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AUTOMORPHISMS OF THE ZERO-DIVISOR GRAPH OVER 2×2 MATRICES

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Abstract. The zero-divisor graph of a noncommutative ring R, denoted by $\Gamma(R)$, is a graph whose vertices are nonzero zero-divisors of R, and there is a directed edge from a vertex x to a distinct vertex y if and only if $xy = 0$. Let $R = M_2(F_q)$ be the 2×2 matrix ring over a finite field F_q. In this article, we investigate the automorphism group of $\Gamma(R)$.

1. Introduction

Zero-divisor graphs have received a lot of attention (see [1, 2, 4, 6, 13]), because they are helpful for revealing the ring-theoretic properties via their graph-theoretic properties. In 1988, I. Beck first introduced the concept of zero-divisor graphs of commutative rings in [7], where all elements of a commutative ring R are defined to be vertices and distinct vertices x and y are adjacent if and only if $xy = 0$. In such a graph, the vertex 0 is adjacent to every other vertex, and non-zero-divisors are adjacent only to 0. In order to better illustrate the zero-divisor structure of a ring, D. F. Anderson and P. S. Livingston [5] redefined the notion of a zero-divisor graph by cutting off 0 and non-zero-divisors from the graph. Let R be a commutative ring (with 1) and let $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph $\Gamma(R)$ of R (defined by D. F. Anderson and P. S. Livingston [5]) is a graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and distinct vertices x and y are adjacent if and only if $xy = 0$. S. P. Redmond [16] further extended the concept of a zero-divisor graph to a noncommutative ring R, also written as $\Gamma(R)$, by taking the vertex set to be $Z(R)^*$, and there is a directed edge from a vertex

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x to a distinct vertex y if and only if $xy = 0$. In [9] and [10] F. DeMeyer et al. introduced the definition of zero-divisor graphs over arbitrary semigroups.

Automorphisms of graphs play an important role both in graph theory and in algebra, and characterization of the full automorphisms of a graph is often a difficult work. Until now, little is known (as far as we know) about the automorphisms of zero-divisor graphs. Now we list some known results related to ours. In 1999, D. F. Anderson and P. S. Livingston [5] proved that if $n \geq 4$ is a non-prime integer, then $\mathrm{Aut}(\Gamma(\mathbb{Z}_n))$ is a direct product of some symmetric groups. In 2002, S. B. Mulay [14] established some group-theoretic properties of the group of graph-automorphisms of zero-divisor graphs over commutative rings. In 2002, F. DeMeyer and K. Schneider [11] studied the relationship between $\mathrm{Aut}(R)$ and $\mathrm{Aut}(\Gamma(R))$ when R is a commutative ring. In 2008, J. Han (see [12, Theorem 3.9]) showed that $\mathrm{Aut}(\Gamma(M_2(\mathbb{Z}_p)))$ is isomorphic to the symmetric group S_{p+1} of degree $p+1$ when p is a prime. In 2011, S. Park and J. Han [15] generalized [12, Theorem 3.9] to an arbitrary finite field F_q with q elements, and proved that $\mathrm{Aut}(\Gamma(M_2(F_q))) \cong S_{q+1}$. When reading the proof of [15, Theorem 3.8], we find some major mistakes there (the automorphism σ they constructed in the proof fails to be a bijection), which inspires us to determine the full automorphisms of $\Gamma(M_2(F_q))$ again. As an application of our main theorem, we show that [12, Theorem 3.9] and [15, Theorem 3.8] are both wrong.

This article is organized as follows. In Section 2, we give some preliminary results and introduce the compressed zero-divisor graph $\Gamma_E(M_2(F_q))$. In Section 3, we determine the automorphisms of $\Gamma_E(M_2(F_q))$. Section 4 is devoted to investigate the automorphism group of $\Gamma(M_2(F_q))$ via what has been obtained in Section 3.

2. Preliminaries and notations

Let Γ be a directed graph with vertex set $V(\Gamma)$. We write $x \to y$ to mean that there is a directed edge from a vertex x to a distinct vertex y, and write $x \leftrightarrow y$ to mean $x \to y$ and $y \to x$. A graph H is called a *subgraph* of Γ if $V(H) \subseteq V(\Gamma)$ and for any $x, y \in V(H)$, $x \to y$ in Γ whenever $x \to y$ in H. Further, H is called an *induced subgraph* of Γ if for any $x, y \in V(H)$, $x \to y$ in Γ if and only if $x \to y$ in H. An induced subgraph K of Γ is called a *clique* if $x \leftrightarrow y$ for any distinct vertices x and y in K. A set of vertices that induces a subgraph with no edges is called an *independent set*. For any vertex x of Γ, $N_l(x) = \{y \in V(\Gamma) \,|\, y \to x\}$ and $N_r(x) = \{y \in V(\Gamma) \,|\, x \to y\}$ are respectively called the *left neighborhood* and the *right neighborhood* of x. For any set X, we denote by $|X|$ the cardinality of X. $|N_l(x)|$ is called the *in-degree* of x, and $|N_r(x)|$ is called the *out-degree* of x. A bijection σ on $V(\Gamma)$ is said to be an *automorphism* of Γ if for any two vertices $x, y \in V(\Gamma)$, $x \to y$ if and only if $\sigma(x) \to \sigma(y)$. Denote by $\mathrm{Aut}(\Gamma)$ the automorphism group of Γ.

Hereafter, R will always denote $M_2(F_q)$, the 2×2 matrix ring over a finite field F_q with $q \geq 2$ elements. By $M_{1\times 2}(F_q)$ we mean the set of all 1×2 matrices over F_q, and by α^t we mean the transpose of $\alpha \in M_{1\times 2}(F_q)$. Let $M^1_{1\times 2}(F_q)$ be a subset of $M_{1\times 2}(F_q)$ consisting of the vectors whose first nonzero component is 1, i.e., $M^1_{1\times 2}(F_q) = \{(0\ 1), (1\ a) \,|\, a \in F_q\}$. By $Z(R)$ we denote the set of all zero-divisors of R, and by $U(R)$ we denote the set of all units of R. Then $R = Z(R) \cup U(R)$. For every matrix $A \in R$, let $\det(A)$ be the determinant of A, and let $r(A)$ be the rank of A. In R, the matrix unit who has 1 in the (i, j) position and 0 elsewhere is denoted by E_{ij}, the zero matrix is denoted by $\mathbf{0}$, and the identity matrix is denoted by I. For any subset X of F_q (resp., $R; M_{1\times 2}(F_q)$), let $X^* = X - \{0\}$. If X is either an element or a subset of R, then the *left annihilator* of X is $ann_l X = \{A \in R \,|\, AX = \mathbf{0}\}$ and the *right annihilator* of X, denoted by $ann_r X$, is similarly defined. As usual, $\Gamma(R)$ denotes the zero-divisor graph over R, i.e., $\Gamma(R)$ is a graph with vertex set $Z(R)^*$, and $A \to B$ if and only if $A \neq B$ and $AB = \mathbf{0}$. By S_{F_q} we mean the symmetric group over F_q, and $S_{F_q^*}$ is similarly defined.

Lemma 2.1. *The following three conditions are equivalent*:
(i) $A \in Z(R)^*$;
(ii) $r(A) = 1$;
(iii) $A = \alpha^t\beta$ *for* $\alpha, \beta \in M_{1\times 2}(F_q)^*$.

Proof. Note that $A \in Z(R)$ if and only if $\det(A) = 0$. Thus $A \in Z(R)^*$ if and only if $r(A) = 1$, since A is a 2×2 matrix. This yields (i)⇔(ii). Clearly, we have (ii)⇔(iii). □

From the lemma above, each vertex A in $\Gamma(R)$ can be written as $A = \alpha^t\beta$ for some $\alpha, \beta \in M_{1\times 2}(F_q)^*$. The set of nonzero scalar multiples of $A \in R$ is denoted by $[A]$, i.e., $[A] = \{aA \,|\, a \in F^*\}$. Then the multiplication $[A][B] = [AB]$ is well-defined, and $[A] = [B]$ if and only if $B = aA$ for some $a \in F_q^*$. If $A \in V(\Gamma(R))$, then there exist unique $\alpha, \beta \in M^1_{1\times 2}(F_q)$ such that $[A] = [\alpha^t\beta]$. We call $\alpha^t\beta$ the *standard representation* of $[A]$.

Lemma 2.2. *If* $\alpha \in M^1_{1\times 2}(F_q)$*, then there exists a unique* $\beta \in M^1_{1\times 2}(F_q)$ *such that* $\alpha\beta^t = 0$*. Meanwhile* $\beta\alpha^t = 0$*, and*
(i) *if* $\alpha = (0\ 1)$*, then* $\beta = (1\ 0)$*,*
(ii) *if* $\alpha = (1\ 0)$*, then* $\beta = (0\ 1)$*,*
(iii) *if* $\alpha = (1\ a)$ *for some* $a \neq 0$*, then* $\beta = (1\ \ {-a^{-1}})$*.*

Proof. It is easy to get these results by direct calculations. □

Lemma 2.3. *If* $A \in Z(R)^*$*, then* $A^2 = \mathbf{0}$ *if and only if* $A \in [E_{12}] \cup [E_{21}] \cup_{a\in F_q^*} \left[\begin{pmatrix} 1 & a \\ -a^{-1} & -1 \end{pmatrix}\right]$.

Proof. Obviously, we get the 'if' part. For the 'only if' part, assume that $\alpha^t\beta$ is the standard representation of $[A]$, where $\alpha, \beta \in M^1_{1\times 2}(F_q)$. Then it follows from $A^2 = \mathbf{0}$ if and only if $[A]^2 = \mathbf{0}$ that $\beta\alpha^t = 0$. By Lemma 2.2, we have

$[A] = [\alpha^t\beta] = [(1\ 0)^t(0\ 1)] = [E_{12}]$, or $[A] = [\alpha^t\beta] = [(0\ 1)^t(1\ 0)] = [E_{21}]$, or $[A] = [\alpha^t\beta] = [(1\ \ -a^{-1})^t(1\ a)] = \left[\left(\begin{smallmatrix} 1 & a \\ -a^{-1} & -1 \end{smallmatrix}\right)\right]$ for some $a \neq 0$. Thus $A \in [A] \subseteq [E_{12}] \cup [E_{21}] \cup_{a\in F_q^*} \left[\left(\begin{smallmatrix} 1 & a \\ -a^{-1} & -1 \end{smallmatrix}\right)\right]$. □

Lemma 2.4. *Let $A, B \in Z(R)^*$. Then $ann_l A \cap ann_l B \neq \{\mathbf{0}\}$ and $ann_r A \cap ann_r B \neq \{\mathbf{0}\}$ if and only if $[A] = [B]$. In particular, $ann_l A = ann_l B$ and $ann_r A = ann_r B$ if and only if $[A] = [B]$.*

Proof. Clearly, "⇐" holds. In order to prove "⇒", assume that $C \neq \mathbf{0}$ and $C \in ann_l A \cap ann_l B$. Then $[C] \in ann_l[A] \cap ann_l[B]$. That means $[C][A] = \mathbf{0}$ and $[C][B] = \mathbf{0}$. Suppose that $\alpha_1^t\alpha_2, \beta_1^t\beta_2, \gamma_1^t\gamma_2$ are the standard representations of $[A], [B], [C]$, respectively. It follows from $[C][A] = \mathbf{0}$ and $[C][B] = \mathbf{0}$ that

$$\gamma_2\alpha_1^t = \gamma_2\beta_1^t = 0. \tag{1}$$

Applying Lemma 2.2 on Eq. (1), we get $\alpha_1 = \beta_1$. Similarly, if $D \neq \mathbf{0}$ and $D \in ann_r A \cap ann_r B$, then $\alpha_2 = \beta_2$. Thus $[A] = [\alpha_1^t\alpha_2] = [\beta_1^t\beta_2] = [B]$. □

S. B. Mulay [14] introduced the compressed zero-divisor graph of a commutative ring while studying automorphisms. The relation on a commutative ring R given by $s \sim t$ if and only if $ann_R(s) = ann_R(t)$ is an equivalence relation. The compressed zero-divisor graph $\Gamma_E(R)$ is the (undirected) graph whose vertices are the equivalence classes induced by $\sim$ other than [0] and [1], such that distinct vertices $[s]$ and $[t]$ are adjacent in $\Gamma_E(R)$ if and only if $st = 0$. This graph was later studied more extensively in [3, 8, 17]. Now, we extend this definition to noncommutative rings.

Definition 2.5. The relation on a noncommutative ring R given by $s \sim t$ if and only if $ann_l s = ann_l t$ and $ann_r s = ann_r t$ is an equivalence relation. The compressed zero-divisor graph of R, denoted by $\Gamma_E(R)$, is the directed graph whose vertices are the equivalence classes induced by $\sim$ other than [0] and [1], such that $[s] \to [t]$ in $\Gamma_E(R)$ if and only if $[s] \neq [t]$ and $st = 0$.

Let $\Gamma_E(R)$ be the compressed zero-divisor graph of $R = M_2(F_q)$, i.e., $\Gamma_E(R)$ is a graph with vertex set $\{[A] \mid A \in Z(R)^*\}$, and there is a directed edge from a vertex $[A]$ to a distinct $[B]$ if and only if $AB = \mathbf{0}$. For any vertex $[A]$ in $\Gamma_E(R)$, suppose that $A = \alpha^t\beta = (a_1\ a_2)^t(b_1\ b_2)$ and the first nonzero component of α (resp., β) is a_i (resp., b_j), and call $[A]$ a vertex of *type* (i, j). Then all vertices in $\Gamma_E(R)$ can be categorized into four types as follows.

type (1,1): $[A] = [(1\ a)^t(1\ b)] = [\left(\begin{smallmatrix} 1 & b \\ a & ab \end{smallmatrix}\right)],\ a, b \in F$,
type (1,2): $[A] = [(1\ a)^t(0\ 1)] = [\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)],\ a \in F$,
type (2,1): $[A] = [(0\ 1)^t(1\ b)] = [\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)],\ b \in F$,
type (2,2): $[A] = [(0\ 1)^t(0\ 1)] = [E_{22}]$.

It is obvious to see that in $\Gamma_E(R)$, there are $q^2, q, q, 1$ vertices in types (1,1), (1,2), (2,1), (2,2) respectively. It follows that there are $(q+1)^2$ vertices in $\Gamma_E(R)$, and $(q-1)(q+1)^2$ vertices in $\Gamma(R)$ since each vertex $[A] \in V(\Gamma_E(R))$ contains $q-1$ different vertices in $\Gamma(R)$.

3. Automorphisms of $\Gamma_E(R)$

In this section, we classify the automorphisms of $\Gamma_E(R)$, which will be helpful for the investigation of $\mathrm{Aut}(\Gamma(R))$ in the next section. First we introduce two automorphisms σ_P and τ_f for $\Gamma_E(R)$, then we prove that any automorphism of $\Gamma_E(R)$ can be expressed as $\sigma_P\tau_f$ and show that $|\mathrm{Aut}(\Gamma_E(R))| = (q+1)!$.

For any $P \in U(R)$, let

$$\sigma_P : [A] \mapsto [PAP^{-1}],\ A \in Z(R)^*.$$

It is easy to check that σ_P is an automorphism of $\Gamma_E(R)$. Let $\mathbf{1}$ be the identity map over any given set, $F_q^*I = \{aI \mid a \in F_q^*\}$, and $\Pi = \{\sigma_P \mid P \in U(R)\}$. Then we prove that Π is a subgroup of $\mathrm{Aut}(\Gamma_E(R))$ and $\Pi \cong U(R)/F_q^*I$.

Lemma 3.1. *σ_P fixes $[E_{11}]$, $[E_{12}]$, $[(0\ 1)^t(1\ 1)]$ if and only if $P \in F_q^*I$. In particular, $\sigma_P = \mathbf{1}$ if and only if $P \in F_q^*I$.*

Proof. Clearly, we get "$\Leftarrow$". Conversely, suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Since $\sigma_P([E_{11}]) = [E_{11}]$, we get

$$[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}] = [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}].$$

Thus $[\begin{pmatrix} aa_1 & ab_1 \\ ca_1 & cb_1 \end{pmatrix}] = [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$. This yields $a \neq 0, a_1 \neq 0$, and $b_1 = c = 0$. Hence

$$P = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ P^{-1} = \begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix}.$$

From $\sigma_P([E_{12}]) = [E_{12}]$, we have

$$[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix}] = [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}].$$

It follows that $[\begin{pmatrix} ac_1 & ad_1 \\ 0 & 0 \end{pmatrix}] = [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]$, and so $ac_1 = 0$. Thus $c_1 = 0$, and therefore

$$P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ P^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}.$$

Since $\sigma_P([(0\ 1)^t(1\ 1)]) = [(0\ 1)^t(1\ 1)]$, it follows that

$$[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}] = [\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}].$$

Thus $[\begin{pmatrix} 0 & 0 \\ 1 & ad^{-1} \end{pmatrix}] = [\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}]$. This gives $ad^{-1} = 1$. Thus $a = d$, and P is a nonzero scalar matrix. □

Lemma 3.2. $\Pi \cong U(R)/F_q^*I$.

Proof. Set

$$\psi : U(R) \mapsto \Pi,\ P \mapsto \sigma_P.$$

Then ψ is well-defined and ψ is a surjection. It is easy to prove that $\sigma_P\sigma_Q = \sigma_{PQ}$, and so ψ is a surjective homomorphism. By Lemma 3.1, we get $\mathrm{Ker}\psi = F_q^*I$. Thus $\Pi \cong U(R)/F_q^*I$. □

Next, let us introduce another automorphism τ_f of $\Gamma_E(R)$.

Definition 3.3. Let $f \in S_{F_q}$ that fixes 0, set $f^* \in S_{F_q}$ as follows, and call f^* the companion permutation *of* f.

$$f^*(a) = \begin{cases} 0, & a = 0, \\ -f(-a^{-1})^{-1}, & a \neq 0. \end{cases}$$

Lemma 3.4. *If* $f, g \in S_{F_q}$ *both fix* 0*, then* $(f^*)^* = f$*,* $(f^{-1})^* = (f^*)^{-1}$*, and* $(fg)^* = f^*g^*$*.*

Proof. Apparently, $(f^*)^*(0) = f(0)$.

In the case $a \neq 0$, it follows from $f^*(-a^{-1})f(a) = -1$ and $(f^*)^*(a)f^*(-a^{-1}) = -1$ that $(f^*)^*(a) = f(a)$. Thus $(f^*)^* = f$.

Clearly, we have $(f^{-1})^*f^*(0) = 0$. For any $a \neq 0$, set $b = f(-a^{-1})$. Then $(f^{-1})^*f^*(a) = (f^{-1})^*(-f(-a^{-1})^{-1}) = (f^{-1})^*(-b^{-1}) = -f^{-1}(b)^{-1} = -(-a^{-1})^{-1} = a$. Thus $(f^{-1})^*f^* = \mathbf{1}$, and so $(f^{-1})^* = (f^*)^{-1}$.

Let $a \in F_q^*$. Set $b = g(-a^{-1})$. Then $(fg)^*(0) = f^*g^*(0)$ and $(fg)^*(a) = -(fg(-a^{-1}))^{-1} = -f(b)^{-1} = f^*(-b^{-1}) = f^*(-g(-a^{-1})^{-1}) = f^*g^*(a)$ yielding $(fg)^* = f^*g^*$. □

Let $f \in S_{F_q}$ that fixes 0, and let $\Sigma = \{\tau_f \mid f \in S_{F_q}, f(0) = 0\}$, where the map $\tau_f : V(\Gamma_E(R)) \mapsto V(\Gamma_E(R))$ is defined as the following.

$$\begin{aligned} \tau_f : &[A] = [(1\ a)^t(1\ b)] \mapsto [(1\ f^*(a))^t(1\ f(b))], a, b \in F_q, \\ &[A] = [(1\ a)^t(0\ 1)] \mapsto [(1\ f^*(a))^t(0\ 1)], a \in F_q, \\ &[A] = [(0\ 1)^t(1\ b)] \mapsto [(0\ 1)^t(1\ f(b))], b \in F_q, \\ &[A] = [(0\ 1)^t(0\ 1)] \mapsto [(0\ 1)^t(0\ 1)]. \end{aligned}$$

Lemma 3.5. (i) $\tau_{fg} = \tau_f\tau_g$, $\tau_{f^{-1}} = \tau_f^{-1}$ *for every* τ_f*,* $\tau_g \in \Sigma$*.*

(ii) Σ *is a group, and* $\Sigma \cong S_{F_q^*}$*.*

Proof. (i) For any $a, b \in F_q$, we have

$$\begin{aligned} \tau_{fg}([(1\ a)^t(1\ b)]) &= [(1\ (fg)^*(a))^t(1\ fg(b))] \\ &= [(1\ f^*g^*(a))^t(1\ fg(b))] \\ &= \tau_f([(1\ g^*(a))^t(1\ g(b))]) \\ &= \tau_f\tau_g([(1\ a)^t(1\ b)]). \end{aligned}$$

Thus

$$\tau_{fg}([A]) = \tau_f\tau_g([A]) \tag{2}$$

for each $[A]$ of type (1,1). Similarly, Eq. (2) holds for any $[A]$ of types (1,2), (2,1), (2,2). Hence $\tau_{fg} = \tau_f\tau_g$, and so $\tau_{f^{-1}}\tau_f = \tau_{f^{-1}f} = \tau_{\mathbf{1}} = \mathbf{1}$.

(ii) By (i) we know Σ is a group. Apparently, $S_{F_q^*}$ is a group and $S_{F_q^*} \cong \{f \in S_{F_q} \mid f(0) = 0\}$. Each $f \in S_{F_q}$ that fixes 0 induces a τ_f in Σ. Let

$$\varphi : \{f \in S_{F_q} \mid f(0) = 0\} \mapsto \Sigma,\ f \mapsto \tau_f.$$

We show that φ is an isomorphism, thus $\Sigma \cong \{\tau_f \mid f \in S_{F_q}, f(0) = 0\} \cong S_{F_q^*}$. Indeed, by (i) we get that the surjection φ satisfies $\varphi(fg) = \varphi(f)\varphi(g)$. In what follows, we prove that φ is injective, and so φ is an isomorphism. In fact, if there exist $f, g \in S_{F_q}$ fix 0 such that $\tau_f = \tau_g$, then for every $b \in F_q$ we have $\tau_f([(0\ 1)^t(1\ b)]) = \tau_g([(0\ 1)^t(1\ b)])$. Thus $f(b) = g(b)$, and so $f = g$. Hence φ is injective. □

Now we show that τ_f is an automorphism of $\Gamma_E(R)$.

Lemma 3.6. *τ_f is an automorphism of $\Gamma_E(R)$, and Σ is a subgroup of* $\mathrm{Aut}(\Gamma_E(R))$.

Proof. Clearly, τ_f preserves the type of each vertex in $\Gamma_E(R)$. Since f and f^* are permutations over F_q, it follows that τ_f is bijective. If we can prove that $[A] \to [B]$ if and only if $\tau_f([A]) \to \tau_f([B])$ for each $[A], [B] \in V(\Gamma_E(R))$, then τ_f is an automorphism of $\Gamma_E(R)$. In fact, it follows from $\tau_f([A]) \to \tau_f([B])$ that $[A] \to [B]$ since $\tau_{f^{-1}}\tau_f = \mathbf{1}$. Conversely, if $[A] \to [B]$, then $[A] \neq [B]$ and $[\alpha_1^t\alpha_2][\beta_1^t\beta_2] = \mathbf{0}$, where $\alpha_1^t\alpha_2$ and $\beta_1^t\beta_2$ are respectively the standard representations of $[A]$ and $[B]$. Thus $\alpha_2\beta_1^t = 0$. Lemma 2.2 yields α_2, β_1 satisfy one of the three cases as follows.

(i) If $\alpha_2 = (0\ 1)$, then $\beta_1 = (1\ 0)$, and so

$$\tau_f([A])\tau_f([B]) = [(\star\ \star)^t(0\ 1)][(1\ 0)^t(\star\ \star)] = \mathbf{0}.$$

(ii) If $\alpha_2 = (1\ 0)$, then $\beta_1 = (0\ 1)$. Now

$$\tau_f([A])\tau_f([B]) = [(\star\ \star)^t(1\ 0)][(0\ 1)^t(\star\ \star)] = \mathbf{0}.$$

(iii) If $\alpha_2 = (1\ a)$ for some $a \neq 0$, then $\beta_1 = (1\ \ -a^{-1})$. Thus

$$\tau_f([A])\tau_f([B]) = [(\star\ \star)^t(1\ f(a))][(1\ f^*(-a^{-1}))^t(\star\ \star)] = \mathbf{0}.$$

By (i)-(iii), we always have $\tau_f([A])\tau_f([B]) = \mathbf{0}$. Note that, $[A] \neq [B]$ and τ_f is bijective, so we get $\tau_f([A]) \neq \tau_f([B])$. Hence $\tau_f([A]) \to \tau_f([B])$. □

Lemma 3.7. *Let $P, Q \in U(R)$, and let $f, g \in S_{F_q}$ that fix 0. Then*

(i) *$\tau_f = \mathbf{1}$ if and only if $f = \mathbf{1}$,*

(ii) *if f fixes 1 and $\sigma_P = \tau_f$, then $\sigma_P = \tau_f = \mathbf{1}$,*

(iii) *if $f, g \in S_{F_q}$ fix 1 and $\sigma_P\tau_f = \sigma_Q\tau_g$, then $f = g$ and $P = aQ$ for some $a \in F^*$.*

Proof. (i) Clearly, if $f = \mathbf{1}$, then $f^* = \mathbf{1}$, and so $\tau_f = \mathbf{1}$. Conversely, by the definition of τ_f, we conclude $f = \mathbf{1}$ if $\tau_f = \mathbf{1}$.

(ii) Note that $\sigma_P([E_{11}]) = \tau_f([E_{11}]) = [E_{11}]$, $\sigma_P([E_{12}]) = \tau_f([E_{12}]) = [E_{12}]$, and $\sigma_P([\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}]) = \tau_f([\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}]) = [\begin{pmatrix} 0 & 0 \\ 1 & f(1) \end{pmatrix}] = [\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}]$, it follows from Lemma 3.1 that $\tau_f = \sigma_P = \mathbf{1}$.

(iii) Since $\sigma_P\tau_f = \sigma_Q\tau_g$, it follows that $\sigma_Q^{-1}\sigma_P = \tau_g\tau_f^{-1}$. Then

$$\sigma_{Q^{-1}P} = \sigma_Q^{-1}\sigma_P = \tau_g\tau_f^{-1} = \tau_{gf^{-1}}. \tag{3}$$

By Eq. (3) and (ii), we have

$$\sigma_{Q^{-1}P} = \tau_{gf^{-1}} = \mathbf{1}. \tag{4}$$

By (i), Eq. (4), and Lemma 3.1, we get $Q^{-1}P$ is a nonzero scalar matrix and $gf^{-1} = \mathbf{1}$. Thus $f = g$, and $P = aQ$ for some $a \in F^*$. □

In the following of this section, we develop some lemmas to show that any automorphism of $\Gamma_E(R)$ can be expressed as $\sigma_P\tau_f$, where σ_P and τ_f are the automorphisms we constructed as above.

Lemma 3.8. *The in-degree and out-degree of each vertex $[A]$ in $\Gamma_E(R)$ are both equal to $q+1$ if $A^2 \neq \mathbf{0}$, and are both equal to q if $A^2 = \mathbf{0}$.*

Proof. For any $[B], B \in Z(R)^*$ satisfies $[B][A] = \mathbf{0}$, assume that $\alpha_1^t\alpha_2, \beta_1^t\beta_2$ are respectively the standard representations of classes $[A], [B]$. Thus $\alpha_1, \alpha_2, \beta_1, \beta_2 \in M^1_{1\times 2}(F_q)$ and $[B][A] = \mathbf{0}$ imply $\beta_2\alpha_1^t = 0$. By Lemma 2.2, β_2 is uniquely determined by α_1. Thus the cardinality of $\{[B] | [B][A] = \mathbf{0}, B \in Z(R)^*\}$ is depending on the choice of $\beta_1 \in M^1_{1\times 2}(F_q)$. Hence there are $q+1$ nonzero classes in $\{[B] | [B][A] = \mathbf{0}, B \in Z(R)^*\}$ since $|M^1_{1\times 2}(F_q)| = q+1$ and different β_1 implies different $[\beta_1^t\beta_2]$. Then $N_l([A]) = \{[B] | [B][A] = \mathbf{0}, B \in Z(R)^*\} - \{[A]\}$, and so the in-degree of $[A]$ is $|N_l([A])| = q$ if $A^2 = \mathbf{0}$, the in-degree of $[A]$ is $|N_l([A])| = q+1$ if $A^2 \neq \mathbf{0}$. By a similar argument, we can get the out-degree of each vertex $[A]$. □

Lemma 3.9. *If σ is an automorphism of $\Gamma_E(R)$ and $\sigma([A]) = [B]$, then $A^2 = \mathbf{0}$ if and only if $B^2 = \mathbf{0}$.*

Proof. It immediately follows from Lemma 3.8 since σ preserves the in-degree and the out-degree of each vertex. □

Lemma 3.10. *If σ is an automorphism of $\Gamma_E(R)$ that fixes $[E_{ii}]$, $i = 1, 2$, then it preserves the type of each vertex.*

Proof. Note that, there is exactly one vertex $[E_{22}]$ in type (2,2), and σ already fixes $[E_{22}]$.

From $[E_{11}] \to [\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)]$ we know $\sigma([E_{11}]) \to \sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)])$. Assume that $\sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)]) = [\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)]$. Then $\sigma([E_{11}])\sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)]) = \mathbf{0}$ implies $E_{11}\left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) = \mathbf{0}$. Thus $b_1 = b_2 = 0$, and hence $\sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)]) = [\left(\begin{smallmatrix} 0 & 0 \\ b_3 & b_4 \end{smallmatrix}\right)]$. Since σ is a bijection and $\sigma([E_{22}]) = [E_{22}]$, we get $b_3 \neq 0$. That means

$$\sigma([\begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}]) = [\begin{pmatrix} 0 & 0 \\ b_3 & b_4 \end{pmatrix}] = [\begin{pmatrix} 0 & 0 \\ 1 & b_3^{-1}b_4 \end{pmatrix}].$$

This yields σ preserves vertices of type (2,1).

After a similar argument on $[\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)] \to [E_{11}]$, we get that σ preserves vertices of type (1,2).

Since σ is bijective and it preserves vertices of types (1,2), (2,1), (2,2), it follows that σ preserves vertices of type (1,1). $\square$

Lemma 3.11. *If σ is an automorphism of $\Gamma_E(R)$ that fixes $[E_{ii}]$, $i = 1, 2$, then there exists an $f \in S_{F_q}$ fixes 0 such that $\sigma = \tau_f$.*

Proof. Since σ is an automorphism of $\Gamma_E(R)$ and fixes $[E_{ii}]$, $i = 1, 2$, it follows from Lemma 3.10 that σ preserves the type of (2,1). For every $a \in F_q$, $\sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}\right)]) = [\left(\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}\right)]$ for some $b \in F_q$. Set $f(a) = b$. Then $f \in S_{F_q}$. By Lemma 3.9, we have $\sigma([E_{21}]) = [E_{21}]$, and so $f(0) = 0$. Thus τ_f is well defined. Now, we show that $\sigma = \tau_f$.

(i) *Clearly, $\sigma([A]) = \tau_f([A])$ holds for $[A]$ of type $(2, 2)$, since $\sigma([E_{22}]) = [E_{22}] = \tau_f([E_{22}])$.*

(ii) $\sigma([A]) = \tau_f([A])$ *holds for $[A]$ of type $(2, 1)$.*

For any $a \in F_q$, we get

$$\sigma([\begin{pmatrix} 0 & 0 \\ 1 & a \end{pmatrix}]) = [\begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}] = [\begin{pmatrix} 0 & 0 \\ 1 & f(a) \end{pmatrix}] = \tau_f([\begin{pmatrix} 0 & 0 \\ 1 & a \end{pmatrix}]).$$

(iii) $\sigma([A]) = \tau_f([A])$ *holds for $[A]$ of type $(1, 2)$.*

Suppose that $[A] = [(1\ a)^t(0\ 1)]$, $a \in F_q$. By Lemma 3.10 we can assume $\sigma([(1\ a)^t(0\ 1)]) = [(1\ \star)^t(0\ 1)]$. Apparently, by Lemma 3.9 we get $\sigma([E_{12}]) = [E_{12}] = [(1\ f^*(0))^t(0\ 1)] = \tau_f([E_{12}])$. In the case $a \neq 0$, we have $[\left(\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}\right)] \to [\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)]$, and so $\sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}\right)]) \to \sigma([\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)])$. By (ii) we know

$$\sigma([\begin{pmatrix} 0 & 0 \\ 1 & -a^{-1} \end{pmatrix}]) = \tau_f([(0\ 1)^t(1\ \ -a^{-1})]) = [\begin{pmatrix} 0 & 0 \\ 1 & f(-a^{-1}) \end{pmatrix}],$$

and so $[\left(\begin{smallmatrix} 0 & 0 \\ 1 & f(-a^{-1}) \end{smallmatrix}\right)]\sigma([\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)]) = \sigma([\left(\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}\right)])\sigma([\left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right)]) = \mathbf{0}$. Thus

$$\sigma([\begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}]) = [(1\ f^*(a))^t(0\ 1)] = \tau_f([\begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}]).$$

(iv) $\sigma([A]) = \tau_f([A])$ *holds for $[A]$ of type $(1, 1)$.*

For each $[A] = [(1\ a)^t(1\ b)], a, b \in F_q$, assume $\sigma([A]) = [(1\ \star)^t(1\ \star)]$. Then

$$\sigma([(1\ a)^t(1\ b)]) = [(1\ f^*(a))^t(1\ f(b))] = \tau_f([(1\ a)^t(1\ b)]), a, b \in F_q.$$

Indeed, if $a = 0$, then applying σ on $[E_{22}] \to [\left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right)]$ we know

$$\sigma([(1\ 0)^t(1\ b)]) = [(1\ 0)^t(1\ \star)] = [(1\ f^*(0))^t(1\ \star)]. \tag{5}$$

In the case $a \neq 0$, from $[(0\ 1)^t(1\ \ -a^{-1})] \to [(1\ a)^t(1\ b)]$ we have $\sigma([(0\ 1)^t(1\ \ -a^{-1})]) \to \sigma([(1\ a)^t(1\ b)])$. Thus $\sigma([(0\ 1)^t(1\ \ -a^{-1})])\sigma([(1\ a)^t(1\ b)]) = \mathbf{0}$. Note that, by (ii) we get $\sigma([(0\ 1)^t(1\ \ -a^{-1})]) = \tau_f([(0\ 1)^t(1\ \ -a^{-1})]) = [(0\ 1)^t(1\ f(-a^{-1})]$, thus

$$\sigma([(1\ a)^t(1\ b)]) = [(1\ f^*(a))^t(1\ \star)]. \tag{6}$$

If $b = 0$, then applying σ on $[(1\ a)^t(1\ 0)] \to [(0\ 1)^t(0\ 1)]$ and by Eqs. (5)-(6), we have

$$\sigma([(1\ a)^t(1\ 0)]) = [(1\ f^*(a))^t(1\ 0)] = \tau_f([(1\ a)^t(1\ 0)]).$$

In the case $b \neq 0$, applying σ on $[(1\ a)^t(1\ b)] \to [(1\ \ -b^{-1})^t(0\ 1)]$ and by Eqs. (5)-(6), we get that

$$\sigma([(1\ a)^t(1\ b)]) = [(1\ f^*(a))^t(1\ f(b))] = \tau_f([(1\ a)^t(1\ b)]).$$

From (i)-(iv), we conclude that $\sigma([A]) = \tau_f([A])$ for any $[A]$ of all types. Thus $\sigma = \tau_f$. □

Theorem 3.12. *If σ is an automorphism of $\Gamma_E(R)$, then $\sigma = \sigma_P\tau_f$, where $\sigma_P \in \Pi$ and $\tau_f \in \Sigma$.*

Proof. The proof is divided into two steps as follows.

Step 1. There exists $P \in U(R)$ such that $\sigma_P\sigma$ fixes $[E_{ii}]$, $i = 1, 2$.

Since the rank of every matrix in $Z(R)^*$ is 1, there exist $P_1, Q_1 \in U(R)$ such that $\sigma([E_{11}]) = [P_1E_{11}Q_1]$. Write $\sigma_1 = \sigma_{P_1^{-1}}\sigma$. Then

$$\sigma_1([E_{11}]) = [P_1^{-1}P_1E_{11}Q_1P_1] = [E_{11}Q_1P_1] = [aE_{11} + bE_{12}]$$

for some $a, b \in F_q$. Since $E_{11}^2 \neq \mathbf{0}$, by Lemma 3.9 we know $(aE_{11} + bE_{12})^2 \neq \mathbf{0}$. Thus $a \neq 0$. Set $P_2 = \left(\begin{smallmatrix} 1 & a^{-1}b \\ 0 & 1 \end{smallmatrix}\right)$ and set $\sigma_2 = \sigma_{P_2}\sigma_1$. We have

$$\sigma_2([E_{11}]) = [P_2 \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix} P_2^{-1}] = [E_{11}].$$

Since $[E_{22}] \in N_l([E_{11}]) \cap N_r([E_{11}])$, we get

$$\begin{aligned} \sigma_2([E_{22}]) &\in N_l(\sigma_2([E_{11}])) \cap N_r(\sigma_2([E_{11}])) \\ &= N_l([E_{11}]) \cap N_r([E_{11}]) \\ &= [\begin{pmatrix} 0 & \star \\ 0 & \star \end{pmatrix}] \cap [\begin{pmatrix} 0 & 0 \\ \star & \star \end{pmatrix}] \\ &= \{[E_{22}]\}. \end{aligned}$$

From what is above, if we set $P = P_2P_1^{-1}$, then $P \in U(R)$ and $\sigma_2 = \sigma_P\sigma$ preserves $[E_{ii}]$, $i = 1, 2$.

Step 2. There exists $f \in S_{F_q}$ that fixes 0 such that $\sigma_P\sigma = \tau_f$.

Thanks to Lemma 3.11, we immediately get this step.

By Steps 1-2, if σ is an automorphism of $\Gamma_E(R)$, then there exist $P \in U(R)$ and $f \in S_{F_q}$ that fixes 0 such that $\sigma_P\sigma = \tau_f$. Rewrite P^{-1} as P. Then $\sigma = \sigma_P\tau_f$. □

In order to calculate the number of automorphisms of $\Gamma_E(R)$ more easily, we develop the following theorem.

Theorem 3.13. *If σ is an automorphism of $\Gamma_E(R)$, then $\sigma = \sigma_P\tau_f$, where $P \in U(R)$ and $f \in S_{F_q}$ that fixes 0 and 1.*

Proof. The proof is divided into three steps as follows.

Step 1. There exists $P \in U(R)$ *such that* $\sigma_1 = \sigma_P \sigma$ *fixes* $[E_{ii}]$, $i = 1, 2$.

See the proof of Step 1 in Theorem 3.12.

Step 2. There exists $Q \in U(R)$ *such that* $\sigma_2 = \sigma_Q \sigma_1$ *preserves* $[E_{ii}] = [E_{ii}] (i = 1, 2)$ *and* $[\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]$.

By Lemmas 3.9-3.10, we know σ_1 preserves the type of each vertex and sends a square-zero class to a square-zero one. Thus $\sigma_1([E_{12}]) = [E_{12}]$ and $\sigma_1([E_{21}]) = [E_{21}]$. By Lemma 2.3, we know $A^2 = \mathbf{0}$ if and only if $A \in [E_{12}] \cup [E_{21}] \cup_{a \in F_q^*} [\begin{pmatrix} 1 & a \\ -a^{-1} & -1 \end{pmatrix}]$. Thus $\sigma_1([\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]) = [\begin{pmatrix} 1 & a \\ -a^{-1} & -1 \end{pmatrix}]$ for some unique $a \in F_q^*$. Set $Q = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ and set $\sigma_2 = \sigma_Q \sigma_1$, then $\sigma_2([E_{ii}]) = [E_{ii}]$ $(i = 1, 2)$ and $\sigma_2([\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]) = [\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]$.

Step 3. There exists $f \in S_{F_q}$ *that fixes* 0 *and* 1 *such that* $\sigma_2 = \tau_f$.

By Lemma 3.11, and note that $\sigma_2([\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]) = [\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]$, we have $f(1) = 1$.

By Steps 1-3, if σ is an automorphism of $\Gamma_E(R)$, then there exist $P, Q \in U(R)$ and $f \in S_{F_q}$ that fixes 0, 1 such that $\sigma_Q \sigma_P \sigma = \tau_f$. Rewrite $P^{-1}Q^{-1}$ as P. Then $\sigma = \sigma_P \tau_f$. ☐

Corollary 3.14. $|\mathrm{Aut}(\Gamma_E(R))| = (q + 1)!$.

Proof. Since $|\Pi| = |U(R)/F_q^* I| = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2 - 1)$, $|\{f \in S_{F_q} \mid f(0) = 0, f(1) = 1\}| = (q - 2)!$, and by Lemma 3.7(iii) and Theorem 3.13 we have $|\mathrm{Aut}(\Gamma_E(R))| = q(q^2 - 1) \cdot (q - 2)! = (q + 1)!$. ☐

4. Automorphisms of $\Gamma(R)$

In this section, we discuss the automorphism group of $\Gamma(R)$. Let σ be an automorphism of $\Gamma(R)$, and let $\sigma[A] = \{\sigma(B) \mid B \in [A]\}$.

Lemma 4.1. *Let* σ *be an automorphism of* $\Gamma(R)$. *Then* $\sigma[A] = [\sigma(A)]$, $A \in Z(R)^*$.

Proof. For every $A \in Z(R)^*$, by Lemma 3.8 we have $|N_l([A])| \geq q$ in $\Gamma_E(R)$, and so $|N_l(A)| \geq q(q - 1) \geq 2$ in $\Gamma(R)$. Thus $N_l(A) \cap N_l(aA) \neq \emptyset$, $a \in F_q^*$. That means $N_l(\sigma(A)) \cap N_l(\sigma(aA)) \neq \emptyset$, and so

$$ann_l \sigma(A) \cap ann_l \sigma(aA) \neq \{\mathbf{0}\}.$$

Similarly, we get

$$ann_r \sigma(A) \cap ann_r \sigma(aA) \neq \{\mathbf{0}\}.$$

By Lemma 2.4, we have $[\sigma(A)] = [\sigma(aA)]$. Thus $\sigma(aA) \in [\sigma(A)]$ for any $a \in F_q^*$, and so $\sigma[A] \subseteq [\sigma(A)]$. Note that, σ is a bijection and $|[A]| = |[\sigma(A)]| = q - 1$, so we get $\sigma[A] = [\sigma(A)]$. ☐

We can assume that $V(\Gamma(R)) = \cup_{i=1}^{(q+1)^2} [A_i]$, since there are $(q+1)^2$ vertices in $\Gamma_E(R)$. Let σ_i be a bijection on $V(\Gamma(R))$ satisfying $\sigma_i[A_i] = [A_i]$ and $\sigma_i(A) = A$ if $A \notin [A_i]$, and let $S_{[A_i]}$ be the set consisting of all such bijections. Clearly,

$S_{[A_i]}$ is isomorphic to the symmetric group over set $[A_i]$. Set $K(\Gamma(R)) = \prod_{i=1}^{(q+1)^2} S_{[A_i]}$. Then $|K(\Gamma(R))| = ((q-1)!)^{(q+1)^2}$.

Lemma 4.2. *$K(\Gamma(R))$ is a subgroup of* $\mathrm{Aut}(\Gamma(R))$.

Proof. Let $\sigma_i \in S_{[A_i]}$. Then $\sigma_i[A_i] = [A_i]$ and σ_i fixes any other vertex in $\Gamma(R)$. Note that, if $A_i^2 \neq \mathbf{0}$, then $[A_i]$ induces an independent set, and each of the vertices in $[A_i]$ have the same left neighborhood and the same right neighborhood. This gives σ_i is an automorphism of $\Gamma(R)$, and $S_{[A_i]}$ is a subgroup of $\mathrm{Aut}(\Gamma(R))$. If $A_i^2 = \mathbf{0}$, then $[A_i]$ induces a clique, and $N_l(A) - [A_i] = N_l(B) - [A_i]$, $N_r(A) - [A_i] = N_r(B) - [A_i]$ for every $A, B \in [A_i]$. Again σ_i is an automorphism of $\Gamma(R)$ and $S_{[A_i]}$ is a subgroup of $\mathrm{Aut}(\Gamma(R))$. Thus $K(\Gamma(R)) = \prod_{i=1}^{(q+1)^2} S_{[A_i]}$ is a subgroup of $\mathrm{Aut}(\Gamma(R))$ since $K(\Gamma(R))$ is generated by $\cup_{i=1}^{(q+1)^2} S_{[A_i]}$. □

Theorem 4.3. $\mathrm{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \mathrm{Aut}(\Gamma_E(R))$.

Proof. If σ is an automorphism of $\Gamma(R)$, then by Lemma 4.1 we have $\sigma[A] = [\sigma(A)]$. And for each $A, B \in Z(R)^*$,

$$A \to B \Leftrightarrow \sigma(A) \to \sigma(B).$$

Thus

$$A \neq B, AB = \mathbf{0} \Leftrightarrow \sigma(A) \neq \sigma(B),\ \sigma(A)\sigma(B) = \mathbf{0},$$

and so

$$A \neq B, [A][B] = \mathbf{0} \Leftrightarrow \sigma(A) \neq \sigma(B),\ [\sigma(A)][\sigma(B)] = \mathbf{0}. \tag{7}$$

Set $[\sigma] : [A] \mapsto [\sigma(A)]$. Then $[\sigma]$ is an automorphism of $\Gamma_E(R)$. Indeed, $[\sigma]$ is a bijection, and by Eq. (7) we know $[A] \to [B]$ in $\Gamma_E(R)$ if and only if $[\sigma]([A]) \to [\sigma]([B])$ in $\Gamma_E(R)$.

Set

$$\phi : \mathrm{Aut}(\Gamma(R)) \mapsto \mathrm{Aut}(\Gamma_E(R)), \sigma \mapsto [\sigma].$$

Then we prove that ϕ is a surjective homomorphism, and show that Ker $\phi = K(\Gamma(R))$. Thus $\mathrm{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \mathrm{Aut}(\Gamma_E(R))$.

(i) ϕ *is surjective.*

For any $\delta \in \mathrm{Aut}(\Gamma_E(R))$, let σ be a bijection on $V(\Gamma(R))$ such that $\sigma[A] = \delta([A])$. Then σ is well defined since $|\sigma[A]| = |\delta([A])| = q - 1$. Now we prove that $\sigma \in \mathrm{Aut}(\Gamma(R))$. (ia) If $A \to B$ in $\Gamma(R)$, then $A, B \in Z(R)^*$ are distinct and satisfy $AB = \mathbf{0}$. If $[A] = [B]$, then from $AB = \mathbf{0}$ we get $A^2 = \mathbf{0}$, and so $\sigma[A] = \delta([A])$ is a square-zero class by Lemma 3.9. This gives $\sigma(A)\sigma(B) \in (\sigma[A])^2 = \mathbf{0}$, and so $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$. If $[A] \neq [B]$, then $\delta([A]) \to \delta([B])$ in $\Gamma_E(R)$ since $[A][B] = \mathbf{0}$. Thus $\sigma(A)\sigma(B) \in \sigma[A]\sigma[B] = \delta([A])\delta([B]) = \mathbf{0}$, and so $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$. (ib) Conversely, if $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$, then $A \neq B$ and

$$\sigma(A)\sigma(B) = \mathbf{0}. \tag{8}$$

Note that $\sigma(A) \in \sigma[A] = \delta([A])$, and so $[\sigma(A)] = \sigma[A] = \delta([A])$. Similarly, we have $[\sigma(B)] = \sigma[B] = \delta([B])$. Thus, by Eq. (8) we get

$$\delta([A])\delta([B]) = \mathbf{0}. \tag{9}$$

If $\delta([A]) \neq \delta([B])$, then $\delta([A]) \to \delta([B])$ in $\Gamma_E(R)$. It follows that $[A] \to [B]$ in $\Gamma_E(R)$. Hence $AB = \mathbf{0}$, and so $A \to B$ in $\Gamma(R)$. If $\delta([A]) = \delta([B])$, then $[A] = [B]$. And by Eq. (9) we have $(\delta([A]))^2 = \mathbf{0}$. By Lemma 3.9 we get $A^2 = \mathbf{0}$. And note that $[A] = [B]$, so we have $AB = \mathbf{0}$. Thus $A \to B$ in $\Gamma(R)$. Hence $\sigma \in \mathrm{Aut}(\Gamma(R))$, and so ϕ is a surjection.

(ii) $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$.

$[\sigma_1\sigma_2][A] = (\sigma_1\sigma_2)[A] = \sigma_1([\sigma_2][A]) = [\sigma_1][\sigma_2][A], A \in Z(R)^*$ yields $[\sigma_1\sigma_2] = [\sigma_1][\sigma_2]$, and so $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$.

(iii) $\mathrm{Ker}\phi = K(\Gamma(R))$.

Write $V(\Gamma(R)) = \cup_{i=1}^{(q+1)^2}[A_i]$. If $[\sigma]$ is the identity automorphism of $\Gamma_E(R)$, then $\sigma[A_i] = [\sigma]([A_i]) = [A_i]$ for any $[A_i] \in V(\Gamma_E(R))$. Thus $\sigma \in K(\Gamma(R))$.

From (i)-(iii), we have $\mathrm{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \mathrm{Aut}(\Gamma_E(R))$. □

Corollary 4.4. $|\mathrm{Aut}(\Gamma(R))| = ((q-1)!)^{(q+1)^2} \cdot (q+1)!$

Proof. It immediately follows from $|K(\Gamma(R))| = ((q-1)!)^{(q+1)^2}$, Corollary 3.14 and Theorem 4.3. □

Remark 4.5. [15, Theorem 3.8] (resp., [12, Theorem 3.9]) said that the automorphism group of $\Gamma(R)$ (resp., $\Gamma(M_2(\mathbb{Z}_q))$ where q is a prime) is isomorphic to the symmetric group S_{q+1} of degree $q+1$, which means $|\mathrm{Aut}(\Gamma(R))| = (q+1)!$ (resp., $\mathrm{Aut}(\Gamma(M_2(\mathbb{Z}_q))) = (q+1)!$). But our result $((q-1)!)^{(q+1)^2} \cdot (q+1)!$ is much greater than $(q+1)!$ in general. Hence [12, Theorem 3.9] and [15, Theorem 3.8] are both incorrect. In fact, the automorphism σ that constructed in [12, Theorem 3.9] (resp., [15, Theorem 3.8]) fails to be a bijection.

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