

On Paraopen Sets and Maps in Topological Spaces

BASAVARAJ M. ITTANAGI*

*Department of Mathematics, Siddaganga Institute of Technology, Tumkur-572 103,
Karnataka State, India*

e-mail: dr.basavarajsit@gmail.com

SHIVANAGAPPA S. BENCHALLI

*Department of Mathematics, Karnatak University, Dharwad-580003 Karnataka
State, India*

e-mail: benchalliss@gmail.com

ABSTRACT. In this paper, we introduce and study the concept of a new class of sets called paraopen sets and paraclosed sets in topological spaces. During this process some of their properties are obtained. Also we introduce and investigate a new class of maps called paracontinuous, $*$ -paracontinuous, parairresolute, minimal paracontinuous and maximal paracontinuous maps and study their basic properties in topological spaces.

1. Introduction

In the years 2001 and 2003, F. Nakaoka and N. Oda ([2], [3] and [4]) introduced and studied minimal open (resp. minimal closed) sets and maximal open (resp. maximal closed) sets, which are subclasses of open (resp. closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. Also in the year 2011, S. S. Benchalli, Basavaraj M. Ittanagi, R. S. Wali [1], introduced and studied minimal open sets and maps in topological spaces.

Definition 1.1.([2]) A proper nonempty open subset U of a topological space X is said to be a minimal open set if any open set which is contained in U is ϕ or U .

Definition 1.2.([3]) A proper nonempty open subset U of a topological space X is said to be maximal open set if any open set which contains U is X or U .

* Corresponding Author.

Received October 30, 2014; accepted February 21, 2015.

2010 Mathematics Subject Classification: 54A05, 54C05.

Key words : Paraopen sets, minimal open sets and maximal open sets.

Definition 1.3.([4]) A proper nonempty closed subset F of a topological space X is said to be a minimal closed set if any closed set which is contained in F is ϕ or F .

Definition 1.4.([4]) A proper nonempty closed subset F of a topological space X is said to be maximal closed set if any closed set which contains F is X or F .

Definition 1.5.([1]) Let X and Y be the topological spaces. A map $f : X \rightarrow Y$ is called

- i) minimal continuous if $f^{-1}(M)$ is an open set in X for every minimal open set M in Y .
- ii) maximal continuous if $f^{-1}(M)$ is an open set in X for every maximal open set M in Y .

The family of all minimal open (resp. minimal closed) sets in a topological space X is denoted by $M_iO(X)$ (resp. $M_iC(X)$). The family of all maximal open (resp. maximal closed) sets in a topological space X is denoted by $M_aO(X)$ (resp. $M_aC(X)$).

2. Paraopen Sets and Some of Their Properties

Definition 2.1. Any open subset U of a topological space X is said to be a paraopen set if it is neither minimal open nor maximal open set. The family of all paraopen sets in a topological space X is denoted by $P_aO(X)$.

Any closed subset F of a topological space X is said to be a paraclosed set if and only if its complement $(X - F)$ is paraopen set. The family of all paraclosed sets in a topological space X is denoted by $P_aC(X)$.

Note that every paraopen set is an open set and every paraclosed set is a closed set but not conversely, which is shown by the following example.

Example 2.2. Let $X = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$. Then $M_iO(X) = \{\{a\}, \{b\}, \{c\}\}$, $M_aO(X) = \{a, b, c\}$, $M_iC(X) = \{d\}$, $M_aC(X) = \{\{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}$, $P_aO(X) = \{\phi, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, $P_aC(X) = \{X, \{c, d\}, \{b, d\}, \{a, d\}, \phi\}$.

Here $\{a\}$ is an open set but not a paraopen set and $\{d\}$ is a closed set but not a paraclosed set.

Remark 2.3. Union and intersection of paraopen (resp. paraclosed) sets need not be a paraopen (resp. paraclosed) set.

Example 2.4. In Example 2.2, we have $\{a, b\}, \{a, c\}$ are paraopen sets but $\{a, b\} \cup \{a, c\} = \{a, b, c\}$ and $\{a, b\} \cap \{a, c\} = \{a\}$ which are not paraopen sets. (resp. $\{c, d\}, \{b, d\}$ are paraclosed sets but $\{c, d\} \cup \{b, d\} = \{b, c, d\}$ and $\{c, d\} \cap \{b, d\} = \{d\}$ which are not paraclosed sets).

Theorem 2.5. Let X be a topological space and U be a nonempty paraopen subset of X . Then there exists a minimal open set N such that $N \subset U$.

Proof. By definition of minimal open set, it is obvious that $N \subset U$. □

Theorem 2.6. *Let X be a topological space and U be a proper paraopen subset of X then there exists a maximal open set M such that $U \subset M$.*

Proof. By definition of maximal open set, it is obvious that $U \subset M$. \square

Theorem 2.7. *Let X be a topological space.*

- i) *Let U be a paraopen and N be a minimal open set then $U \cap N = \phi$ or $N \subset U$.*
- ii) *Let U be a paraopen and M be a maximal open set then $U \cup M = X$ or $U \subset M$.*
- iii) *Intersection of paraopen sets is either paraopen or minimal open set.*

Proof. i) Let U be a paraopen and N be a minimal open set in X . Then $U \cap N = \phi$ or $U \cap N \neq \phi$. If $U \cap N = \phi$ then there is nothing to prove. Suppose $U \cap N \neq \phi$. Now we have $U \cap N$ is an open set and $U \cap N \subset N$. Hence $N \subset U$.

ii) Let U be a paraopen and M be a maximal open set in X . Then $U \cup M = X$ or $U \cup M \neq X$. If $U \cup M = X$ then there is nothing to prove. Suppose $U \cup M \neq X$. Now we have $U \cup M$ is an open set and $M \subset U \cup M$. Since M is maximal open set, $U \cup M = M$ which implies $U \subset M$.

iii) Let U and V be paraopen sets in X . If $U \cap V$ is a paraopen set then there is nothing to prove. Suppose $U \cap V$ is not a paraopen set. Then by definition, $U \cap V$ is a minimal open or maximal open set. If $U \cap V$ is a minimal open set then there is nothing to prove. Suppose $U \cap V$ is a maximal open set. Now $U \cap V \subset U$ and $U \cap V \subset V$ which contradicts the fact that U and V are paraopen sets. Therefore $U \cap V$ is not a maximal open set. That is $U \cap V$ must be a minimal open set. \square

Theorem 2.8. *Let X be a topological space. A subset F of X is paraclosed if and only if it is neither maximal closed nor minimal closed set.*

Proof. The proof follows from the definition and fact that the complement of minimal open set is maximal closed set and the complement of maximal open set is minimal closed set. \square

Theorem 2.9. *Let X be a topological space and F be a nonempty paraclosed subset of X then there exists a minimal closed set N such that $N \subset F$.*

Proof. By definition of minimal closed set, it is obvious that $N \subset F$. \square

Theorem 2.10. *Let X be a topological space and F be a proper paraclosed subset of X then there exists a maximal closed set M such that $F \subset M$.*

Proof. By definition of maximal closed set, it is obvious that $F \subset M$. \square

Theorem 2.11. *Let X be a topological space.*

- i) *Let F be paraclosed and N be a minimal closed sets then $F \cap N = \phi$ or $N \subset F$.*
- ii) *Let F be paraclosed and M be a maximal closed sets then $F \cup M = X$ or $F \subset M$.*
- iii) *Intersection of paraclosed sets is either paraclosed or minimal closed set.*

Proof. i) Let F be a paraclosed and N be a minimal closed sets in X . Then $(X - F)$ is paraopen and $(X - N)$ is maximal open sets in X . Then by Theorem 2.7(ii) we have $(X - F) \cup (X - N) = X$ or $(X - F) \subset (X - N)$ which implies $X - (F \cap N) = X$ or $N \subset F$. Therefore $F \cap N = \phi$ or $N \subset F$.

ii) Let F be a paraclosed and M be a maximal closed sets in X . Then $(X - F)$ is

paraopen and $(X - M)$ is minimal open sets in X . Then by Theorem 2.7(i) we have $(X - F) \cap (X - M) = \phi$ or $(X - M) \subset (X - F)$ which implies $X - (F \cup M) = \phi$ or $F \subset M$. Therefore $F \cup M = X$ or $F \subset M$.

iii) Let U and V be paraclosed sets in X . If $U \cap V$ is a paraclosed set then there is nothing to prove. Suppose $U \cap V$ is not a paraclosed set. Then by definition, $U \cap V$ is a minimal closed or maximal closed set. If $U \cap V$ is a minimal closed set then there is nothing to prove. Suppose $U \cap V$ is a maximal closed set. Now $U \subset U \cap V$ and $V \subset U \cap V$ which contradicts the fact that U and V are paraclosed sets. Therefore $U \cap V$ is not a maximal closed set. That is $U \cap V$ must be a minimal closed set. \square

3. Paracontinuous Maps and Some of Their Properties

Definition 3.1. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is called

- i) paracontinuous (briefly p_a -continuous) if $f^{-1}(U)$ is an open set in X for every paraopen set U in Y .
- ii) *-paracontinuous (briefly $*-p_a$ -continuous) if $f^{-1}(U)$ is paraopen set in X for every open set U in Y .
- iii) parairresolute (briefly p_a -irresolute) if $f^{-1}(U)$ is paraopen set in X for every paraopen set U in Y .
- iv) minimal paracontinuous (briefly min- p_a -continuous) if $f^{-1}(M)$ is paraopen set in X for every minimal open set M in Y .
- v) maximal paracontinuous (briefly max- p_a -continuous) if $f^{-1}(M)$ is paraopen set in X for every maximal open set M in Y .

Theorem 3.2. *Every continuous map is paracontinuous but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a continuous map. To prove f is paracontinuous. Let U be any paraopen set in Y . Since every paraopen set is an open set, U is an open set in Y . Since f is continuous, $f^{-1}(U)$ is an open set in X . Hence f is a paracontinuous. \square

Example 3.3. Let $X = Y = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mu = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is paracontinuous but it is not a continuous map, since for the open set $\{b\}$ in Y , $f^{-1}(\{b\}) = \{b\}$ which is not an open set in X .

Theorem 3.4. *Every *-paracontinuous map is continuous map but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a *-paracontinuous map. To prove f is continuous map. Let U be an open set in Y . Since f is *-paracontinuous, $f^{-1}(U)$ is a paraopen set in X . Since every paraopen set is an open set, $f^{-1}(U)$ is an open set in Y . Hence f is a continuous map. \square

Example 3.5. Let $X = Y = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mu = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is

a continuous map but it is not a $*$ -paracontinuous map, since for the open set $\{a\}$ in Y , $f^{-1}(\{a\}) = \{a\}$ which is not a paraopen set in X .

Theorem 3.6. *Every $*$ -paracontinuous map is paracontinuous map but not conversely.*

Proof. The proof follows from the Theorems 3.2 and 3.4. \square

Example 3.7. In Example 3.5, f is a paracontinuous map but it is not a $*$ -paracontinuous map.

Theorem 3.8. *Every parairresolute map is paracontinuous map but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a parairresolute map. To prove f is paracontinuous map. Let U be any paraopen set in Y . Since f is parairresolute, $f^{-1}(U)$ is a paraopen set in X . Since every paraopen set is an open set, $f^{-1}(U)$ is an open set in X . Hence f is a paracontinuous map. \square

Example 3.9. Let $X = Y = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a, b\}, \{a, b, c\}, X\}$ and $\mu = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is a paracontinuous map but it is not a parairresolute map, since for the paraopen set $\{a, b\}$ in Y , $f^{-1}(\{a, b\}) = \{a, b\}$ which is not a paraopen set in X .

Theorem 3.10. *Every $*$ -paracontinuous map is parairresolute map but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a $*$ -paracontinuous map. To prove f is parairresolute map. Let U be any paraopen set in Y . Since every paraopen set is an open set, U is an open set in Y . Since f is $*$ -paracontinuous, $f^{-1}(U)$ is a paraopen set in X . Hence f is a parairresolute map. \square

Example 3.11. In Example 3.5, f is a parairresolute map but it is not a $*$ -paracontinuous map.

Remark 3.12. Parairresolute and continuous maps are independent of each other.

Example 3.13. Let $X = Y = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a\}, \{a, c\}, \{a, b, c\}, X\}$ and $\mu = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f : X \rightarrow Y$ be a map defined by $f(a) = a$, $f(b) = a$, $f(c) = c$ and $f(d) = d$. Then f is a continuous map but it is not a parairresolute map, since for the paraopen set $\{a, b\}$ in Y , $f^{-1}(\{a, b\}) = \{a\}$ which is not a paraopen set in X . In Example 3.3, f is a parairresolute map but it is not continuous map.

Theorem 3.14. *Every minimal paracontinuous map is minimal continuous but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a minimal paracontinuous map. To prove f is minimal continuous. Let N be any minimal open set in Y . Since f is minimal paracontinuous, $f^{-1}(N)$ is a paraopen set in X . Since every paraopen set is an open set, $f^{-1}(N)$ is an open set in X . Hence f is a minimal continuous. \square

Example 3.15. In Example 3.5, f is a minimal continuous but it is not a minimal paracontinuous, since for the minimal open set $\{a\}$ in Y , $f^{-1}(\{a\}) = \{a\}$ which is not a paraopen set in X .

Remark 3.16. Minimal paracontinuous and paracontinuous (resp. continuous) maps are independent of each other.

Example 3.17. Let $X = Y = \{a, b, c, d, e\}$ be with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$, and $\mu = \{\phi, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, Y\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is a minimal paracontinuous but it is not a paracontinuous (resp. continuous), since for the paraopen (resp. open) set $\{a, b, d\}$ in Y , $f^{-1}(\{a, b, d\}) = \{a, b, d\}$ which is not an open set in X . In Example 3.5, f is a paracontinuous (resp. continuous) but it is not minimal paracontinuous.

Theorem 3.18. *Every maximal paracontinuous map is maximal continuous but not conversely.*

Proof. Let $f : X \rightarrow Y$ be a maximal paracontinuous map. To prove f is maximal continuous. Let M be any maximal open set in Y . Since f is maximal paracontinuous, $f^{-1}(M)$ is a paraopen set in X . Since every paraopen set is an open set, $f^{-1}(M)$ is an open set in X . Hence f is a maximal continuous.

Example 3.19. In Example 3.5, f is a maximal continuous but it is not maximal paracontinuous, since for the maximal open set $\{a, b, c\}$ in Y , $f^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not a paraopen set in X .

Remark 3.20. Maximal paracontinuous and paracontinuous (resp. continuous) maps are independent of each other.

Example 3.21. Let $X = Y = \{a, b, c, d, e\}$ be with $\tau = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$, and $\mu = \{\phi, \{a\}, \{a, c\}, \{a, b, c\}, Y\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is a maximal paracontinuous map but it is not a paracontinuous (resp. continuous), since for the paraopen (resp. open) set $\{a, c\}$ in Y , $f^{-1}(\{a, c\}) = \{a, c\}$ which is not an open set in X . In Example 3.5, f is a paracontinuous (resp. continuous) but it is not a maximal paracontinuous.

Remark 3.22. Minimal paracontinuous and Maximal paracontinuous maps are independent of each other.

Example 3.23. In Example 3.17, f is a minimal paracontinuous but it is not a maximal paracontinuous. In Example 3.21, f is a maximal paracontinuous but it is not a minimal paracontinuous.

Theorem 3.24. *Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a paracontinuous if and only if the inverse image of each paraclosed set in Y is a closed set in X .*

Proof. The proof follows from the definition and fact that the complement of paraopen set is paraclosed set. \square

Theorem 3.25. *Let X and Y be topological spaces and A be a nonempty subset of X . If $f : X \rightarrow Y$ is paracontinuous then the restriction map $f_A : A \rightarrow Y$ is a paracontinuous.*

Proof. Let $f : X \rightarrow Y$ be a paracontinuous map and $A \subset X$. To prove f_A is a paracontinuous. Let U be a paraopen set in Y . Since f is paracontinuous, $f^{-1}(U)$ is an open set in X . By definition of relative topology $f_A^{-1}(U) = A \cap f^{-1}(U)$. Therefore $A \cap f^{-1}(U)$ is an open set in A . Hence f_A is a paracontinuous. \square

Remark 3.26. The composition of paracontinuous maps need not be paracontinuous.

Example 3.27. Let $X = Y = Z = \{a, b, c, d, e\}$ be with $\tau = \{\phi, \{a, b, c\}, X\}$, $\mu = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, Y\}$ and $\eta = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Z\}$. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are identity maps. Then f and g are paracontinuous maps but $g \circ f : X \rightarrow Z$ is not a paracontinuous, since for the paraopen set $\{a, b\}$ in Z , $(g \circ f)^{-1}(\{a, b\}) = \{a, b\}$ which is not an open set in X .

Theorem 3.28. *If $f : X \rightarrow Y$ is continuous and $g : Y \rightarrow Z$ is paracontinuous maps. Then $g \circ f : X \rightarrow Z$ is a paracontinuous.*

Proof. Let U be any paraopen set in Z . Since g is paracontinuous, $g^{-1}(U)$ is an open set in Y . Again since f is continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is an open set in X . Hence $g \circ f$ is a paracontinuous. \square

Theorem 3.29. *Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a $*$ -paracontinuous if and only if the inverse image of each closed set in Y is a paraclosed set in X .*

Proof. The proof follows from the definition and fact that the complement of paraopen set is paraclosed set. \square

Remark 3.30. Let X and Y be topological spaces. If $f : X \rightarrow Y$ is $*$ -paracontinuous then $f_A : A \rightarrow Y$ is need not be a $*$ -paracontinuous.

Example 3.31. Let $X = Y = \{a, b, c, d\}$ be with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mu = \{\phi, \{a, b\}, Y\}$. Let $A = \{a, c, d\}$ be with $\tau_A = \{\phi, \{a\}, \{a, c\}, A\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is a $*$ -paracontinuous map but $f_A : A \rightarrow Y$ is not a $*$ -paracontinuous, since for the open set $\{a, b\}$ in Y , $f_A^{-1}(\{a, b\}) = A \cap \{a, b\} = \{a\}$ which is not a paraopen set in A .

Theorem 3.32. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are $*$ -paracontinuous maps. Then $g \circ f : X \rightarrow Z$ is a $*$ -paracontinuous.*

Proof. Let U be an open set in Z . Since g is $*$ -paracontinuous, $g^{-1}(U)$ is a paraopen set in Y . Since every paraopen set is an open, $g^{-1}(U)$ is an open set in Y . Again since f is $*$ -paracontinuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a paraopen set in X . Hence $g \circ f$ is a $*$ -paracontinuous. \square

Theorem 3.33. *If $f : X \rightarrow Y$ is paracontinuous and $g : Y \rightarrow Z$ is $*$ -paracontinuous maps. Then $g \circ f : X \rightarrow Z$ is a paracontinuous (resp. continuous) map.*

Proof. Let U be any paraopen (resp. open) set in Z . Since every paraopen set is an open set, U is an open set in Z . Since g is $*$ -paracontinuous, $g^{-1}(U)$ is a paraopen set in Y . Again since f is paracontinuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is an open set in X . Hence $g \circ f$ is a paracontinuous (resp. continuous) map. \square

Theorem 3.34. *Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a parairresolute if and only if the inverse image of each paraclosed set in Y is paraclosed set in X .*

Proof. The proof follows from the definition and fact that the complement of paraopen set is paraclosed set. \square

Remark 3.35. Let X and Y be topological spaces. If $f : X \rightarrow Y$ is a parairresolute then $f_A : A \rightarrow Y$ is need not be a parairresolute.

Example 3.36. In Example 3.5, let $A = \{a, c, d\}$ be with $\tau_A = \{\phi, \{a\}, \{a, c\}, A\}$. Let $f : X \rightarrow Y$ be an identity map. Then f is a parairresolute map but $f_A : A \rightarrow Y$ is not a parairresolute, since for the paraopen set $\{a, b\}$ in Y , $f_A^{-1}(\{a, b\}) = A \cap \{a, b\} = \{a\}$ which is not a paraopen set in A .

Theorem 3.37. *If $f : X \rightarrow Y$ is paracontinuous and $g : Y \rightarrow Z$ is parairresolute maps. Then $g \circ f : X \rightarrow Z$ is a paracontinuous map.*

Proof. Let U be any paraopen set in Z . Since g is parairresolute, $g^{-1}(U)$ is a paraopen set in Y . Again since f is paracontinuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is an open set in X . Hence $g \circ f$ is a paracontinuous map. \square

Theorem 3.38. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are parairresolute maps. Then $g \circ f : X \rightarrow Z$ is a parairresolute map.*

Proof. Let U be any paraopen set in Z . Since g is parairresolute, $g^{-1}(U)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a paraopen set in X . Hence $g \circ f$ is a parairresolute map. \square

Theorem 3.39. *If $f : X \rightarrow Y$ is $*$ -paracontinuous and $g : Y \rightarrow Z$ is parairresolute maps. Then $g \circ f : X \rightarrow Z$ is a parairresolute map.*

Proof. Let U be any paraopen set in Z . Since g is parairresolute, $g^{-1}(U)$ is a paraopen set in Y . Since every paraopen set is an open, we have $g^{-1}(U)$ is an open set in Y . Again since f is $*$ -paracontinuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a paraopen set in X . Hence $g \circ f$ is a parairresolute map. \square

Theorem 3.40. *If $f : X \rightarrow Y$ is parairresolute and $g : Y \rightarrow Z$ is $*$ -paracontinuous maps. Then $g \circ f : X \rightarrow Z$ is a parairresolute map.*

Proof. Let U be any paraopen set in Z . Since every paraopen set is an open set, U is an open set in Z . Since g is $*$ -paracontinuous, $g^{-1}(U)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a paraopen set in X . Hence $g \circ f$ is a parairresolute map. \square

Theorem 3.41. *Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a minimal paracontinuous if and only if the inverse image of each maximal closed set in Y is*

a paraclosed set in X .

Proof. The proof follows from the definition and fact that the complement of minimal open set is maximal closed set and the complement of paraopen set is paraclosed set.

Remark 3.42. The composition of minimal paracontinuous maps need not be a minimal paracontinuous.

Example 3.43. Let $X = Y = Z = \{a, b, c, d, e\}$ be with $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$, $\mu = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, Y\}$ and $\eta = \{\phi, \{a, b, c\}, \{a, b, c, d\}, Z\}$. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be identity maps. Then f and g are minimal paracontinuous maps but $g \circ f : X \rightarrow Z$ is not a minimal paracontinuous, since for the minimal open set $\{a, b, c\}$ in Z , $(g \circ f)^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not a paraopen set in X .

Theorem 3.44. *If $f : X \rightarrow Y$ is parairresolute and $g : Y \rightarrow Z$ is minimal paracontinuous maps, then $g \circ f : X \rightarrow Z$ is a minimal paracontinuous.*

Proof. Let N be any minimal open set in Z . Since g is minimal paracontinuous, $g^{-1}(N)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is a paraopen set in X . Hence $g \circ f$ is a minimal paracontinuous. \square

Theorem 3.45. *If $f : X \rightarrow Y$ is paracontinuous and $g : Y \rightarrow Z$ is minimal paracontinuous maps, then $g \circ f : X \rightarrow Z$ is a minimal continuous.*

Proof. Let N be any minimal open set in Z . Since g is minimal paracontinuous, $g^{-1}(N)$ is a paraopen set in Y . Again since f is paracontinuous, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is an open set in X . Hence $g \circ f$ is a minimal paracontinuous. \square

Theorem 3.46. *If $f : X \rightarrow Y$ is parairresolute and $g : Y \rightarrow Z$ is $*$ -paracontinuous maps, then $g \circ f : X \rightarrow Z$ is a minimal paracontinuous.*

Proof. Let N be any minimal open set in Z . Since every minimal open set is an open set, N is an open set in Z . Since g is $*$ -paracontinuous, $g^{-1}(N)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is a paraopen set in X . Hence $g \circ f$ is a minimal paracontinuous. \square

Theorem 3.47. *Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a maximal paracontinuous if and only if the inverse image of each minimal closed set in Y is paraclosed set in X .*

Proof. The proof follows from the definition and fact that the complement of maximal open set is minimal closed set and the complement of paraopen set is paraclosed set. \square

Remark 3.48. The composition of maximal paracontinuous maps need not be a maximal paracontinuous.

Example 3.49. Let $X = Y = \{a, b, c, d, e\}$ be with $\tau = \{\phi, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}, X\}$, $\mu = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$ and $\eta = \{\phi, \{a\}, \{a, b\}, Z\}$. Let

$f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are identity maps. Then f and g are maximal paracontinuous maps but $g \circ f : X \rightarrow Z$ is not maximal paracontinuous, since for the maximal open set $\{a, b\}$ in Z , $(g \circ f)^{-1}(\{a, b\}) = \{a, b\}$ which is not a paraopen set in X .

Theorem 3.50. *If $f : X \rightarrow Y$ is parairresolute and $g : Y \rightarrow Z$ is maximal paracontinuous maps. Then $g \circ f : X \rightarrow Z$ is a maximal paracontinuous.*

Proof. Let M be any maximal open set in Z . Since g is maximal paracontinuous, $g^{-1}(M)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(M)) = (g \circ f)^{-1}(M)$ is a paraopen set in X . Hence $g \circ f$ is a maximal paracontinuous. \square

Theorem 3.51. *If $f : X \rightarrow Y$ is paracontinuous and $g : Y \rightarrow Z$ is maximal paracontinuous maps, then $g \circ f : X \rightarrow Z$ is a maximal continuous.*

Proof. Let M be any maximal open set in Z . Since g is maximal paracontinuous, $g^{-1}(M)$ is a paraopen set in Y . Again since f is paracontinuous, $f^{-1}(g^{-1}(M)) = (g \circ f)^{-1}(M)$ is an open set in X . Hence $g \circ f$ is a maximal continuous. \square

Theorem 3.52. *If $f : X \rightarrow Y$ is parairresolute and $g : Y \rightarrow Z$ is *-paracontinuous maps, then $g \circ f : X \rightarrow Z$ is a maximal paracontinuous.*

Proof. Let M be any maximal open set in Z . Since every maximal open set is an open set, M is an open set in Z . Since g is *-paracontinuous, $g^{-1}(M)$ is a paraopen set in Y . Again since f is parairresolute, $f^{-1}(g^{-1}(M)) = (g \circ f)^{-1}(M)$ is a paraopen set in X . Hence $g \circ f$ is a maximal paracontinuous. \square

References

- [1] S. S. Benchalli, Basavaraj M. Ittanagi and R. S. Walli, *On minimal open sets and maps in topological spaces*, Journal of Computer and Mathematical Sciences, **2(2)**(2011), 208–220.
- [2] F. Nakaoka and N. Oda, *Some Applications of Minimal Open Sets*, Interident national Journal of Mathematics and Mathematical Sciences, **27(8)**(2011), 471–476.
- [3] F. Nakaoka and N. Oda, *Some Properties of Maximal Open Sets*, International Journal of Mathematics and Mathematical Sciences, **21**(2003), 1331–1340.
- [4] F. Nakaoka and N. Oda, *Minimal closed sets and maximal closed sets*, International Journal of Mathematics and Mathematical Sciences, (2006), 1–8.