in (t-2\delta, t-\delta)$. For $v \in L^\infty(0,T; J^2(\Omega)) \cap L^2(0,T; J(\Omega))$, it is clear that

$$\nabla \cdot \Psi_\delta(v) = 0 \quad \text{a.e.} \quad x \in \Omega,$$

$$\sup_{0 \leqslant t \leqslant T} \int_\Omega |\Psi_\delta(v)|^2(x,t) dx \leqslant N \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v|^2 dx,$$

$$\int_\Omega |\nabla \Psi_\delta(v)|^2 dx \leqslant N \int_\Omega |\nabla v|^2 dx.$$

Such (u_N, p_N) exist by Lemma 4.1 inductively on each time interval $(m\delta, (m+1)\delta)$, $0 \leqslant m \leqslant N-1$.

By $\frac{d}{dt}\int_\Omega |u|^2 dx = 2\int_\Omega (u_t, u) dx$, we have

$$\int_{\Omega \times \{t\}} |u_N|^2 dxds + 2\int_0^t \int_\Omega |\nabla u_N|^2 dxds = \int_\Omega |u_0|^2 dx + 2\int_0^t \int_\Omega f \cdot u_N dxds$$

for $0 < t < T$. Therefore, we have

$$\int_{\Omega \times \{t\}} |u_N|^2 dxds + \int_0^t \int_\Omega |\nabla u_N|^2 dxds \leq \int_\Omega |u_0|^2 dx + \int_0^t \|f\|^2_{H^{-1}} d\tau ds.$$

In particular,

$$u_N \text{ stays bounded in } L^\infty(0,T; L^2) \cap L^2(0,T; H^1),$$

$$\frac{d}{dt} u_N \text{ stays bounded in } L^2(0,T; H_0^{-2})$$

and hence,

$$\{u_N\} \text{ stays bounded in } L^2(Q_T).$$

From Stokes estimate,

$$\{\mathrm{p}_N\} \text{ stays bounded in } L^{\frac{5}{3}}(Q_T).$$

Thus, there exist their limits $(u_\star, p_\star)$ such that

$$u_N \to u_\star \begin{cases} \text{Strongly in } L^q(Q_T), \quad 2 \le q < \frac{10}{3}, \\ \text{weakly in } L^2(0,T;J(\Omega)), \\ \text{weak-star in } L^\infty(0,T;J^2(\Omega)), \end{cases}$$

$$\mathrm{p}_N \to \mathrm{p}_\star \quad \text{weakly in } L^{\frac{5}{3}}(0,T;J(\Omega)).$$

We note that $(u_\star, p_\star)$ is a suitable weak solution of the Navier-Stokes equations (33). The remaining parts of the proof are similar to that of [4].

4.2. Stokes estimates

Here we sketch the local boundary estimate for the Stokes system with slip boundary conditions in [31]. Let,

$$\begin{aligned} \langle D_t\rangle^{1/2}u(t) &= \mathscr{F}_\xi^{-1}[(1+s^2)^{\frac{1}{4}}\mathscr{F}_\xi u(s)](t), \\ H_q^{1/2}(\mathbb{R},X) &= \{u \in L_q(\mathbb{R},X) | \langle D_t\rangle^{1/2}u(t) \in L_q(\mathbb{R},X)\}, \\ \|u\|_{H_q^{1/2}(\mathbb{R},X)} &= \|u\|_{L_q(\mathbb{R},X)} + \|\langle D_t\rangle^{1/2}u\|_{L_q(\mathbb{R},X)}, \end{aligned}$$

where $\mathscr{F}_\xi$ and $\mathscr{F}_\xi^{-1}$ denote the Fourier transform and its inverse formula, respectively. Set

$$\begin{aligned} H_{p,q}^{1,1/2}(\mathbb{R}_+^3 \times \mathbb{R}_+) &= H_q^{1/2}(\mathbb{R}_+, L_p(\mathbb{R}_+^3)) \cap L_q(\mathbb{R}_+, W_p^1(\mathbb{R}_+^3)), \\ \|u\|_{H_{p,q}^{1,1/2}(\mathbb{R}_+^3\times\mathbb{R}_+)} &= \|u\|_{H_q^{1/2}(\mathbb{R}_+,L_p(\mathbb{R}_+^3))} + \|u\|_{L_q(\mathbb{R}_+,W_p^1(\mathbb{R}_+^3))}. \end{aligned}$$

Moreover,

$$\begin{aligned} \widehat{W}_q^1(x) &= \left\{u \in W_q^1(X) | \int_X u(x)dx = 0\right\}, \\ \widehat{W}_q^{-1}(x) &= [\widehat{W}_{q'}^1(x)]^*, \quad q' = q/q-1, \quad 1 < q < \infty, \\ \|u\|_{\widehat{W}_q^{-1}(x)} &= \sup_{0\neq v\in \widehat{W}_{q'}^1(x)} \frac{|[u,v]|}{\|\nabla v\|_{L_{q'}(X)}}, \end{aligned}$$

where $[\cdot,\cdot]$ denotes the duality of $\widehat{W}_q^{-1}(x)$ and $\widehat{W}_{q'}^1(x)$.

Lemma 4.2. *Let $1 < p,q < \infty$, $2r < \rho$ and $Q_\rho^+ = B_\rho^+ \times (-\rho^2, 0)$. Suppose that $v \in L_t^q W_x^{2,p}(Q_\rho^+)$, $v_t \in L_t^q L_x^p(Q_\rho^+)$ and $\pi \in L_t^q W_x^{1,p}(Q_\rho^+)$ such that (v,π) solves the following Stokes system:*

$$(34) \quad \begin{cases} v_t - \widehat{\Delta}v + \widehat{\nabla}\pi = g, \quad \widehat{\nabla}\cdot v = 0 & \text{in } Q_\rho^+, \\ v_3 = 0, \quad \partial_3 v_1 = \varphi_{x_1}\partial_3 v_3, \quad \partial_3 v_2 = \varphi_{x_2}\partial_3 v_3 & \text{on } Q_\rho \cap \{x_3 = 0\}, \end{cases}$$

where φ is given in Assumption 2.1, and $\widehat{\Delta}$, $\widehat{\nabla}$ are differential operators in Section 2. Then (v,π) satisfies

$$\begin{aligned} &\|v_t\|_{L^{p,q}(Q_r^+)} + \|v\|_{L^q((-r^2,0),W_p^2(B_r^+))} + \|\nabla\pi\|_{L^{p,q}(Q_r^+)} \\ &\le N\left(\|g\|_{L^{p,q}(Q_\rho^+)} + \|v\|_{L^{p,q}(Q_\rho^+)} + \|\nabla v\|_{L^{p,q}(Q_\rho^+)} + \|\pi\|_{L^{p,q}(Q_\rho^+)}\right). \end{aligned}$$

Proof. Let ξ be a standard cut-off function satisfying:

$$\begin{aligned}&\xi \in C_0^\infty(\mathbb{R}^3), \quad 0 \le \xi \le 1 \text{ in } \mathbb{R}^3,\\ &\xi \equiv 1 \text{ in } B_r, \quad \xi = 0 \text{ outside on } B_\rho,\\ &|\widehat{\nabla}\xi| < \frac{c}{\rho - r}, \quad |\widehat{\nabla}^2\xi| < \frac{c}{(\rho-r)^2}.\end{aligned}$$

Take $\nu = v\xi$, $\Pi = \pi\xi$. Then,

$$\begin{cases} \nu_t + \nu - \widehat{\Delta}\nu + \widehat{\nabla}\Pi = G, \quad \widehat{\nabla}\cdot\nu = d & \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+,\\ \nu_3 = 0, \quad \partial_3\nu_1 = h_1, \quad \partial_3\nu_2 = h_2 & \text{on } \partial\mathbb{R}^3_+ \times \mathbb{R}_+, \end{cases} \tag{35}$$

where

$$\begin{aligned}&G = \nu - 2\widehat{\nabla}v\widehat{\nabla}\xi - v\widehat{\Delta}\xi + \pi\widehat{\nabla}\xi + \xi g, \quad d = v\cdot\widehat{\nabla}\xi,\\ &h_1 = v_1\partial_3\xi + \xi\varphi_{x_1}\partial_3 v_3, \quad h_2 = v_2\partial_3\xi + \xi\varphi_{x_2}\partial_3 v_3.\end{aligned}$$

Then (35) can be expressed:

$$\begin{cases} \nu_t + \nu - \Delta\nu + \nabla\Pi = G^*, \quad \nabla\cdot\nu = d^* & \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+,\\ \nu_3 = 0, \quad \partial_3\nu_1 = h_1, \quad \partial_3\nu_2 = h_2 & \text{on } \partial\mathbb{R}^3_+ \times \mathbb{R}_+, \end{cases} \tag{36}$$

where

$$\begin{aligned}&G^* = G + \Delta'\nu - \nabla'\Pi, \quad d^* = d - \nabla'\nu,\\ &\Delta' = \widehat{\Delta} - \Delta = -2\varphi_{x_1}\partial_{x_1x_3} - 2\varphi_{x_2}\partial_{x_2x_3} + (\varphi_{x_1})^2\partial_{x_3x_3}\\ &\qquad + (\varphi_{x_2})^2\partial_{x_3x_3} - \varphi_{x_1x_1}\partial_{x_3} - \varphi_{x_2x_2}\partial_{x_3},\\ &\nabla' = \widehat{\nabla} - \nabla = (-\varphi_{x_1}\partial_{x_3}, -\varphi_{x_2}\partial_{x_3}, 0).\end{aligned}$$

Using the maximal estimate for Stokes system with slip boundary [31, Theorem 5.1], we get

$$\begin{aligned}&\|\nu_t\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\nu\|_{L^q(\mathbb{R}_+,W^2_p(\mathbb{R}^3_+))} + \|\nabla\Pi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\\ &\le N\bigg(\|G^*\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|d^*\|_{L^q(\mathbb{R}_+,\hat{W}^{-1}_p(\mathbb{R}^3_+))} + \|d^*_t\|_{L^q(\mathbb{R}_+,\hat{W}^{-1}_p(\mathbb{R}^3_+))}\\ &\qquad + \|d^*\|_{L^q(\mathbb{R}_+,W^1_p(\mathbb{R}^3_+))} + \|h\|_{H^{1,1/2}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\bigg).\end{aligned}$$

Then, the following estimates hold:

$$\begin{aligned}&\|\Delta'\nu\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\epsilon\|\nu\|_{L^q(\mathbb{R}_+,W^2_p(\mathbb{R}^3_+))},\\ &\|\nabla'\Pi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le \epsilon\|\nabla\Pi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}.\end{aligned}$$

Thus, choosing ϵ small enough, we have

$$\begin{aligned}&\|\nu_t\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\nu\|_{L^q(\mathbb{R}_+,W^2_p(\mathbb{R}^3_+))} + \|\nabla\Pi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\\ &\le N\bigg(\|G\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|d\|_{L^q(\mathbb{R}_+,\hat{W}^{-1}_p(\mathbb{R}^3_+))} + \|d_t\|_{L^q(\mathbb{R}_+,\hat{W}^{-1}_p(\mathbb{R}^3_+))}\\ &\qquad + \|d\|_{L^q(\mathbb{R}_+,W^1_p(\mathbb{R}^3_+))} + \|h\|_{H^{1,1/2}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\bigg).\end{aligned}$$

From [31], we get following estimate:

$$\begin{aligned}
&\|d\|_{L^q(\mathbb{R}_+,\hat{W}_p^{-1}(\mathbb{R}^3_+))} + \|d_t\|_{L^q(\mathbb{R}_+,\hat{W}_p^{-1}(\mathbb{R}^3_+))} \\
&\le N\left(\left\|v\cdot\widehat{\nabla}\xi\right\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \left\|v_t\cdot\widehat{\nabla}\xi\right\|_{L^q(\mathbb{R}_+,W_p^{-1}(\mathbb{R}^3_+))}\right) \\
&\le N\biggl(\left\|v\cdot\widehat{\nabla}\xi\right\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \varepsilon\left\|\nabla^2(v\cdot\widehat{\nabla}\xi)\right\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\qquad + \left\|v\cdot\widehat{\nabla}\xi\right\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} + \left\|g\cdot\widehat{\nabla}\xi\right\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\biggr),
\end{aligned}$$

$$\begin{aligned}
&\|h\|_{H^{1,1/2}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\le N\left(\|v\nabla\xi\|_{H^{1,1/2}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\xi\nabla\varphi\nabla v\|_{H^{1,1/2}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\right) \\
&\le N\biggl(\|v\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\langle D_t\rangle^{\frac{1}{2}}(v\nabla\xi)\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|v\nabla\xi\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} \\
&\qquad + \|\xi\nabla\varphi\nabla v\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \varepsilon_0\|\langle D_t\rangle^{\frac{1}{2}}(\xi\nabla\varphi\nabla v)\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\qquad + \varepsilon_0\|\xi\nabla\varphi\nabla v\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))}\biggr) \\
&\le N\biggl(\|v\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|v\nabla\xi\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} + \|\xi\nabla\varphi\nabla v\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\qquad + \varepsilon_0\|\xi\nabla\varphi\nabla v\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} + R^{-\frac{1}{2}}\|v_t\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\qquad + R^{\frac{1}{2}}\|v\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \varepsilon_0\|\xi v\|_{W^{2,1}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\biggr).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\|v_t\xi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} - R^{-\frac{1}{2}}\|v_t\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|v\xi\|_{L^q(\mathbb{R}_+,W_p^2(\mathbb{R}^3_+))} \\
&- \varepsilon\|\nabla^2(v\cdot\widehat{\nabla}\xi)\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|\nabla(\pi\xi)\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&- \varepsilon_0\|\xi v\|_{W^{2,1}_{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} - \varepsilon_0\|\xi\nabla\varphi\nabla v\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} \\
&\le N\biggl(\|G\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|v\cdot\widehat{\nabla}\xi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|g\cdot\widehat{\nabla}\xi\|_{L^{p,q}(\mathbb{R}^3_+\times\mathbb{R}_+)} \\
&\qquad + \|v\cdot\widehat{\nabla}\xi\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} + \|v\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + \|v\nabla\xi\|_{L^q(\mathbb{R}_+,W_p^1(\mathbb{R}^3_+))} \\
&\qquad + \|\xi\nabla\varphi\nabla v\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)} + R^{\frac{1}{2}}\|v\nabla\xi\|_{L^{p\cdot q}(\mathbb{R}^3_+\times\mathbb{R}_+)}\biggr).
\end{aligned}$$

Therefore, choosing R large enough, ε and ε_0 small enough, recalling that $\xi\equiv 1$ on B_r and $\xi=0$ outside on B_ρ, and $G=\nu-2\widehat{\nabla}v\widehat{\nabla}\xi - v\widehat{\Delta}\xi+\pi\widehat{\nabla}\xi+\xi g$,

$$\begin{aligned}
&\|v_t\|_{L^{p,q}(Q_r^+)} + \|v\|_{L^q((-r^2,0),W_p^2(B_r^+))} + \|\nabla\pi\|_{L^{p,q}(Q_r^+)} \\
&\le N\left(\|g\|_{L^{p,q}(Q_\rho^+)} + \|v\|_{L^{p,q}(Q_\rho^+)} + \|\nabla v\|_{L^{p,q}(Q_\rho^+)} + \|\pi\|_{L^{p,q}(Q_\rho^+)}\right)
\end{aligned}$$

holds. □

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