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Polarized Super Torus

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Abstract. For the complex torus, there is the Riemann condition on the polarization for it to be an abelian variety. We extend this condition on the super torus for super abelian variety.

1. Introduction

A complex torus M is a real torus $T^{2g} = S^1 \times \cdots \times S^1$ ($2g$ times) which has a complex structure. That is expressed as V/Λ, where V is $\mathbb{C}^g(\cong \mathbb{R}^{2g})$ and Λ is an embedded lattice of rank $2g$ such that the quotient space M is compact[5, 6, 8]. Kodaira embedding theorem says that M is algebraic if and only if there exists an integral 2 form ω which is closed and positive of type $(1,1)$. Algebraic torus is also called abelian variety. Such ω can be expressed as

$$\omega = -\sum_{i=1} \delta_i \, dx_i \wedge dx_{i+g}, \text{ with } \delta_i | \delta_{i+1},$$

where δ_i is a positive integer. In this paper we assume $\delta_i = 1$ for all i. The argument for general δ_i is essentially same as $\delta_i = 1$.

Let $\{\lambda_1, \ldots, \lambda_{2g}\}$ be an integral basis for Λ and $\{x_1, \ldots, x_{2g}\}$ is the dual coordinate of V such that $\int_{\lambda_i} dx_j = \delta_{ij}$. Let $\{e_1, \ldots, e_g\}$ be a complex basis for V, where $e_i = \lambda_{g+i}$. Then we have the period matrix Ω of $\Lambda \subset V$, which is $g \times 2g$ matrix such that

$$\lambda_j = \sum_i \Omega_{ij} e_i, \quad i = 1, \ldots, g, \ j = 1, \ldots, 2g.$$

Then we have $\Omega = (T, I_g)$, where T is a $g \times g$ complex matrix and I_g is the identity

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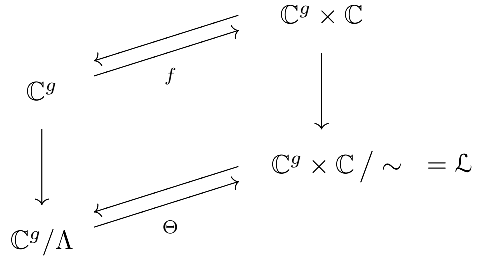
matrix of size g. Then we get

$$dz_i = \sum_j \Omega_{ij}\, dx_j,$$
$$d\bar{z}_i = \sum_j \overline{\Omega}_{ij}\, dx_j,$$

so that the matrix $\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}$ gives the change of basis from $\{dx_j\}$ to $\{dz_i, d\bar{z}_i\}$. Then the Kodaira embedding criteria is just

$$T^t = T,$$
$$\operatorname{Im} T > 0.$$

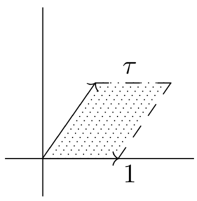
We can find a line bundle $\mathcal{L}$ on M with $c_1(\mathcal{L}) = \omega$. This can be expressed as a diagram



$$\Lambda \cong \mathbb{Z}^{2g} \ (= \mathbb{Z}^g \times T\mathbb{Z}^g)$$

$$T \in M_{g\times g}(\mathbb{C}),$$
$$T^t = T,$$
$$\operatorname{Im} T > 0.$$

For $g = 1$, it is called an elliptic curve M determined by $\{1, \tau\}$,



$$\begin{matrix} & 1 & 1 \\ 1 & \tau & 1 \end{matrix} \ : \ \mathbb{C}/(\mathbb{Z} \times \tau\mathbb{Z}).$$

For higher g, the matrix is as follows,

$$\begin{matrix} & g & g \\ g & T & I \end{matrix} \ : \ \mathbb{C}^g/(\mathbb{Z}^g \times T\mathbb{Z}^g).$$

Here the equivalence $\sim$ is defined as follows. For $n \in \mathbb{Z}^g$, $z \in \mathbb{C}^g$, $v \in \mathbb{C}$,

$$\begin{aligned}(z, v) &\sim (z + n, v)\\ &\sim (z + Tn, e^{-\pi i n^t T n}\, e^{-2\pi i n^t z}\, v).\end{aligned}$$

Then

$$\Theta_T(z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i m^t T m}\, e^{2\pi i m^t z}$$

satisfies

$$\begin{aligned}\Theta_T(z + n) &= \Theta_T(z),\\ \Theta_T(z + Tn) &= e^{-\pi i n^t T n}\, e^{-2\pi i n^t z}\, \Theta_T(z).\end{aligned}$$

So, Θ_T is a section of the line bundle $\mathfrak{L}$ on M.

2. Polarized Supertorus

The concept of supergeometry is systematically developed mathematically [2, 3, 4, 7] and is widely used also in physics [1, 9]. In this section, we want to find such a criteria for the polarization in the case of super torus. Let $\mathbb{R}^{2g|2k}$ is a super Euclidean space over $\mathbb{R}$ which has $2g$ even functions and $2k$ odd functions. Even functions are commuting ordinary functions and odd functions are anticommuting functions. Even functions and odd functions are commuting. A super torus $T^{2g|2k}$ is the quotient space $\mathbb{R}^{2g|2k}/\Lambda$, where Λ is an embedded lattice in $\mathbb{R}^{2g|2k}$ of rank $2g$ and the projection of Λ in $\mathbb{R}^{2g}$ is also an embedded lattice of the full rank $2g$.

Let ω be an integral 2 form of even type on the supertorus $T^{2g|2k}$ such that with respect to some coordinates x_i, y_j, η_k, ε_l, ω can be expressed as

$$\omega = -\sum_{i=1}^{g} dx_i \wedge dy_i + \sum_{j=1}^{k} (d\eta_j \otimes d\eta_j + d\varepsilon_j \otimes d\varepsilon_j),$$

where $\{x_i, y_i\}$ are even functions and $\{\eta_j, \varepsilon_j\}$ are odd functions. We want to find a complex structure on $T^{2g|2k}$, by setting

$$\begin{aligned}z^t &= x^t \cdot T - \eta^t \cdot \nabla + y^t \cdot I_g - \varepsilon^t \cdot 0\\ \theta^t &= x^t \cdot \Delta + \eta^t \cdot S + y^t \cdot 0 + \varepsilon^t \cdot I_k,\end{aligned}$$

where $z \in \mathbb{C}^g$, $\theta \in \mathbb{C}^k$, $T \in M_{g\times g}(\mathbb{C})$, $S \in M_{k\times k}(\mathbb{C})$, $\Delta \in M_{g\times k}(\mathbb{C})$, $\nabla \in M_{k\times g}(\mathbb{C})$, and I_g, I_k are identity matrices of size g and k. Note that Δ and ∇ are odd elements. Then

$$\begin{aligned}dz &= T^t\, dx + \nabla^t\, d\eta + I\, dy + 0\, d\varepsilon\\ d\theta &= \Delta^t\, dx + S^t\, d\eta + 0\, dy + I\, d\varepsilon\\ d\bar{z} &= \overline{T}^t\, dx + \overline{\nabla}^t\, d\eta + I\, dy + 0\, d\varepsilon\\ d\bar{\theta} &= \overline{\Delta}^t dx + \bar{S}^t\, d\eta + 0\, dy + I\, d\varepsilon.\end{aligned}$$

It can be written by matrix form

$$\begin{pmatrix} dz \\ d\theta \\ d\bar{z} \\ d\bar{\theta} \end{pmatrix} = \begin{pmatrix} \Omega \\ \hline \overline{\Omega} \end{pmatrix} \begin{pmatrix} dx \\ d\eta \\ dy \\ d\varepsilon \end{pmatrix},$$

where

$$\widetilde{\Omega} = \begin{pmatrix} \Omega \\ \hline \overline{\Omega} \end{pmatrix} = \begin{pmatrix} T^t & \nabla^t & I & 0 \\ \Delta^t & S^t & 0 & I \\ \overline{T}^t & \overline{\nabla}^t & I & 0 \\ \overline{\Delta}^t & \overline{S}^t & 0 & I \end{pmatrix}.$$

Here we use the supertranspose of a supermatrix $\begin{pmatrix} A & C \\ D & B \end{pmatrix}$ with A, B even and C, D odd,

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}^{st} = \begin{pmatrix} A^t & -D^t \\ C^t & B^t \end{pmatrix},$$

where A^t is the ordinary transpose of A. We also deal with x, y, z, $d\eta$, $d\varepsilon$, $d\theta$ as even variables and η, ε, θ, dx, dy, dz as odd variables. Conversely we can write

$$\begin{pmatrix} dx \\ d\eta \\ dy \\ d\varepsilon \end{pmatrix} = \begin{pmatrix} \widetilde{\Pi} \end{pmatrix} \begin{pmatrix} dz \\ d\theta \\ d\bar{z} \\ d\bar{\theta} \end{pmatrix} = \begin{pmatrix} \Pi & \overline{\Pi} \end{pmatrix} \begin{pmatrix} dz \\ d\theta \\ d\bar{z} \\ d\bar{\theta} \end{pmatrix}.$$

Then $\widetilde{\Omega}\widetilde{\Pi} = I$. Here

$$\omega = (-dx^t, d\eta^t, -dy^t, d\varepsilon^t) \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} dx \\ d\eta \\ dy \\ d\varepsilon \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix}.$$

Then

$$\omega = \left(-dz^t, d\theta^t, -d\bar{z}^t, d\bar{\theta}^t\right) \widetilde{\Pi}^{st} Q \widetilde{\Pi} \begin{pmatrix} dz \\ d\theta \\ d\bar{z} \\ d\bar{\theta} \end{pmatrix}.$$

Since ω is positive of $(1,1)$ type, we have

$$\widetilde{\Pi}^{st} Q \overline{\widetilde{\Pi}} = \left(\begin{array}{c} \Pi^{st} \\ \hline \overline{\Pi}^{st} \end{array}\right) \left(\begin{array}{c} Q \end{array}\right) \left(\begin{array}{c|c} \overline{\Pi} & \Pi \end{array}\right) = \left(\begin{array}{c|c} H & 0 \\ \hline 0 & -H^{st} \end{array}\right),$$

where iH is positive. Then its inverse is

$$\left(\begin{array}{c} \overline{\Omega} \\ \hline \Omega \end{array}\right) \left(\begin{array}{c} Q^{-1} \end{array}\right) \left(\begin{array}{c|c} \Omega^{st} & \overline{\Omega}^{st} \end{array}\right) = \left(\begin{array}{c|c} H^{-1} & 0 \\ \hline 0 & -H^{-st} \end{array}\right),$$

where $-iH^{-1}$ is also positive. Here

$$\begin{aligned}
&\left(\begin{array}{c} \overline{\Omega} \\ \hline \Omega \end{array}\right) \left(\begin{array}{c} Q^{-1} \end{array}\right) \left(\begin{array}{c|c} \Omega^{st} & \overline{\Omega}^{st} \end{array}\right) \\
&= \begin{pmatrix} \overline{T}^t & \overline{\nabla}^t & I & 0 \\ \overline{\Delta}^t & \overline{S}^t & 0 & I \\ T^t & \nabla^t & I & 0 \\ \Delta^t & S^t & 0 & I \end{pmatrix} \begin{pmatrix} & & -1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} T & -\Delta & \overline{T} & -\overline{\Delta} \\ \nabla & S & \overline{\nabla} & \overline{S} \\ I & 0 & I & 0 \\ 0 & I & 0 & I \end{pmatrix} \\
&= \left(\begin{array}{cc|cc} T - \overline{T}^t + \overline{\nabla}^t \nabla, & -\Delta + \overline{\nabla}^t S & \overline{T} - \overline{T}^t + \overline{\nabla}^t \overline{\nabla}, & -\overline{\Delta} + \overline{\nabla}^t \overline{S} \\ -\overline{\Delta}^t + \overline{S}^t \nabla, & \overline{S}^t S + I & -\overline{\Delta}^t + \overline{S}^t \overline{\nabla}, & \overline{S}^t \overline{S} + I \\ \hline T - T^t + \nabla^t \nabla, & -\Delta + \nabla^t S & \overline{T} - T^t + \nabla^t \overline{\nabla}, & -\overline{\Delta} + \nabla^t \overline{S} \\ -\Delta^t + S^t \nabla, & S^t S + I & -\Delta^t + S^t \overline{\nabla}, & S^t \overline{S} + I \end{array}\right).
\end{aligned}$$

Since ω is of $(1,1)$ type, we have

$$\begin{gathered} T - T^t + \nabla^t \nabla = 0, \\ S^t S + I = 0, \end{gathered}$$

and

$$-\Delta^t + S^t \nabla = 0.$$

Since ω is positive, we have

$$-i \begin{pmatrix} T - \overline{T}^t + \overline{\nabla}^t \nabla & -\Delta + \overline{\nabla}^t S \\ -\overline{\Delta}^t + \overline{S}^t \nabla & \overline{S}^t S + I \end{pmatrix} > 0.$$

So, we have proved the following theorem.

Theorem.

$$H_{g,k} = \left\{ \begin{pmatrix} T & \Delta \\ \nabla & S \end{pmatrix} \middle| \begin{array}{l} T \in M_{g\times g}(\mathbb{C}), S \in M_{k\times k}(\mathbb{C}), \\ \Delta \in M_{g\times k}(\mathbb{C}), \nabla \in M_{k\times g}(\mathbb{C}), \\ T - T^t + \nabla^t \nabla = 0, S^t S + I = 0, -\Delta^t + S^t \nabla = 0, \\ -i \begin{pmatrix} T - \overline{T}^t + \overline{\nabla}^t \nabla & -\Delta + \overline{\nabla}^t S \\ -\overline{\Delta}^t + \overline{S}^t \nabla & \overline{S}^t S + I \end{pmatrix} > 0 \end{array} \right\}$$

is the period domain of the principally polarized super abelian variety of even type for super torus $T^{2g|2k}$. Here T, S are even elements and Δ, ∇ are odd elements.

As in the case of ordinary torus, we can define a line bundle on the supertorus, where there is no ∇ and S, whose section is of the form[2]

$$\Theta_{T,\Delta}(z,\theta) = \prod_{\alpha} \left(\theta_\alpha + \frac{1}{2\pi i} \Delta_\alpha^j \frac{\partial}{\partial z^j} \right) \Theta_T(z).$$

Then for $n \in \mathbb{Z}^g$,

$$\Theta_{T,\Delta}\left(z+n,\theta\right) = \Theta_{T,\Delta}\left(z,\theta\right),$$
$$\Theta_{T,\Delta}\left(z+Tn,\theta+\Delta^t n\right) = e^{-\pi i n^t T n}\, e^{-2\pi i n^t z}\, \Theta_{T,\Delta}\left(z,\theta\right).$$

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